SUM SIGNED GRAPHS – II

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Abstract: In this paper, the study of sum signed graphs is continued. The balancing and switching nature of the graphs are analyzed. The concept of rna number is revisited and an important relation between the number and its complement is established.

Keywords: Balanced signed graph, rna number, rna complement number.

1. Introduction

Consider a scenario in which there exists a road of width 5 units. Clearly, vehicles of width 1 & 2, 1 & 3, 1 & 4 and 2 & 3 pass through the road with ease whereas the vehicles of width 2 & 4 and 3 & 4 find it difficult to pass through the road simultaneously. Also, the vehicle with width 5 cannot pass with any other given vehicles. There can exist situations in which the vehicle of width 5 can pass through the road alone which is depicted in the first figure. Also, there can be situations in which such a vehicle cannot pass through the road or may not exist at all and this idea is represented in the second figure. These two situations are modelled below in which the vertices are the vehicles and the corresponding width is given as labels for the vertices. The dashed line represents the case in which two vehicles travelling together are not possible and these types of edges are the negative edges. The other case in which two vehicles can travel together is represented by the bold line or positive edge.
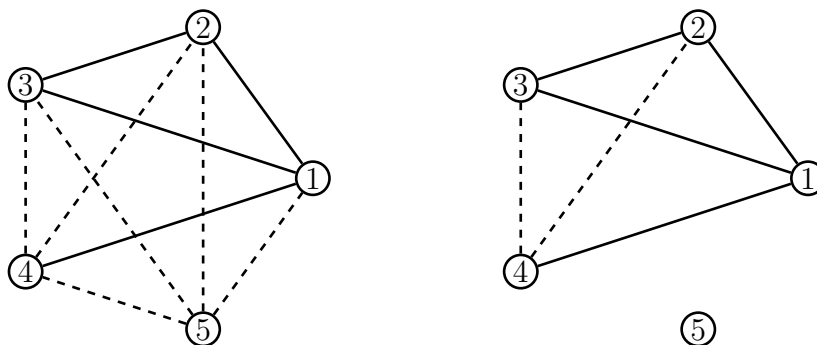

Figure 1. The illustration above gives us a picture of which all vehicles can travel together or not.

The motivation for the concept of sum signed labeling arose from the above idea that the vehicles passing through a road are sometimes restricted in terms of width of the road from a smooth, continuous ride. Introductory studies on sum signed graphs is available in [7]. In this paper, we extend the studies on sum signed graphs.

In Section 2, the criteria for switching from one sum signed graph to another will be discussed. The balance theory in the context of sum signed graphs is discussed in Section 3. An algorithm is provided in Section 4 to find the *rna* number for trees. Finally, the Section 5 introduces the concept of *rna* complement and discusses some of its properties.

2. Preliminaries

All the underlying graphs considered in this paper are simple, unless mentioned otherwise. We give the definition of the sum signed graph now.

Definition 1 [7]. *A sum signed graph is a trio, $S = (G, f, \sigma)$ where G is a graph,*

$$f : V(G) \longrightarrow \{1, 2, \ldots, \mid V(G) \mid\}$$

is a bijective function and $\sigma : E(G) \longrightarrow \{+, -\}$ is a mapping such that $\sigma(uv) = +$, whenever $f(u) + f(v) \leq n$ and $\sigma(uv) = -$, whenever $f(u) + f(v) > n$.

An edge receiving '+' sign is said to be a positive edge and the one receiving '−' sign is said to be a negative edge. A sum signed graph is said to be homogeneous, if all the edges receive either positive or negative sign, or else the signed graph will be a heterogeneous one. For basic terminologies on unsigned graphs and signed graphs, we refer to [5] and [9].

Definition 2 [7]. *The smallest number of negative edges among all sum signed labeling of its underlying graph G is called the rna number. It is denoted by $\sigma^-(G)$.*

From a socio-psychological point of view, negation [6] of a signed graph to another is an important operation. The negation $\eta(S)$ of a signed graph S is obtained by negating every edge of S that is, by changing the sign of each edge to its opposite. When this process is done, surely a signed graph is obtained but the question arises when will that signed graph become a sum signed graph. This question is answered in the next theorem.

Theorem 1. *A sum sign graph $S_1 = (G, f, \sigma)$ can be negated to another sum signed graph $S_2 = (G, g, \sigma)$ whenever*

$$g(u) = n + 1 - f(u), \quad u \in V(G).$$

P r o o f. Consider that there exists a sum signed graph $S_1 = (G, f, \sigma)$ for an underlying graph G with n vertices which can be switched to $S_2 = (G, g, \sigma)$ without the stated relation between f and g. In particular, assume that the stated relation does not exist for the vertex v_i having the label n i.e., all the edges incident to the vertex v_i in S_1 is negative. Then, v_i can be assigned the label $k > 1$ in S_2 such that all edges incident to v_i in is positive. Now, consider the same procedure in which the edges incident to the vertex receive opposite signs, for the vertex v_j having the label $n - 1$ in S_1. Proceeding like this, a contradiction will be reached at some vertex u in S_1, where at least one of the edges of the vertex u will have the same sign in S_1 and S_2 by not admitting the condition of sum signed labeling. □

Remark 1. In the case of $(n, 1)$-shovel graphs [8], none of its sum signed graphs can be switched to another sum signed graph.
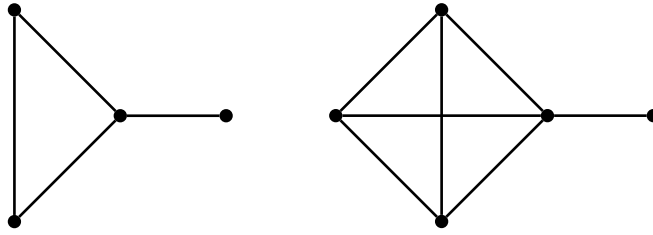
Figure 2. $(3, 1)$ and $(4, 1)$ shovel graphs.

3. Balanced sum signed graphs

Before diving into proving the balanced nature of sum signed graphs, we shall have a look at the topic of energy in sum signed graphs [4]. Let $v_1, v_2, ..., v_n$ be the vertices of a graph G and S be its sum signed graph. Then, the $n \times n$ matrix $A(S) = a_{ij}$ known as the adjacency matrix is defined as,

$$a_{ij} = \begin{cases} \sigma(v_i, v_j) & \text{if } v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A(S)$ that are also the eigenvalues of S, are real in nature. The spectrum is the set of distinct eigenvalues along with their multiplicities. The graphs with same spectrum are known as cospectral. Upon observation, some interesting characteristics were seen.

Theorem 2. *Every sum signed graph of the underlying graph G which satisfies the switching function are cospectral if and only if the spectrum is symmetric about the origin.*

P r o o f. Let S be a sum signed graph of G which can be switched to another sum signed graph S_1 using the switching function above. Clearly, $S_1 = -S$ where $-S$ is the sum signed graph obtained by negating each sign of S. Then, the statement holds as proved in [4]. $\square$

The concept of balance in signed graphs was introduced by Harary in [6]. This concept is very relevant while applying the ideas related to signed graph in a society. A signed graph is balanced if all the cycles present in the graph are positive. Otherwise, the signed graph is said to be unbalanced. In [1], a criterion was formulated to prove the balanced nature of signed graphs in terms of spectrum. The same condition is satisfied in the case of sum signed graphs.

Theorem 3. *A sum signed graph is balanced if and only if it is cospectral with the underlying graph.*

Theorem 4. *Every sum signed complete graph K_n, where $n \geq 4$ is unbalanced.*

P r o o f. Let K_n where $n \geq 4$, be a complete graph with n vertices having a sum signed labeling. Every vertex of K_n is adjacent to every other vertex of the graph. Thus, there exists only one sum signed labeling of the graph. For $n = 3$, the number of negative edges in K_3 is 2 as given [7]. When $n \geq 4$, the minimum size of cycle present in K_n is that of 3. Consider a cycle between the vertices having the labels $n, n-1$ and $n-2$ and such a C_3 exists since all the vertices are adjacent. The product of signs of that cycle is negative, i.e., that cycle is unbalanced. In every K_n, $n \geq 4$ contains such a C_3 making the graph unbalanced. $\square$

Theorem 5. *For every graph that is neither a complete graph nor a tree there exists at least one sum signed labeling which is unbalanced.*

P r o o f. Let G be a connected graph which is neither complete nor a tree, with n vertices and m edges. Any sum signed labeling S of G will have $\sigma^-(G)$ and $\sigma_+(G)$ negative and positive edges respectively since they are the minimum in number. Thus, there exists $m - (\sigma^-(G) + \sigma_+(G))$ edges that can vary in signs. Accordingly, for each variation a sum signed graph is obtained which contains at least one cycle with odd number of negative edges making the graph unbalanced. $\square$

Figure 3 represents two distinct sum signed graphs of the underlying graph G. The solid line represents positive edge and dashed line represents negative edge. Clearly, the former is balanced and the latter is unbalanced.
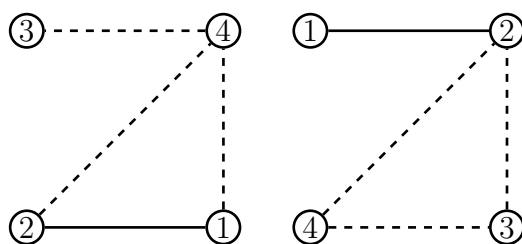

Figure 3. A graph with 4 vertices having two distinct sum signed labelling.

4. Algorithm for *rna* number of a tree

The concept of *rna* number in parity signed graphs was introduced in [2] and was studied in [3]. In [7], the *rna* number of sum signed graphs was introduced and has been found for various types of graphs. Here, the concept will be generalized in particular for acyclic graphs.

Consider a tree T with n vertices that has k pendant vertices. For finding the minimum number of negative edges in T, let T be drawn in such a way that the degrees of the vertices are increasing in each level by placing the pendant vertices in the lower level and the root in the upper level.

- Label the k pendant vertices $v_1, v_2, \ldots, v_k$ as $n, n-1, \ldots, n-k-1$, respectively. Thus, the first level of vertices in tree is labeled.

- Except the vertex adjacent to v_1 labeled as n, mark the j adjacent vertices of $v_2, v_3, \ldots, v_k$ with $1, 2, \ldots, j$. If any of the two pendant vertices have the same adjacent vertex v_i then, mark v_i with $\min\{1, 2, \ldots, j\}$ which is not repeated. If v_1 is adjacent to a vertex which is adjacent to any other pendant vertex the, label it accordingly as mentioned previously. Otherwise, label the adjacent vertex to v_1 with the renewed $\min\{1, 2, \ldots, j\}$ or $j + 1$.

- Label the next level of l vertices with the minimum label from the set $\{1, 2, \ldots, n\}$ which has not been repeated yet and also maintains the condition that the sum of the labels of the adjacent vertices never exceeds n.

- In a similar way, label all the remaining levels of the tree T.

Labeling a tree T in this way ensures that $\sigma^-(T) = 1$.

Pictorial representation of the algorithm is given in Fig. 4 for a tree with 23 vertices. In the figure, solid line represents positive edge and dashed line represents negative edge.
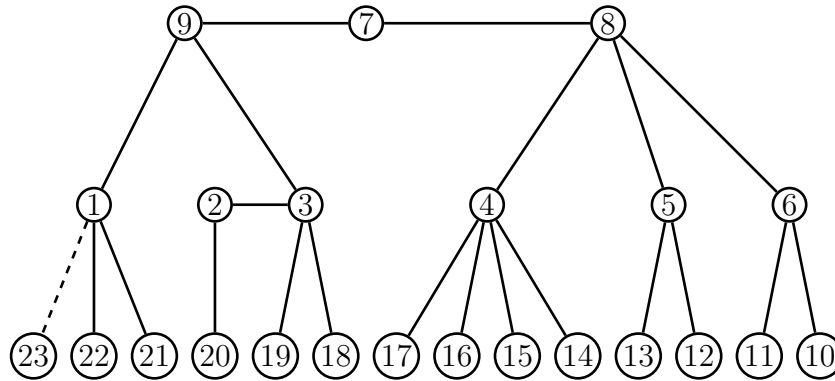
Figure 4. A tree with 23 vertices satisfying the above algorithm.

5. The *rna* complement number

The notion of *rna* number is mainly associated with that of negativity or negative people in a society. In contrast to that another number is introduced which deals with the minimization of positivity or positive people in a society. This number is known as *rna complement*.

Definition 3. *The 'rna complement' number of a sum signed graph G is the minimum number of positive edges among all the sum signed labelings of G. It is denoted by $\sigma_+(G)$.*

Theorem 6. *For every tree T, $\sigma_+(T) = 0$.*

Proceeding as in the algorithm for $\sigma_+(T)$ by changing the lowest labels to the highest ones and and vice versa, we have the result.

Theorem 7. *For any graph G containing no pendant vertices, $\sigma_+(G) \geq 1$.*

P r o o f. Consider a graph G with n vertices containing no pendant vertices. Assign the minimum label *1* to the vertex v_j with degree $\delta(G)$ so that minimum number of positive edges is obtained for the graph. Then, there exists at least $\delta(G) - 1$ positive edges. Since, G is a graph with no pendant vertices, $\delta(G) \geq 2$. Then, $\sigma_+(G) \geq \delta(G) - 1$ can be written as $\sigma_+(G) \geq 1$. □

From the above observations, we can conclude the following theorem.

Theorem 8. *For any graph G, $\sigma_+(G) < \sigma^-(G)$.*

P r o o f. Let G be a graph with n number of vertices and let v_j be the vertex with degree $\delta(G)$. Let S_1 and S_2 be two sum signed graphs of G such that in S_1, the vertex v_j with degree $\delta(G)$ is given the lowest label *1* and in S_2, v_j is given the highest label n. Let the two graphs satisfy the condition that $|E^+(S_1)| > |E^-(S_2)|$. So, S_1 has the minimum number of negative edges as compared to S_2 and S_2 has the minimum number of positive edges as compared to S_1. So,

$$|E^+(S_1)| = \delta(G) - 1 + k, \quad |E^-(S_2)| = \delta(G) + m,$$

where k and m are two positive integers. We need to minimise k and m such that $k \leq m$.

In S_1, we need to minimise the number of positive edges, i.e., to maximise the number of negative edges. For this, we will exchange labels of any two vertices except v_j such that number of negative edges is maximised. Then,

$$|E^+(S_1)| = \delta(G) - 1 + k - a.$$

This process is continued for every vertex till the maximum number of negative edges is obtained in S_1. Thus, $|E^+(S_1)| = \sigma_+(G)$. Similarly, we proceed with S_2. After the process we will see that $|E^+(S_1)| \leq |E^-(S_2)|$.

The determination of *rna* and *rna complement* number is mainly depended on the assignment of maximum and minimum labels existing for the graph G to the vertex v_j. When these two labels, i.e., *n* and *1* are assigned to the vertex v_j in two different graphs S_3 and S_4, it will be observed that in S_3, there exists $\delta(G)$ negative edges for v_j and in S_4 and $\delta(G) - 1$ positive edges associated with the vertex v_j. This difference in the number of negative and positive edges goes on increasing as the labeling of the graph proceeds to find *rna* and *adhika* number. Hence, it can be concluded that $\sigma_+(G) \neq \sigma^-(G)$.   □

Theorem 9. *The number of distinct signed graphs satisfying sum signed labeling for a connected graph G is at most $|E(G)| - \sigma^-(G) - \sigma_+(G)$.*

P r o o f. Let G be a connected graph with n vertices and m edges. Any sum signed labeling S of G will have $\sigma^-(G)$ and $\sigma_+(G)$ negative and positive edges since they are the minimum in number. Thus, there exist $m - (\sigma^-(G) + \sigma_+(G))$ edges which can vary in signs. For each variation in the sign of the edge, we will obtain a sum signed labeling of G which will be different from the previous ones. Hence, there exist at most $m - \sigma^-(G) - \sigma_+(G)$ distinct sum signed graphs for a graph G.   □

6. Conclusion

The recently introduced signed graphs open a wide variety of interesting problems in Graph Theory. In this paper, we have explored the balance nature, switching property, number of negative edges etc. of sum signed graph. The balanced nature of the sum signed graphs are studied using the concept of energy of signed graphs. An algorithm relating to the *rna* number of trees has been discussed. The concept of *rna* complement has been introduced and analyzed to some extent. There is enough scope for further studies. One such concept is that of energy of sum signed graphs and the other one is that of *rna* complement which has only been grazed.

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