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CAN A FINITE DEGENERATE 'STRING' HEAR ITSELF? NUMERICAL SOLUTIONS TO A SIMPLIFIED IBVP

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Abstract. Some discrete models for a simplified (compared to that published earlier in JODEA, **28** (1) (2020), 1 − 42) initial boundary value problem for a 1D linear degenerate wave equation, posed in a space-time rectangle and solved earlier exactly (JODEA, **30** (1) (2022), 89 − 121), have been considered. It has been demonstrated that the correct evaluation of the degenerate grid flux can be possible.

Key words: degenerate wave equation, vibrating string, separation of variables, transmission condition, travelling wave, degenerate flux.

2010 Mathematics Subject Classification: 35C10, 35L05, 35L10, 35L20, 35L80.

1. Introduction and the problem formulation

The current study complementes [5], dealing with the following 1-parameter simplified initial boundary value problem (IBVP) for the degenerate wave equation, posed in the space-time rectangle $[0,T]\times[-1,+1]\subset\mathbb{R}_t^+\times\mathbb{R}_x$ wrt $u(t,x;\alpha)$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right)=0\,, & (t,|x|)\in(0,T]\times(0,1)\,,\\ \left.\begin{aligned}\dfrac{\partial u(0,x;\alpha)}{\partial t}&=\overset{**}{u}(x;\alpha)\\ u(0,x;\alpha)&=\overset{*}{u}(x;\alpha)\end{aligned}\right\}, & x\in[-1,+1]\,,\\ \left.\begin{aligned}u(t,-1;\alpha)&=h_1(t;\alpha)\\ u(t,+1;\alpha)&=h_2(t;\alpha)\end{aligned}\right\}, & t\in[0,T]\,,\end{cases} \tag{1.1}$$

where known control functions $h_{1,2}(t;\alpha)\in\mathscr{C}^1[0,T]\bigcap\mathscr{C}^2(0,T]$ obey the compatibility conditions: $h_1(0;\alpha)=\overset{*}{u}(-1;\alpha)$, $h_1'(0;\alpha)=\overset{**}{u}(-1;\alpha)$, $h_2(0;\alpha)=\overset{*}{u}(+1;\alpha)$, $h_2'(0;\alpha)=\overset{**}{u}(+1;\alpha)$, and the 1-parameter family of coefficient functions is defined as follows

$$a(x;\alpha)=|x|^\alpha,\qquad x\in[-1,+1]\,, \tag{1.2}$$

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the parameter of degeneracy $\alpha \in (0,2)$, and all the variables are nondimensional. The point $x=0$, where the coefficient (1.2) vanishes, is referred below to as the degeneracy point, whereas $[0,T]\times[-1,+1]\supset[0,T]\times\{0\}$ is referred to as the degeneracy segment, or the dividing segment of the space-time rectangle. Dealing with (1.1), (1.2), we distinguish between the cases of: 1) weak degeneracy, $\alpha\in(0,1)$, 2) strong degeneracy, $\alpha\in(1,2)$, and 3) non-degeneracy, $\alpha=0$ (the limiting case).

The following matching conditions must be imposed on the required solution to the IBVP at the degeneracy segment

$$\begin{cases} u(t,x;\alpha)\big|_{x=0-0} = u(t,x;\alpha)\big|_{x=0+0}\,, \\ f(t,x;\alpha)\big|_{x=0-0} = f(t,x;\alpha)\big|_{x=0+0}\,, \end{cases} \qquad t\in[0,T]\,. \tag{1.3}$$

where there is used a notion of the flux

$$f(t,x;\alpha) = a(x;\alpha)\,\frac{\partial u(t,x;\alpha)}{\partial x}\,. \tag{1.4}$$

The exact series solution to the IBVP (1.1) was obtained in [5], therefore we follow the notations, terminology and even an analogy of the problem and its solution to an imaginary 'string', wherever it is convenient.

In the current study our concern is numerical solving the IBVP (1.1). Formally, a proper grid approximation of the IBVP is not a problem. Nevertheless, any attempt to implement directly a numerical procedure to the IBVP involves a bulk of nested problems having relation to evaluating the flux (1.4) at the degeneracy, segment, where the flux degenerates. It was proved [2, 4, 5], using series solutions to the IBVP and to the degenerate wave equation alone, that the flux at the degeneracy segment does not vanish and is continuous. From this it immediately appears a problem to retain the above properties for the grid flux. Note, that in our previous study [1] we discussed some auxiliary problems arising in computational procedures applied to the IBVP. For example, it was attempted to introduce a regularization of the IBVP, unfortunately the convergence of the numerical solutions to the regularized problem was found not to exist.

The goal of the current study is:

1) to complement our previous study [5] in terms of suitable Bessel functions being linearly independent (refer to Sect.2);

2) to demonstrate that correct evaluating the degenerate flux on the grid is possible (refer to Sects. 3, 4, 5).

2. Some notes on separation of variables

Separation of variables applied to the original IBVP (1.1) is known [2–5] to involve us into solving the following boundary-value problem

$$\begin{cases} D'(x;\alpha) + \lambda(\alpha)\,X(x;\alpha) = 0\,, \qquad 0<|x|<1\,, \\ a)\ X(\mp1;\alpha)=0\,, \qquad b)\ X(x;\alpha)\big|_{x=0-0} = X(x;\alpha)\big|_{x=0+0}\,, \\ \qquad\qquad\qquad\qquad c)\ D(x;\alpha)\big|_{x=0-0} = D(x;\alpha)\big|_{x=0+0}\,, \end{cases} \tag{2.1}$$

where $D(x;\alpha)=a(x;\alpha)\,X'(x;\alpha)$ is the flux of the solution $X(x;\alpha)$, referred to as the eigenfunction, whereas $\lambda(\alpha)$ is referred to as the eigenvalue.

To simplify further discussion, it is convenient to introduce the following α-dependent quantities

$$\nu(\alpha) = 1-\alpha\,, \qquad \theta(\alpha) = 2-\alpha\,, \qquad \varrho(\alpha) = \frac{\nu}{\theta} = \frac{1-\alpha}{2-\alpha}\,, \tag{2.2}$$

then in the case of weak degeneracy: 1) the eigenvalues $\lambda_{k,\mu}(\alpha)$ and the eigenfunctions $X_{k,\mu}(x;\alpha)$ of the problem (2.1) of the two kinds (marked with $k\in\{1,2\}$) are defined as follows

$$\begin{cases} \lambda_{1,\mu}(\alpha) = \left(\dfrac{\theta}{2}\, s_{1,\mu}\right)^2, & X_{1,\mu}(x;\alpha) = \qquad Z_{1,\mu}(x;\alpha), \\[2ex] \lambda_{2,\mu}(\alpha) = \left(\dfrac{\theta}{2}\, s_{2,\mu}\right)^2, & X_{2,\mu}(x;\alpha) = \operatorname{sgn} x\; Z_{2,\mu}(x;\alpha), \end{cases} \tag{2.3}$$

where $\varrho\notin\mathbb{Z}$, $\{s_{k,\mu}\}_{\mu=1}^{\infty}$ are the unbounded monotonically increasing sequences of the zeros of the linearly independent Bessel functions $\mathtt{J}_{\mp\varrho}(s)$ of the first kind and orders $\mp\varrho$ [7], and

$$\begin{cases} Z_{1,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\; \mathtt{J}_{-\varrho}\!\left(s_{1,\mu}\,|x|^{\frac{\theta}{2}}\right), \\[2ex] Z_{2,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\; \mathtt{J}_{+\varrho}\!\left(s_{2,\mu}\,|x|^{\frac{\theta}{2}}\right). \end{cases} \tag{2.4}$$

The Bessel functions $\mathtt{J}_{\mp\varrho}(s)$ satisfy the ordinary differential equation

$$\mathtt{Z}''_{\mp\varrho}(s) + \frac{1}{s}\,\mathtt{Z}'_{\mp\varrho}(s) - \left(\frac{\varrho^2}{s^2} - 1\right)\mathtt{Z}_{\mp\varrho}(s) = 0 \tag{2.5}$$

and have the following power series representations

$$\mathtt{J}_{\mp\varrho}(s) = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{\gamma!\,\Gamma(1\mp\varrho+\gamma)} \left(\frac{s}{2}\right)^{2\gamma}. \tag{2.6}$$

Now we shortly recall the underlying idea [2] to reduce the governing equation of the BVP (2.1) to the Bessel equation (2.5). To this end, we introduce the following ansatz

$$Z(x;\alpha) = x^{o}\, V_\alpha(r)\,, \qquad r = x^{\omega}, \tag{2.7}$$

where o, ω are undetermined real exponents, and for the sake of brevity we assume that $x>0$. Substituting the ansatz (2.7) and its flux

$$D(x;\alpha) = aZ'(x;\alpha) = o\,x^{o+\alpha-1} V_\alpha(r) + \omega\, x^{o+\omega+\alpha-1} V'_\alpha(r) \tag{2.8}$$

into the equation of the IBVP gives the following relation involving the undetermined exponents and the degeneracy parameter α

$$D' + \lambda Z = \omega^2 x^{o+2\omega+\alpha-2} V_\alpha'' + \omega\,[2o + \omega + \alpha - 1]\, x^{o+\omega+\alpha-2} V_\alpha' + o\,[o + \alpha - 1]\, x^{o+\alpha-2} V_\alpha + \lambda\, x^o\, V_\alpha = 0\,.$$

Dividing by x^o simplifies the above relation to the following one

$$\omega^2 x^{2\omega-\theta} V_\alpha'' + \omega\,[2o + \omega + \alpha - 1]\, x^{\omega-\theta} V_\alpha' + o\,[o + \alpha - 1]\, x^{-\theta} V_\alpha + \lambda\, V_\alpha = 0\,, \quad (2.9)$$

where quantities (2.2) are used. To agree (2.9) with the Bessel equation (2.6), we assume that $2\omega - \theta = 0$, then (2.9) simplifies as follows

$$\left(\frac{\theta}{2}\right)^2 V_\alpha'' + \frac{\theta}{2}\left(2o + \frac{\theta}{2} - \nu\right) x^{-\frac{\theta}{2}} V_\alpha' + o\,(o - \nu)\, x^{-\frac{\theta}{2}2} V_\alpha + \lambda\, V_\alpha = 0\,,$$

and we have to assume that $2o - \nu$ to complete the agreement with the Bessel equation (2.6) in the form

$$\left(\frac{\theta}{2}\right)^2 V_\alpha'' + \left(\frac{\theta}{2}\right)^2 x^{-\frac{\theta}{2}} V_\alpha' - \left(\frac{\nu}{2}\right)^2 x^{-\frac{\theta}{2}2} V_\alpha + \lambda\, V_\alpha = 0\,. \quad (2.10)$$

From (2.10) we immediately find the eigenvalues and the eigenfunctions (2.3), where the functions $Z_{1,\mu}(x;\alpha)$, $Z_{2,\mu}(x;\alpha)$ are defined in (2.4) and are linearly independent, provided that $\varrho \notin \mathbb{Z}$. In the case $\varrho \in \mathbb{Z}$ we should take following pairs of the eigenvalues and the eigenfunctions

$$\lambda_{3,\mu}(\alpha) = \left(\frac{\theta}{2}\, s_{3,\mu}\right)^2, \qquad \lambda_{4,\mu}(\alpha) = \left(\frac{\theta}{2}\, s_{4,\mu}\right)^2, \quad (2.11)$$

$$\begin{cases} Z_{3,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{Y}_\varrho\!\left(s_{3,\mu}\, |x|^{\frac{\theta}{2}}\right), \\ Z_{4,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{J}_\varrho\!\left(s_{4,\mu}\, |x|^{\frac{\theta}{2}}\right), \end{cases} \quad (2.12)$$

where $\mathtt{Y}_\varrho(s)$ are the Bessel functions of the second kind and orders $\mp\varrho$ [7] (referred to as the Neumann functions), $\{s_{k,\mu}\}_{\mu=1}^{\infty}$, $k = 3, 4$, are the unbounded monotonically increasing sequences of the zeros of the linearly independent functions $\mathtt{Y}_\varrho(s)$, $\mathtt{J}_\varrho(s)$, and it is evident that $s_{1,\mu} = s_{3,\mu}$, $\sigma_{1,\mu} = \sigma_{3,\mu}$, $X_{2,\mu} = X_{4,\mu}$.

The above reducing (2.7) to the Bessel equation (2.5) is not valid for the intermediate case $\alpha = 1$, therefore we repeat reducing especially for the case. Again taking the ansatz of the form (2.7)

$$Z(x;1) = x^\sigma\, V_\alpha\,(r)\,, \qquad r = x^\omega, \quad (2.13)$$

calculating its flux

$$D(x;1) = aZ'(x;1) = \sigma\, x^{\sigma-1} V_0\,(r) + \sigma\omega\, x^{\sigma+\omega} V_0'\,(r)\,, \tag{2.14}$$

and substituting into the equation of the IBVP gives the following relation involving the undetermined exponents

$$D' + \lambda X = \omega^2 x^{\sigma+2\omega-1} V_0''\,(r) + \omega\,[2\sigma+\omega]\, x^{\sigma+\omega-1} V_0'\,(r)$$
$$+ \sigma^2 x^{\sigma-1} V_0\,(r) + \lambda x^{\sigma} V_0\,(r) = 0\,.$$

Dividing by x^o yields to the simplified relation

$$\omega^2 x^{2\omega-1} V_0''(r) + \omega\,[2\sigma+\omega]\, x^{\omega-1} V_0'(r) + \sigma^2 x^{-1} V_0(r) + \lambda V_0(r) = 0\,, \tag{2.15}$$

where we have to assume $2\omega - 1 = 0$, $\sigma = 0$, to obtain the required equation

$$\frac{1}{4}\, V_0''(r) + \frac{1}{4}\,\frac{1}{\sqrt{x}}\, V_0'(r) + \lambda V_0(r) = 0\,, \tag{2.16}$$

consistent with the Bessel equation (2.5) of the order zero, provided $s = 2\sqrt{\lambda x}\,$. In this case we have the following pairs of the eigenvalues and the eigenfunctions.

$$\lambda_{5,\mu}(\alpha) = s_{5,\mu}^2\,, \qquad \lambda_{6,\mu}(\alpha) = s_{6,\mu}^2\,, \tag{2.17}$$

$$\begin{cases} Z_{5,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{Y}_0\Big(2\, s_{5,\mu}\, |x|^{\frac{1}{2}}\Big), \\ X_{6,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\, \mathtt{J}_0\Big(2\, s_{6,\mu}\, |x|^{\frac{1}{2}}\Big), \end{cases} \tag{2.18}$$

where notation used is exactly the same as that in (2.11), (2.12).

The Neumann function has a series representation different from that for the Bessel function of the first kind, for example in the case of the zero order it reads

$$\mathtt{Y}_0(s) = \frac{2}{\pi}\Big(C + \ln\frac{s}{2}\Big)\,\mathtt{J}_0(s) - \frac{2}{\pi}\sum_{\gamma=0}^{\infty}\frac{(-1)^{\gamma}\,\Phi(\gamma)}{(\gamma!)^2}\Big(\frac{s}{2}\Big)^{2\gamma}, \quad \Phi(\gamma) = \sum_{\rho=1}^{\gamma}\frac{1}{\rho}\,, \tag{2.19}$$

where $C = 0.5772\ldots$ is the Euler constant and $\Phi(0) = 0$, nevertheless all the properties of the solution expressed in terms of (2.12) are exactly the same as those expressed in terms of (2.4).

3. Discrete formulation of the problem

In order to develop discrete models of the IBVP(1.1), we first introduce an orthogonal grid with space-time nodes $\big(x_k, t^n\big)$, $k = 1, \ldots, K$, $n = 0, \ldots, N$, in rectangle $[-1, +1] \times [0, T]$. Nodes t^n are distributed uniformly on segment $[0, T]$, whereas spatial nodes x_k cluster in some way on segment $[-1,+1]$ towards the midpoint $x=0$. Second, we introduce the following grid operators

$$\begin{aligned} \Delta_k^{\mp} x_k &:= \mp\left(x_{k\mp1} - x_k\right), \\ 2\Delta_k^0 x_k &:= \Delta_k^- x_k + \Delta_k^+ x_k = \left(\Delta_k^- + \Delta_k^+\right) x_k, \\ \Delta_{\mp}^n t^n &:= \mp\left(t^{n\mp1} - t^n\right) \equiv \Delta t. \end{aligned} \tag{3.1}$$

Third, we integrate the degenerate wave equation over the cell, centered at an arbitrary interior node $\left(x_k, t^n\right)$ ($x_{k-h} \leqslant x \leqslant x_{k+h}$, $t^{n-h} \leqslant t^{n+h}$, $h = \frac{1}{2}$) of the grid

$$0 = \iint_{\omega_k^n} \left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right)\right] \mathrm{d}x\,\mathrm{d}t = \int_{x_{k-h}}^{x_{k+h}} \int_{t^{n-h}}^{t^{n+h}} \left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(a\,\frac{\partial u}{\partial x}\right)\right] \mathrm{d}x\,\mathrm{d}t$$

$$= \int_{x_{k-h}}^{x_{k+h}} \left(\frac{\partial u}{\partial t}\right)\Bigg|_{t^{n-h}}^{t^{n+h}} \mathrm{d}x - \int_{t^{n-h}}^{t^{n+h}} \left(a\,\frac{\partial u}{\partial x}\right)\Bigg|_{x_{k-h}}^{x_{k+h}} \mathrm{d}t$$

and evaluate the integrals by applying the midpoint rule of Calculus as follows

$$\Delta_k^0 x_k \left(\frac{\partial u}{\partial t}\right)\Bigg|_{(k,n-h)}^{(k,n+h)} = \Delta t\left(a\,\frac{\partial u}{\partial x}\right)\Bigg|_{(k-h,n)}^{(k+h,n)}, \tag{3.2}$$

where notations $(k, n \mp h) = \left(x_k, t^{n\mp h}\right)$, $(k \mp h, n) = \left(x_{k\mp h}, t^n\right)$ are used for the sake of brevity. Fourth, using spatial averaging, introduce grid functions

$$u_k^n = \frac{1}{\Delta_k^0 x_k} \int_{x_{k-h}}^{x_{k+h}} u(t^n, x; \alpha)\,\mathrm{d}x, \tag{3.3}$$

$$f_{k\mp h}^n = \frac{\mp 1}{\Delta_k^{\mp} x_k} \int_{x_k}^{x_{k\mp h}} a(x;\alpha)\,\frac{\partial u(t^n, x; \alpha)}{\partial x}\,\mathrm{d}x, \tag{3.4}$$

then integration in (3.2) yields to

$$\Delta_k^0 x_k \left(\frac{\Delta_+^n u_k^n}{\Delta t} - \frac{\Delta_-^n u_k^n}{\Delta t}\right) = \Delta t\left(f_{k+h}^n - f_{k-h}^n\right). \tag{3.5}$$

From (3.5) it follows the required explicit computational formula for finding the values of grid function u_k^n at the upper time level $n+1$, being a well known three-layer finite-difference scheme [6]

$$u_k^{n+1} = 2\,u_k^n - u_k^{n-1} + \sigma_k\left(f_{k+h}^n - f_{k-h}^n\right), \qquad \sigma_k = \frac{(\Delta t)^2}{\Delta_k^0 x_k}\,. \tag{3.6}$$

4. Calculation of the fluxes

The inter-cell fluxes, playing the key role in (3.6), can be evaluated in various ways, but the most obvious one reads

$$f^n_{k\mp h} = a_{k\mp h}\,\frac{\Delta^{\mp}_k\, u^n_k}{\Delta^{\mp}_k\, x_k}\,, \tag{4.1}$$

where the inter-cell coefficients $a_{k\mp h}$ are not determined uniquely. For example, they can be directly taken as the inter-cell values of the coefficient function $a(x;\alpha)$, as follows

$$a_{k\mp h} = a(x_{k\mp h};\alpha) = |x_{k\mp h}|^{\alpha}. \tag{4.2}$$

Simplicity of the direct approach is in contrast to nature of the phenomenon being under consideration. Indeed, in original IBVP (1.1), the flux on the degeneracy segment was proved [1, 5] not to vanish, whereas the grid flux computed due to (4.1), (4.2) on the degeneracy segment, vanishes whatever values grid function u^n_k takes.

To overcome this fault of the direct approach due to (4.2), we refer to 'the best scheme' [6]. Following [6], we resolve the definition of the flux(1.4) wrt to 'the string' inclination

$$\frac{\partial u(t,x;\alpha)}{\partial x} = \frac{f(t,x;\alpha)}{a(x;\alpha)}$$

and integrate the above relation over segment $[x_k, x_{k+h}]$ at instant t^n

$$\int_{x_k}^{x_{k+h}} \frac{\partial u(t^n,x;\alpha)}{\partial x}\,\mathrm{d}x = \int_{x_k}^{x_{k+h}} \frac{f(t^n,x;\alpha)}{a(x;\alpha)}\,\mathrm{d}x\,.$$

Applying the fundamental and the midpoint theorems of Calculus to the above relation and dividing both sides of the resulting equality by the length of the segment yields to

$$\frac{u(t^n,x_{k+1};\alpha) - u(t^n,x_k;\alpha)}{\Delta^{+}_k\, x_k} = f(t^n,x_{k+h};\alpha)\frac{1}{\Delta^{+}_k\, x_k}\int_{x_k}^{x_{k+h}} \frac{\mathrm{d}x}{a(x;\alpha)}\,.$$

Comparing the obtained equation with (4.1) prompts the way used in 'the best scheme' to calculate the inter-cell coefficient

$$a_{k\mp h} = \mathring{a}_{k\mp h} = \left[\frac{\mp 1}{\Delta^{\mp}_k x_k}\int_{x_k}^{x_{k\mp h}} \frac{\mathrm{d}x}{a(x;\alpha)}\right]^{-1}. \tag{4.3}$$

In further discussion we will distinguish between the approaches to calculate the inter-cell fluxes outside the degeneracy segment (the regular fluxes, or the fluxes at the regular inter-cells) and exactly on the the degeneracy segment (the degenerate flux).

For example, the first approach, based on (4.2), is applicable only for the regular fluxes, whereas the second one, based on (4.3), is valid for fluxes of both kinds. Unfortunately, the second approach can not be applied in the case of strong degeneracy, therefore we consider some other approaches to evaluate the inter-cell fluxes.

Again, refer to the definition of the flux (1.4), written in its original form at instant t^n, integrate it over the same segment

$$\int_{x_k}^{x_{k+h}} f(t^n, x; \alpha)\,\mathrm{d}x = \int_{x_k}^{x_{k+h}} a(x; \alpha)\,\frac{\partial u(t^n, x; \alpha)}{\partial x}\,\mathrm{d}x\,,$$

apply the midpoint theorem of Calculus, divide both sides of the resulting equality by the length of the segment, and account for (3.4), to obtain

$$f^n_{k+h} = \left(\frac{\partial u(t^n, x; \alpha)}{\partial x}\right)\Bigg|_{x=x_{k+h}} \frac{1}{\Delta^+_k x_k} \int_{x_k}^{x_{k+h}} a(x; \alpha)\,\mathrm{d}x\,.$$

Evaluating the inter-cell inclination of 'the string' similarly to (4.1), we easily obtain one more approach for the inter-cell coefficients

$$a_{k\mp h} = a^*_{k\mp h} = \frac{\mp 1}{\Delta^\mp_k x_k} \int_{x_k}^{x_{k\mp h}} a(x; \alpha)\,\mathrm{d}x\,. \tag{4.4}$$

Other approaches to evaluate the inter-cell fluxes, we are going to discuss, refer to the degenerate flux and do not involve any direct way to calculate the inter-cell coefficient $a_{k\mp h}$. The first group of such approaches utilizes the continuity of the flux across the degenerate segment, for example, the simplest averaging of the regular fluxes calculated at the inter-cells adjacent to the degenerate inter-cell $k+h$

$$f_{k+h} = \frac{1}{2}\left(f_{k-h} + f_{k+3h}\right). \tag{4.5}$$

A more sophisticated approach, utilizing the flux continuity, reads as follows

$$f_{k+h} = \frac{1}{2}\left(f^-_{k+h} + f^+_{k+h}\right), \tag{4.6}$$

where $f^\mp_{k+h}$ are 'one-sided' values of the required degenerate flux, obtained using extrapolation

$$\begin{cases} f^-_{k+h} = f_{k-1h} + (\Delta x)_k \quad D^{-m}_k f_{k-1h}\,, \\ f^+_{k+h} = f_{k+3h} - (\Delta x)_{k+1}\, D^{+m}_k f_{k+3h}\,, \end{cases} \tag{4.7}$$

where $(\Delta x)_k = x_{k+h} - x_{k-h}$, $(\Delta x)_{k+1} = x_{k+1+h} - x_{k+h}$, $D^{\mp m}_k$, $m \geqslant 2$, are one-sided m-nodal grid operators of the first order differentiation (involving the regular grid fluxes calculated at m nodes). For example, the left (or backward) operator reads

$$D^{-m}_k f_{k-1h} = b_{k-1h}\, f_{k-1h} + b_{k-3h}\, f_{k-3h} + \ldots + b_{k-(2m-1)h}\, f_{k-(2m-1)h}\,, \tag{4.8}$$

where the coefficients b_{k-1h}, etc, are undetermined. The proper well-conditioned linear algebraic system $m \times m$ wrt the coefficients can be set up and solved easily.

The second group of approaches utilizes the possibility of building the solution to the original IBVP (1.1) in space-time rectangle $[-1,+1] \times [0,T]$ using matching the solutions to the auxiliary IBVPs posed in subrectangles $[-1,0] \times [0,T]$, $[0,+1] \times [0,T]$. Proper matching may involve other conditions in addition to (1.3). For example, it was shown [2] that the flux across the degenerate segment can be continuously differentiable. This property can be easily implemented to evaluate the degenerate flux. Indeed, using the above one-sided grid operators $D_k^{\mp m}$, we can represent the property as the equality of two one-sided derivatives of the first order at both sides of the degenerate inter-cell

$$D_k^{-m} f_{k+h} = D_k^{+m} f_{k+h}\,, \tag{4.9}$$

involving the required degenerate flux f_{k+1h}. The above equality is nothing but the linear algebraic equation wrt the required flux f_{k+h}.

5. The test case of the problem

To estimate and compare the approaches of Sect. 4 for the flux evaluation, we refer to test case A of [5], as a benchmark. Recall that in that test case: 1) the initially ($t=0$) disturbed 'string' is at rest

$$\overset{**}{u}(x;\alpha) \equiv \overset{**}{u}_0 = 0\,, \quad \overset{*}{u}(x;\alpha) = \begin{cases} 0\,, & \left|x - x_0\right| > \delta\,, \\ \overset{*}{u}_0\,, & \left|x - x_0\right| \leqslant \delta\,, \end{cases} \qquad x \in [-1,+1]\,; \tag{5.1}$$

and 2) both ends of the 'string' are fixed

$$u(-1,t;\alpha) = u(+1,t;\alpha) = 0\,, \qquad t \in [0,T]\,, \tag{5.2}$$

i. e., both controls are not applied: $h_1(t;\alpha) = h_2(t;\alpha) \equiv 0$. The initial step function was smoothed using a mollifier.

To resolve the structure of the grid solutions to the IBVP near the degenerate segment we first introduce the uniform grid on $[-1,+1] \subset \mathbb{R}_\xi$ with spacing $\Delta\xi$ between the nodes, the coordinates ξ_k of the nodes are calculated as follows

$$(K-1)\,\Delta\xi = 2\,, \qquad \xi_k = -1 + (k-1)\,\Delta\xi\,, \qquad k = 1,\ldots,K\,,$$

N being the number of the nodes. In the case of even K, two central nodes are biased wrt the degeneracy point $\xi = 0$ by half of $\Delta\xi$. Second, a nonlinear transformation $\xi \to x$ is applied to calculate the coordinates x_k of the grid nodes on segment $[-1,+1] \subset \mathbb{R}_x$, for the nodes to cluster near the degeneracy point $x = 0$. To obtain the results partially presented below in Figs. 5.1 − 5.6, we assigned the number 2000 to K, and two values 0.25 and 0.75 to the parameter of degeneracy. Exact solutions to the test case A [5] are drawn as dashed lines. We will not give any comments to the behavior of the solution plots, since any curve should be studied individually to evaluate the possibilities of the approaches used to model vibrations of 'the damaged string'. The results, we believe, will be useful to develop proper discrete models for the case of strong degeneracy as well.

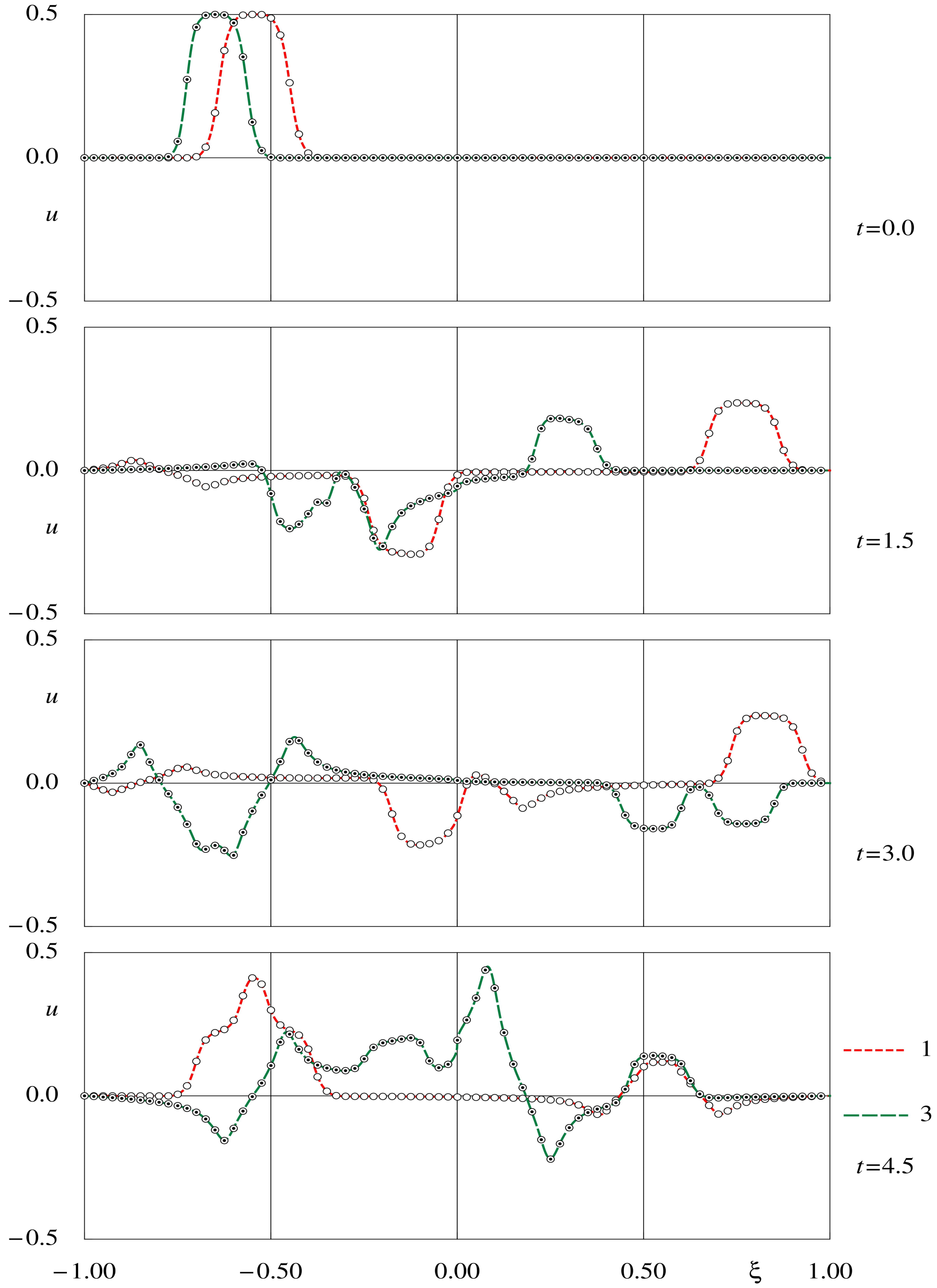

Fig. 5.1. Test case A: the regular fluxes are calculated due to (4.1), (4.3) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

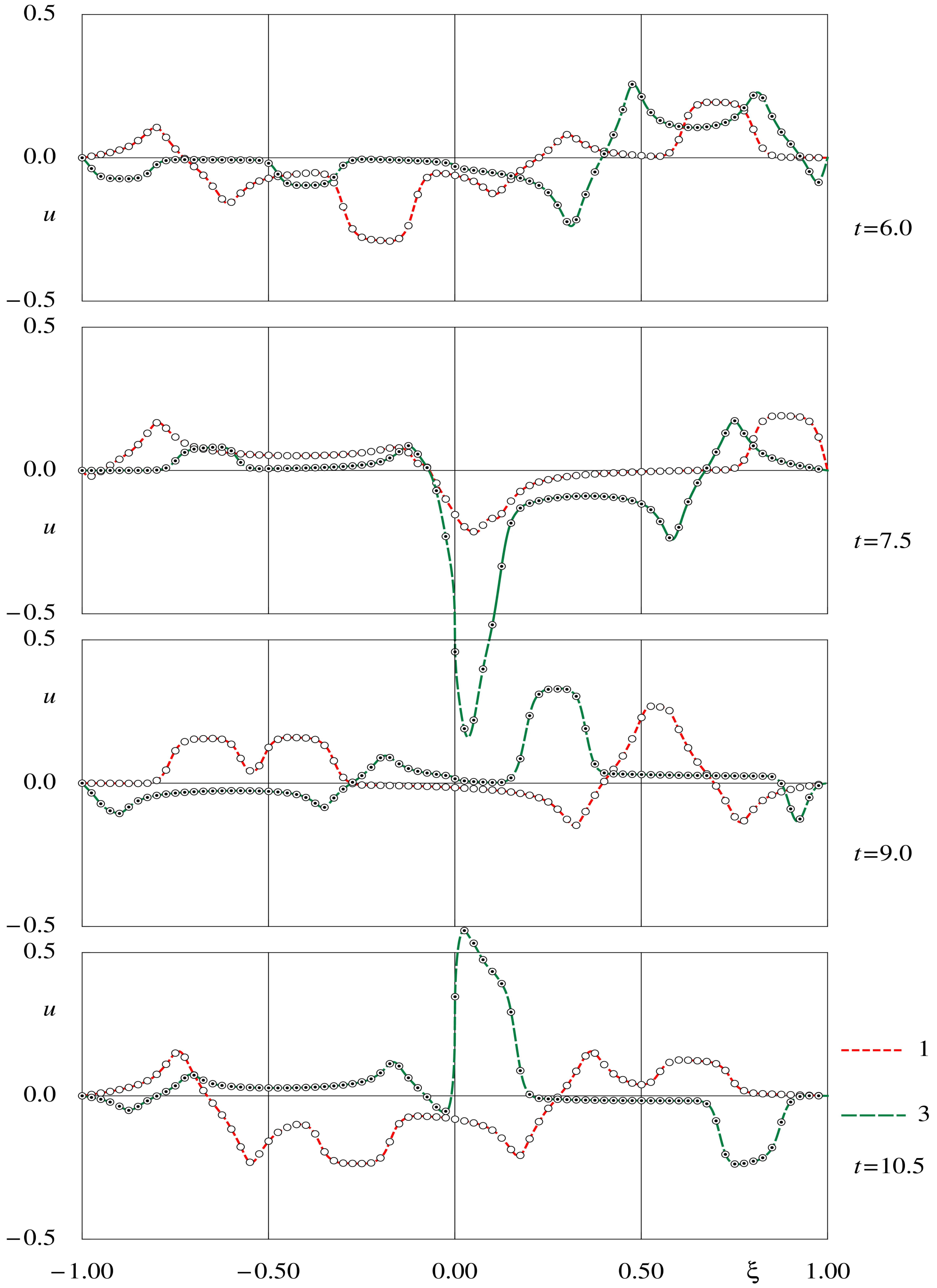

Fig. 5.2. Test case A: the regular fluxes are calculated due to (4.1), (4.3) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

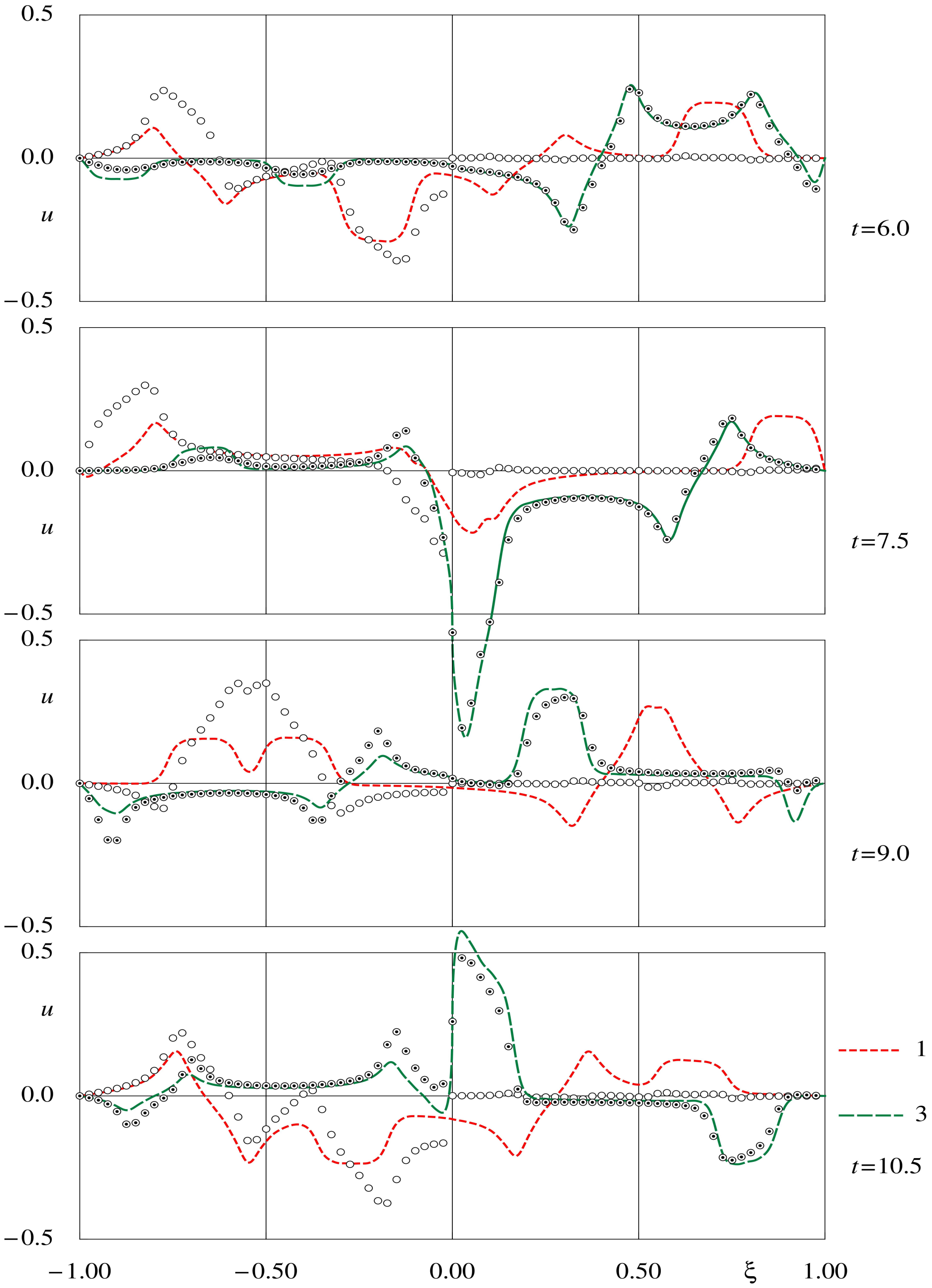

Fig. 5.3. Test case A: the regular fluxes are calculated due to (4.1), (4.3), whereas the degenerate fluxes are calculated using averaging (4.5) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

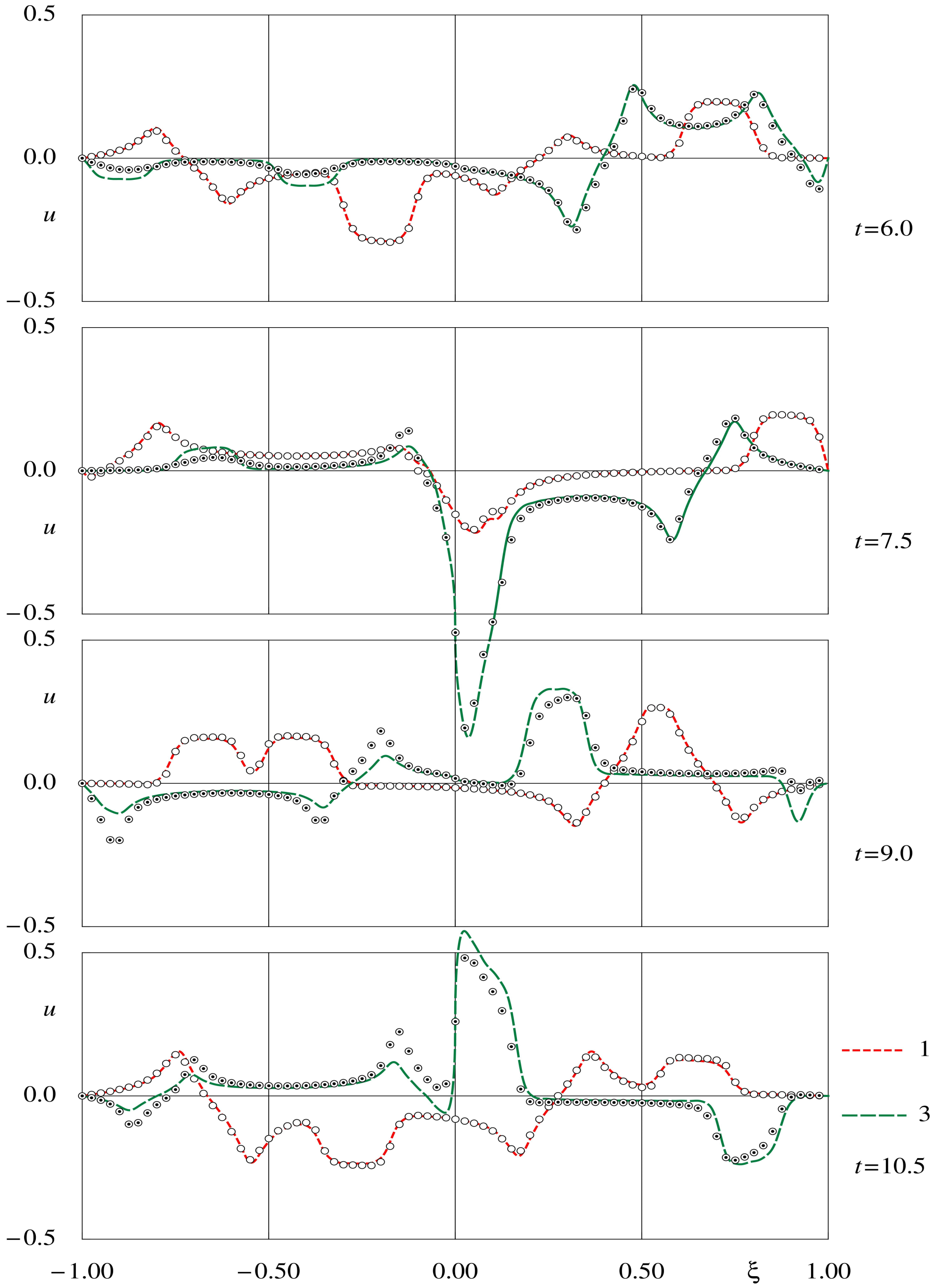

Fig. 5.4. Test case A: the regular fluxes are calculated due to (4.1), (4.3), whereas the degenerate fluxes are calculated using 2-nodal grid operators $D_k^{\mp m}$ and equality (4.9) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

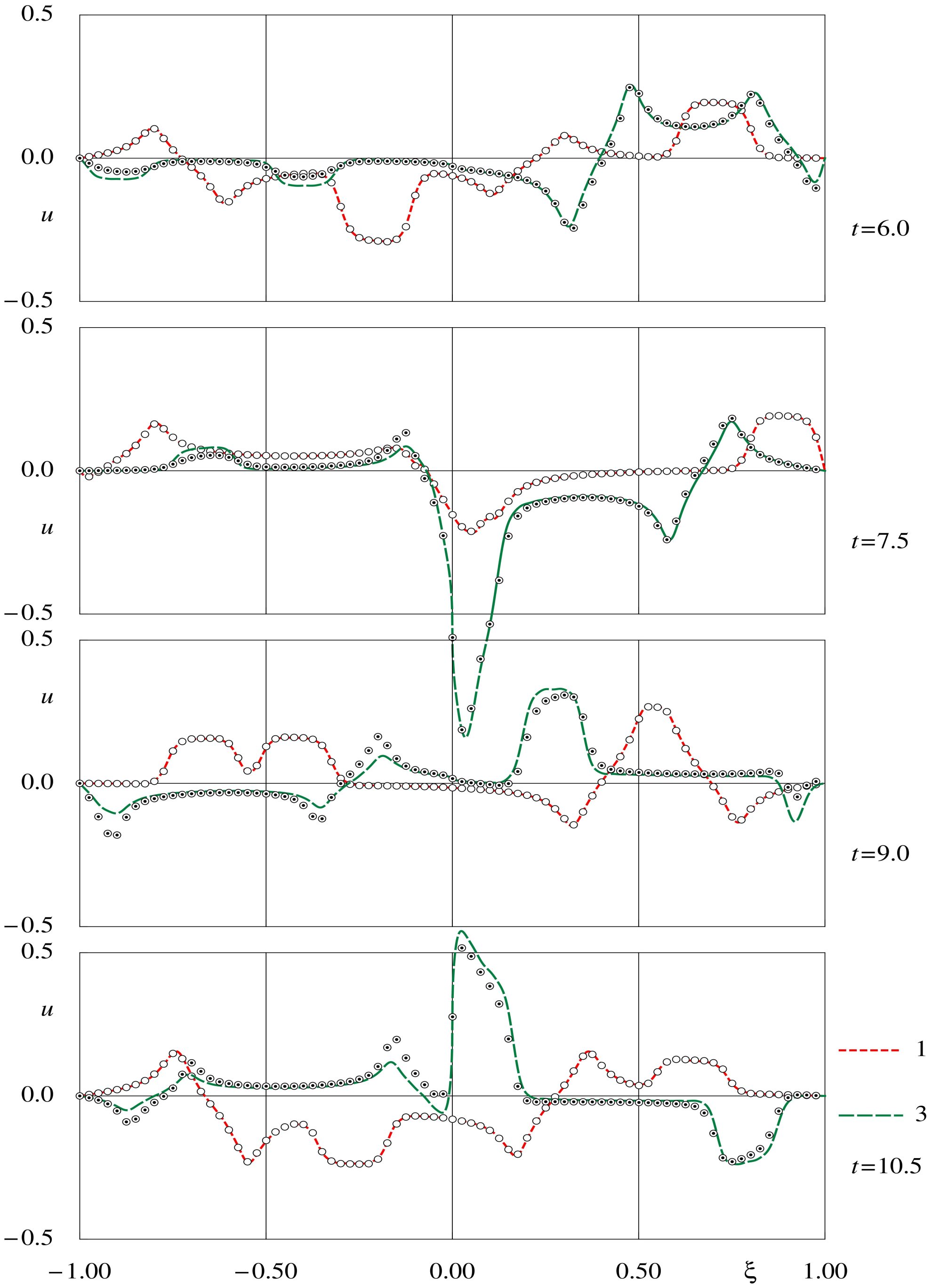

Fig. 5.5. Test case A: the regular fluxes are calculated due to (4.1), (4.3), whereas the degenerate fluxes are calculated using 3-nodal grid operators $D_k^{\mp m}$ and equality (4.9) ($\alpha = 0.25$ — curves 1 of short dashes, $\alpha = 0.75$ — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

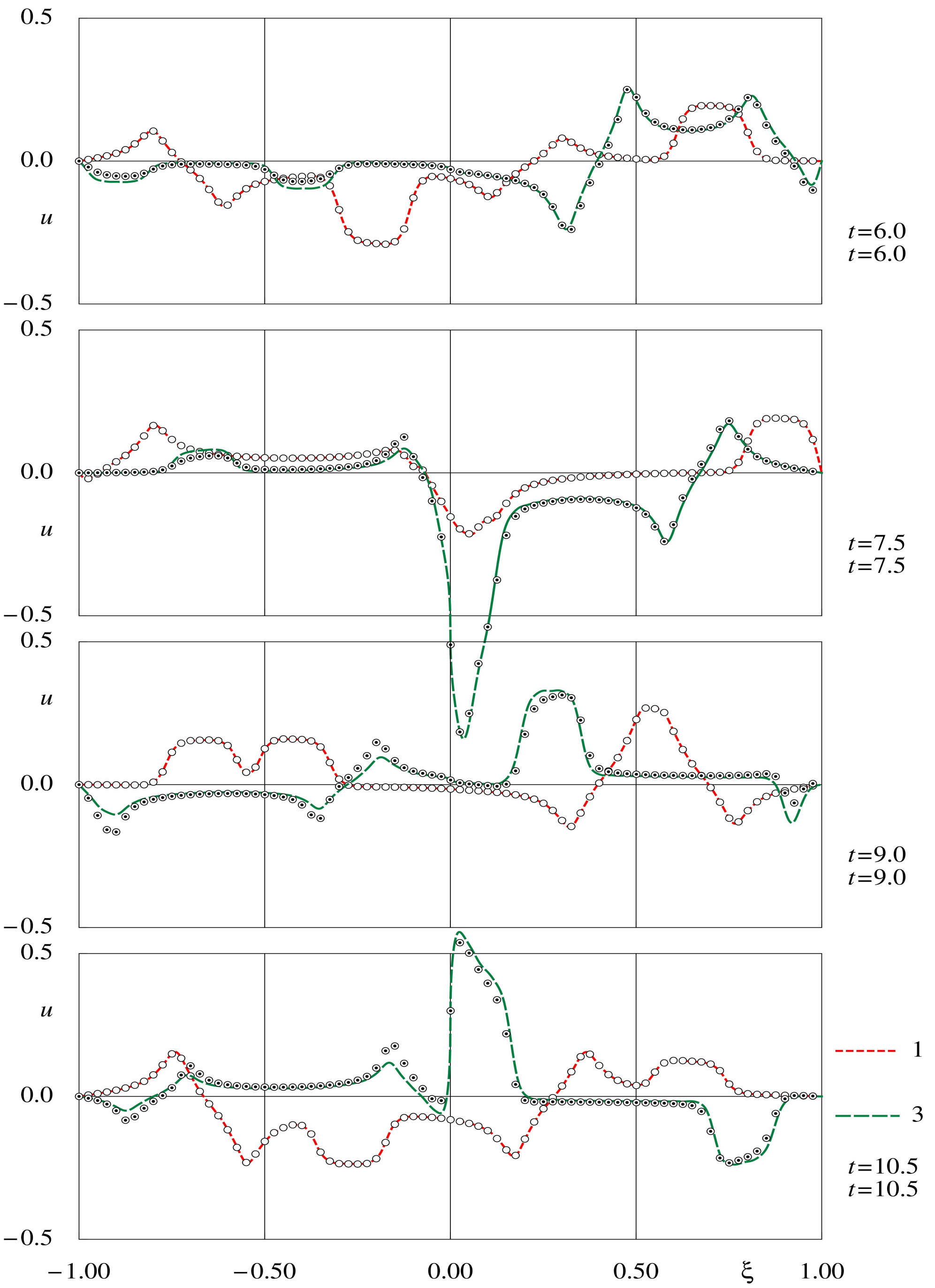

Fig. 5.6. Test case A: the regular and the degenerate fluxes are calculated due to (4.1), (4.4) (α = 0.25 — curves 1 of short dashes, α = 0.75 — curves 2 of long dashes, each 25 th node point of the grid solution is shown)

6. Conclusions

We have demonstrated for test case A [5], treated as a benchmark, that the problem of correct evaluating the inter-cell fluxes at the degeneracy segment can be solved using various approaches. The first group of approaches is based on a proper (or efficient) calculation of the coefficient function $a(x;\alpha)$ at the degenerate inter-cell. The second and third group utilize respectively the properties of the flux continuity and continuous differentiability across the degeneracy segment and do not involve any calculation of the coefficient function $a(x;\alpha)$ at the degenerate inter-cell.

The preliminary results of the current study will be further used to develop discrete models of the IBVP(1.1) in the case of strong degeneracy.

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