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ZAGREB INDICES OF A NEW SUM OF GRAPHS

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Abstract: The first and second Zagreb indices, since its inception have been subjected to extensive research in the physio-chemical analysis of compounds. In [5], Hanyuan Deng *et al.* computed the first and second Zagreb indices of four new operations on a graph defined by M. Eliasi, B. Taeri [6]. Motivated by [6], in this paper we define a new operation on graphs and compute the first and second Zagreb indices of the resultant graph. We illustrate the results with some examples.

Keywords: First Zagreb index $M_1(G)$, Second Zagreb index $M_2(G)$, F^* sum.

1. Introduction

A graph without loops and also without parallel edges is called a simple graph and if all the pairs of vertices of the graph are connected by a path then it is said to be connected. Throughout our discussion, we consider only connected simple graphs. The degree-based structural descriptors have been a subject of detailed study since their induction from the first degree-based topological index in 1972 by I. Gutman, N. Trinajstić [11]. Later, in 1975 I. Gutman, B. Rusćić, N. Trinajstić, C.F. Wilcox [12] defined another degree based index in connection with studying physical properties of chemical compounds. At first, both these indices were named as Zagreb group indices [3], but later I. Gutman named them as first and second Zagreb indices. The first Zagreb index $M_1(G)$ is defined as the sum of squares of degrees of all the vertices and the second Zagreb index $M_2(G)$ is defined as the sum of product of degrees of end vertices of all the edges. That is,

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2, \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

Various physical applications of these indices can be found in [8–10, 13, 19, 20]. A more unified and general approach on degree based indices of graphs were considered by X. Li, H. Zhao in [17, 18] which lead in defining generalized Zagreb index as

$$M_\alpha(G) = \sum_{u\in V(G)} d_G(u)^\alpha.$$

Various particular cases for this generalized Zagreb index were considered separately, one among them is the Forgotten index $F(G)$ (when $\alpha = 3$) defined in 1972 [11] but resurged in 2015 through the works of B. Furtula, I. Gutman [7]. For more works on topological indices, see [2, 15, 17, 18, 21]. The degree based topological indices of graph operations have been a subject of detailed study recently [1, 5]. In [16], M.H. Khalifeh, H. Yousefi–Azari, A.R. Ashrafi computed the first and second Zagreb indices of graph operations such as cartesian product, composition, join, disjunction and symmetric difference of graphs. In [6], M. Eliasi, B. Taeri defined four new operations of graphs related to subdivisions and computed the Wiener index. Motivated by [6], in this paper we define a new sum related to the four subdivison graphs and compute the first and second Zagreb indices of the new sum. We also find the Zagreb indices of some chemical structures and some classes of bridge graphs using the expressions obtained. We refer to this new sum as F^* sums of graphs.

2. F^* sums of graphs

Let G_1, G_2 be two graphs with vertex set V_1, V_2 and edge set E_1, E_2 respectively. The four subdivision graphs $S(G_1)$, $R(G_1)$, $Q(G_1)$, $T(G_1)$ are defined as follows in [4]:

1. $S(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a vertex and making the new vertex adjacent to the corresponding end vertices of e_i for each $e_i \in E_1$. That is, $S(G_1)$ is a graph with vertex set $V(S(G_1)) = V_1 \bigcup V_1^*$ where V_1^* is the collection of new vertices and the edge set

$$E(S(G_1)) = \left\{(v,h),(u,h) : e = vu \in E_1, h \in V_1^*\right\}.$$

2. $R(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a vertex and making new vertex adjacent to the corresponding end vertices of e_i for each $e_i \in E_1$ also keeping every edge in G_1 as well. That is, $R(G_1)$ is a graph with vertex set $V(R(G_1)) = V_1 \bigcup V_1^*$ where V_1^* is the collection of new vertices and edge set

$$E(R(G_1)) = \left\{(v,h),(u,h) : e = vu \in E_1, h \in V_1^*\right\} \cup E_1.$$

3. $Q(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a vertex and making new vertex adjacent to the corresponding end vertices of e_i for each $e_i \in E_1$ along with edges joining vertex in the ith copy of V_1^* to the vertex in the jth copy of V_1^* whenever e_i adjacent to e_j in G_1. That is, $Q(G_1)$ is a graph with vertex set $V(Q(G_1)) = V_1 \bigcup V_1^*$ where V_1^* is the collection of new vertices and edge set

$$E(Q(G_1)) = \left\{(v,h),(u,h) : e = vu \in E_1, h \in V_1^*\right\} \cup E_1^*,$$
$$E_1^* = \left\{(u_i,u_j) : e_i \textit{ adjacent to } e_j \textit{ in } E_1, u_i, u_j \in V_1^*\right\},$$

where u_i, u_j are the vertices corresponding to the edges $e_i, e_j \in E_1$.

4. $T(G_1)$ is the graph obtained from G_1 by replacing each edge e_i of G_1 with a vertex and making new vertex adjacent to the corresponding end vertices of e_i for each $e_i \in E_1$ along with edges joining vertex in the ith copy of V_1^* to the vertex in the jth copy of V_1^* whenever e_i adjacent to e_j in G_1 and keeping every edge of G_1 as well. That is, $T(G_1)$ is a graph with vertex set $V(T(G_1)) = V_1 \bigcup V_1^*$ where V_1^* is the collection of new vertices and edge set

$$E(T(G_1)) = \{(v,h),(u,h) : e = vu \in E_1, h \in V_1^*\} \cup E_1^*,$$
$$E_1^* = \{(u_i,u_j) : e_i \textit{ adjacent to } e_j \textit{ in } E_1, u_i, u_j \in V_1^*\} \cup E_1,$$

where u_i, u_j are the vertices corresponding to the edges $e_i, e_j \in E_1$.

In each of these new subdivision graphs the vertices V_1 can be termed as black vertices and the vertices V_1^* can be termed as white vertices. In [6], M. Eliasi, B. Taeri defined four new sums called F sums with the operation cartesian product on black vertices on copies of subdivison graphs. Motivated by this we define a sum on copies of white vertices related to the cartesian product. Let F be any one of the symbols S, R, Q, T, then the F^* sum of two graphs G_1 and G_2 is denoted by $G_1 *_F G_2$, is a graph with the vertex set $V(G_1 *_F G_2) = V(F(G_1)) \times V_2$ and the edge set

$$E(G_1 *_F G_2) = \big\{(a,b)(c,d) : a = c \in V_1^* \text{ and } bd \in E_2 \text{ or } ac \in E(F(G_1)) \text{ and } b = d \in V_2\big\}.$$

Fig. 1 is an example with $G_1 = P_4$, $G_2 = P_6$.

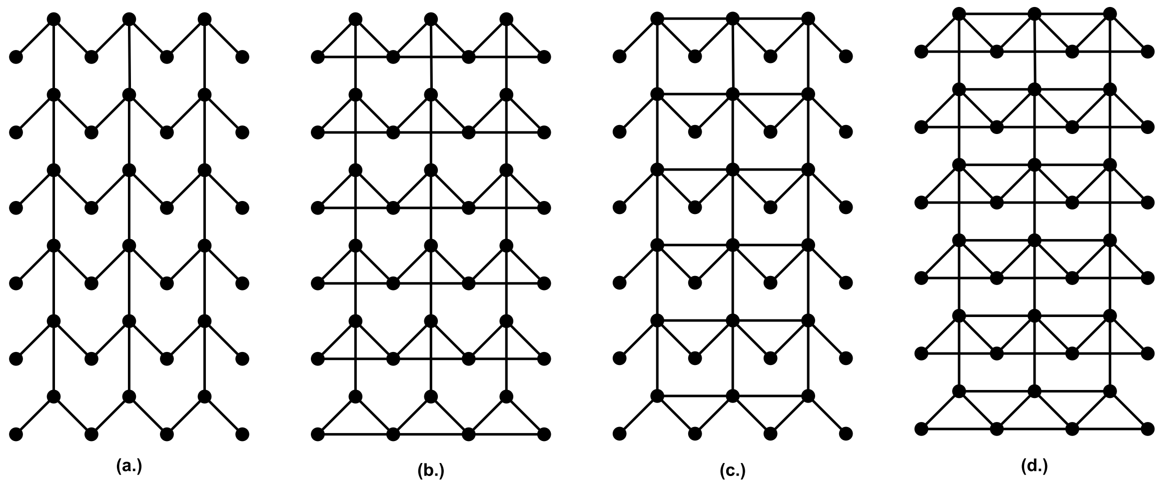

Figure 1. (a) $P_4 *_S P_6$, (b) $P_4 *_R P_6$, (c) $P_4 *_Q P_6$, (d) $P_4 *_T P_6$.

3. Zagreb index of F^* sum

In this section we compute the first and second Zagreb indices of F^* sums of graphs.

Theorem 1. *Let G_1 and G_2 be two connected graphs, then*

$$(a) \quad M_1(G_1 *_S G_2) = |V_2|M_1(G_1) + |E_1|M_1(G_2) + 4|E_1|\left(2|E_2| + |V_2|\right),$$
$$(b) \quad M_2(G_1 *_S G_2) = 2|E_1|M_1(G_2) + |E_1|M_2(G_2) + 4|E_1|\left(2|E_2| + |V_2|\right).$$

P r o o f. From the definition of first Zagreb index, we have

$$M_1(G_1 *_S G_2) = \sum_{(a,b)\in V(G_1 *_S G_2)} \big(d_{(G_1 *_S G_2)}(a,b)\big)^2$$
$$= \sum_{(u,v)(x,y)\in E(G_1 *_S G_2)} \big(d_{(G_1 *_S G_2)}(u,v) + d_{(G_1 *_S G_2)}(x,y)\big)$$
$$= \sum_{u\in V_1^*} \sum_{vy\in E_2} \big(d_{(G_1 *_S G_2)}(u,v) + d_{(G_1 *_S G_2)}(u,y)\big)$$
$$+ \sum_{v\in V_2} \sum_{ux\in E(S(G_1)} \big(d_{(G_1 *_S G_2)}(u,v) + d_{(G_1 *_S G_2)}(x,v)\big).$$

Now we separately find the values of the each parts in the sum. Firstly we consider the sum in which $u \in V_1^*$ and $vy \in E_2$

$$\begin{aligned}
&\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1*_SG_2)}(u,v)+d_{(G_1*_SG_2)}(u,y)\big)\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big[d_{S(G_1)}(u)+d_{G_2}(v)+\big(d_{S(G_1)}(u)+d_{G_2}(y)\big)\big]\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big[2d_{S(G_1)}(u)+d_{G_2}(v)+d_{G_2}(y)\big]\\
&=\sum_{u\in V_1^*}\big[4|E_2|+M_1(G_2)\big]=4|E_1||E_2|+|E_1|M_1(G_2).
\end{aligned}$$

Now for each edge $ux \in E(S(G_1))$, $v \in V_2$

$$\begin{aligned}
&\sum_{v\in V_2}\sum_{ux\in E(S(G_1))}\big(d_{(G_1*_SG_2)}(u,v)+d_{(G_1*_SG_2)}(x,v)\big)\\
&=\sum_{v\in V_2}\sum_{\substack{ux\in E(S(G_1)\\ u\in V_1,\, x\in V_1^*}}\big[d_{S(G_1)}(u)+\big(d_{G_2}(v)+d_{S(G_1)}(x)\big)\big]\\
&=\sum_{v\in V_2}\big(2|E_1|d_{G_2}(v)+M_1(G_1)+4|E_1|\big)=4|E_1||E_2|+|V_2|M_1(G_1)+4|E_1||V_2|.
\end{aligned}$$

From the expressions we obtain

$$M_1(G_1 *_S G_2) = |V_2|M_1(G_1)+|E_1|M_1(G_2)+4|E_1|\left(2|E_2|+|V_2|\right).$$

Next consider

$$\begin{aligned}
M_2(G_1 *_S G_2) &= \sum_{(u,v)(x,y)\in E(G_1*_SG_2)}\big(d_{G_1*_SG_2}(u,v)d_{G_1*_SG_2}(x,y)\big)\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{G_1*_SG_2}(u,v)d_{G_1*_SG_2}(u,y)\big)+\sum_{v\in V_2}\sum_{ux\in E(S(G_1)}\big(d_{G_1*_SG_2}(u,v)d_{G_1*_SG_2}(x,v)\big)\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big[d_{S(G_1)}(u)+d_{G_2}(v)\big]\big[d_{S(G_1)}(u)+d_{G_2}(y)\big]\\
&\quad+\sum_{v\in V_2}\sum_{\substack{ux\in E(S(G_1)),\\ u\in V_1,\, x\in V_1^*}}\big[d_{S(G_1)}(u)\big(d_{G_2}(v)+d_{S(G_1)}(x)\big)\big]\\
&=\sum_{u\in V_1^*}\sum_{vy\in E_2}\left[4+2\left(d_{G_2}(v)+d_{G_2}(y)\right)+d_{G_2}(v)d_{G_2}(y)\right]+\sum_{v\in V_2}\left(2\left(|E_1|\right)d_{G_2}(v)+4|E_1|\right)\\
&=4|E_1||E_2|+2|E_1|M_1(G_2)+|E_1|M_2(G_2)+4|E_1|\left(|E_2|+|V_2|\right).
\end{aligned}$$

Thus,

$$M_2(G_1 *_S G_2) = 2|E_1|M_1(G_2)+|E_1|M_2(G_2)+4|E_1|\left(2|E_2|+|V_2|\right).$$

□

Theorem 2. *Let G_1 and G_2 be two connected graphs, then*

(a) $M_1(G_1 *_R G_2) = 4|V_2|M_1(G_1) + |E_1|M_1(G_2) + 4|E_1|(2|E_2| + |V_2|)$,

(b) $M_2(G_1 *_R G_2) = 4M_1(G_1)(1 + |E_2|) + M_2(G_2)(4|V_2| + |E_1|) + 2|E_1|M_1(G_2) + 4|E_1||E_2|$.

P r o o f. We have

$$M_1(G_1 *_R G_2) = \sum_{(a,b)\in V(G_1 *_R G_2)} \big(d_{(G_1 *_R G_2)}(a,b)\big)^2$$
$$= \sum_{(u,v)(x,y)\in E(G_1 *_R G_2)} \big(d_{(G_1 *_R G_2)}(u,v) + d_{(G_1 *_R G_2)}(x,y)\big)$$
$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} \big(d_{(G_1 *_R G_2)}(u,v) + d_{(G_1 *_R G_2)}(u,y)\big) + \sum_{v\in V_2}\sum_{ux\in E(R(G_1)} \big(d_{(G_1 *_R G_2)}(u,v) + d_{(G_1 *_R G_2)}(x,v)\big).$$

Now we separately find the values of each part in the sum. First we consider the sum in which $u \in V_1^*$ and $vy \in E_2$

$$\sum_{u\in V_1^*}\sum_{vy\in E_2} \big(d_{(G_1 *_R G_2)}(u,v) + d_{(G_1 *_R G_2)}(u,y)\big)$$
$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} \big[\big(d_{R(G_1)}(u) + d_{G_2}(v)\big) + \big(d_{R(G_1)}(u) + d_{G_2}(y)\big)\big]$$
$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} [4 + d_{G_2}(v) + d_{G_2}(y)] = \sum_{u\in V_1^*} [4|E_2| + M_1(G_2)] = 4|E_1||E_2| + |E_1|M_1(G_2).$$

Now for each edge $ux \in E(R(G_1))$, $v \in V_2$

$$\sum_{v\in V_2}\sum_{ux\in E(R(G_1))} \big(d_{(G_1 *_S G_2)}(u,v) + d_{(G_1 *_S G_2)}(x,v)\big)$$
$$= \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u,x\in V_1}} \big(d_{(G_1 *_S G_2)}(u,v) + d_{(G_1 *_S G_2)}(x,v)\big)$$
$$+ \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u\in V_1,\, x\in V_1^*}} \big(d_{(G_1 *_S G_2)}(u,v) + d_{(G_1 *_S G_2)}(x,v)\big).$$

Now we calculate the each sum separately

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u,x\in V_1}} \big(d_{(G_1 *_R G_2)}(u,v) + d_{(G_1 *_R G_2)}(x,v)\big)$$
$$= \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u,x\in V_1^*}} \big(d_{R(G_1)}(u) + d_{R(G_1)}(x)\big)$$
$$= \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u,x\in V_1}} 2\big(d_{G_1}(u) + d_{G_1}(x)\big) = 2|V_2|M_1(G_1).$$

By considering the case where $ux \in E(R(G_1))$, $u \in V_1$, $x \in V_1^*$

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u\in V_1^*,\, x\in V_1^*}}\big(d_{(G_1*_RG_2)}(u,v)+d_{(G_1*_RG_2)}(x,v)\big)=\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u\in V_1,\, x\in V_1^*}} d_{R(G_1)}(u)+(d_{G_2}(v)\ +2)$$

$$=\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u\in V_1,\, x\in V_1^*}} 2d_{G_1}(u)+d_{G_2}(v)+2=2|V_2|M_1(G_1)+4|E_1|\big(|E_2|+|V_2|\big).$$

Thus we obtain

$$M_1(G_1 *_R G_2) = 4|V_2|M_1(G_1) + |E_1|M_1(G_2) + 4|E_1|\big(2|E_2| + |V_2|\big).$$

Similarly,

$$M_2(G_1 *_R G_2) = \sum_{(u,v)(x,y)\in E(G_1*_RG_2)} \big(d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(x,y)\big)$$

$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} (d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(u,y)) + \sum_{v\in V_2}\sum_{ux\in E(R(G_1)}\big(d_{(G_1*_RG_2}(u,v)d_{(G_1*_RG_2}(x,v)\big).$$

Now we find the sums separately

$$\sum_{u\in V_1^*}\sum_{vy\in E_2} (d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(u,y)) = \sum_{u\in V_1^*}\sum_{vy\in E_2}\left[\big(d_{R(G_1)}(u)+d_{G_2}(v)\big)\left(d_{R(G_1)}(u)+d_{G_2}(y)\right)\right]$$

$$= \sum_{u\in V_1^*}\sum_{vy\in E_2}\left[d_{R(G_1)}(u)^2 + d_{R(G_1)}(u)\left(d_{G_2}(v)+d_{G_2}(y)\right) + d_{G_2}(v)d_{G_2}(y)\right]$$

$$= \sum_{u\in V_1^*}\sum_{vy\in E_2}\left[4+2\left(d_{G_2}(v)+d_{G_2}(y)\right)+d_{G_2}(v)d_{G_2}(y)\right] = 4|E_1||E_2| + 2|E_1|M_1(G_2) + |E_1|M_2(G_2).$$

Also,

$$\sum_{v\in V_2}\sum_{ux\in E(R(G_1))}\big(d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(x,v)\big) = \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u,x\in V_1}}\big(d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(x,v)\big)$$

$$+\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1)),\\ u\in V_1,\, x\in V_1^*}}\big(d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(x,v)\big).$$

Finding the sums separately, we get

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1))\\ u,x\in V_1}}\big(d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(x,v)\big) = \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1))\\ u,x\in V_1}}\big(d_{R(G_1)}(u)d_{R(G_1)}(x)\big)$$

$$= \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1))\\ u,x\in V_1}} 4d_{G_1}(u)d_{G_1}(x) = 4|V_2|M_2(G_2).$$

Now,

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1))\\ u\in V_1,\, x\in V_1^*}}\big(d_{G_1*_RG_2}(u,v)d_{G_1*_RG_2}(x,v)\big) = \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1))\\ u\in V_1,\, x\in V_1^*}} d_{R(G_1)}(u)\big(d_{G_2}(v)+d_{R(G_1)}(x)\big)$$

$$= \sum_{v\in V_2}\sum_{\substack{ux\in E(R(G_1))\\ u\in V_1,\, x\in V_1^*}} 4d_{G_1}(u) + 2d_{G_1}(u)d_{G_2}(v) = 4M_1(G_1) + 4|E_2|M_1(G_1).$$

Now collecting all the previous terms, we get

$$M_2(G_1 *_R G_2) = 4M_1(G_1)(1+|E_2|) + M_2(G_2)(4|V_2|+|E_1|) + 2|E_1|M_1(G_2) + 4|E_1||E_2|.$$

□

Theorem 3. *Let G_1 and G_2 be two connected graphs, then*

(a) $M_1(G_1 *_Q G_2) = (|V_2|+2|E_2|)M_1(G_1) + |E_1|M_1(G_2) + 2|V_2|M_2(G_1) + |V_2|F(G_1) + 2|E_2|\big(2|E(Q(G_1))| + 3|E_1|\big)$,

(b) $M_2(G_1 *_Q G_2) = |E_2|M_2(G_1) + |E_1|M_2(G_2) + M_2(G_1)M_2(G_2) + 2|E_2|M_1(G_1) + \frac{1}{2}\big[|V_2|M_4(G_1) + (2|E_2| + |V_2|F(G_1))\big] + |V_2|\Big(\sum_{u_i,u_j\in V_1} r_{ij}d_{G_1}(u_i)d_{G_1}(u_j) + \sum_{u_j\in V_1} d_{G_1}(u_j)^2 \sum_{u_i\in V_1,\, u_iu_j\in E_1} d_{G_1}(u_i)\Big)$,

where r_{ij} denotes the number of neighbouring common vertices adjacent to both u_i and u_j.

Proof. We have

$$M_1(G_1 *_Q G_2) = \sum_{(a,b)\in V(G_1 *_S G_2)} \big(d_{(G_1 *_Q G_2)}(a,b)\big)^2$$

$$= \sum_{(u,v)(x,y)\in E(G_1 *_Q G_2)} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(x,y)\big)$$

$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(u,y)\big)$$

$$+ \sum_{v\in V_2}\sum_{ux\in E(Q(G_1))} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(x,v)\big).$$

First we consider the sum in which $u \in V_1^*$ and $vy \in E_2$

$$\sum_{u\in V_1^*}\sum_{vy\in E_2} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(u,y)\big)$$

$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} \big[\big(d_{Q(G_1)}(u) + d_{G_2}(v)\big) + \big(d_{Q(G_1)}(u) + d_{G_2}(y)\big)\big]$$

$$= \sum_{u\in V_1^*}\sum_{vy\in E_2} \big[2d_{Q(G_1)}(u) + d_{G_2}(v) + d_{G_2}(y)\big]$$

$$= \sum_{e=pq\in E_1} 2|E_2|(d_{G_1}(p) + d_{G_1}(q)) + \sum_{u\in V_1^*} M_1(G_2) = 2|E_2|M_1(G_1) + |E_1|M_1(G_2).$$

For each edge $ux \in E(Q(G_1))$ and the vertex $v \in V_2$

$$\sum_{v\in V_2}\sum_{ux\in E(Q(G_1))} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(x,v)\big)$$

$$= \sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,\, x\in V_1^*}} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(x,v)\big)$$

$$+ \sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}} \big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(x,v)\big).$$

Now we separately find both the sums. First,

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1))\\ u\in V_1,\, x\in V_1^*}}\Big(d_{(G_1*_QG_2)}(u,v)+d_{(G_1*_QG_2)}(x,v)\Big)$$

$$=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1))\\ u\in V_1,\, x\in V_1^*}} d_{Q(G_1)}(u)+\big(d_{G_2}(v)+d_{Q(G_1)}(x)\big)=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1))\\ u\in V_1,\, x\in V_1^*}} d_{G_1}(u)+d_{G_2}(v)+d_{Q(G_1)}(x)$$

$$=\sum_{v\in V_2} M_1(G_1)+2|E_1|d_{G_2}(v)+2\sum_{v\in V_2}\sum_{\substack{e=u_iv_i\in E(G_1)\\ u_i,\, v_i\in V_1}}(d_{G_1}(u_i)+d_{G_1}(v_i))$$

$$=|V_2|M_1(G_1)+4|E_1||E_2|+2|V_2|M_1(G_1).$$

The second part of the sum is the following

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}\big(d_{(G_1*_QG_2)}(u,v)+d_{(G_1*_QG_2)}(x,v)\big)$$

$$=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}\big(d_{Q(G_1)}(u)+d_{G_2}(v)+d_{Q(G_1)}(x)+d_{G_2}(v)\big)$$

$$=\sum_{v\in V_2}\Big(\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}2d_{G_2}(v)\Big)+\sum_{v\in V_2}\Big(\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}\big(d_{Q(G_1)}(u)+d_{Q(G_1)}(x)\big)\Big)$$

$$=\sum_{v\in V_2}\Big(\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}2d_{G_2}(v)\Big)+\sum_{v\in V_2}\Big(\sum_{u_iu_j,u_ju_k\in E_1}\big(d_{G_1}(u_i)+d_{G_1}(u_j)+d_{G_1}(u_j)+d_{G_1}(u_k)\big)\Big)$$

$$=4(|E(Q(G_1))|-2|E_1|)|E_2|+|V_2|\Big(2\sum_{u_j\in V_1}C^2_{d_{G_1}(u_j)}d_{G_1}(u_j)+\sum_{u_j\in V_1}(d_{G_1}(u_j)-1)\sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}}d_{G_1}(u_i)\Big)$$

$$=4(|E(Q(G_1))|-2|E_1|)|E_2|+|V_2|\Big(\sum_{u_j\in V_1}(d_{G_1}(u_j)^3-d_{G_1}(u_j)^2)+\sum_{u_j\in V_1}(d_{G_1}(u_j)-1)\sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}}d_{G_1}(u_i)\Big)$$

$$=4\big(|E(Q(G_1))|-2|E_1|\big)|E_2|+|V_2|\big(F(G_1)+2M_2(G_1)-2M_1(G_1)\big).$$

Here u_iu_j is the edge corresponding to the vertex u and u_ju_k is the edge corresponding to the vertex x.

Thus we obtain

$$M_1(G_1*_Q G_2)=(|V_2|+2|E_2|)M_1(G_1)+|E_1|M_1(G_2)+2|V_2|M_2(G_1)$$
$$+|V_2|F(G_1)+2|E_2|(2|E(Q(G_1))|+3|E_1|).$$

Similarly,

$$M_2(G_1*_Q G_2)=\sum_{(u,v)(x,y)\in E(G_1*_QG_2)}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(x,y)\big)$$

$$=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(u,y)\big)+\sum_{v\in V_2}\sum_{ux\in E(Q(G_1)}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(x,v)\big).$$

Now we separately find the values of each part in the sum

$$\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(u,y)\big)=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big[\big(d_{Q(G_1)}(u)+d_{G_2}(v)\big)\big(d_{Q(G_1)}(u)+d_{G_2}(y)\big)\big]$$

$$=\sum_{u\in V_1^*}\sum_{vy\in E_2}\big[d_{Q(G_1)}(u)^2+d_{Q(G_1)}(u)\left(d_{G_2}(v)+d_{G_2}(y)\right)+d_{G_2}(v)d_{G_2}(y)\big]$$

$$=\sum_{vy\in E_2}\sum_{u_iu_j\in E_1}(d_{G_1}(u_i)+d_{G_1}(u_j))^2+\sum_{u_iu_j\in E_1}\sum_{vy\in E_2}(d_{G_1}(u_i)+d_{G_1}(u_j))(d_{G_2}(v)+d_{G_2}(y))$$

$$+\sum_{u_iu_j\in E_1}\sum_{vy\in E_2}d_{G_2}(v)d_{G_2}(y)$$

$$=\sum_{vy\in E_2}\sum_{u_iu_j\in E_1}(d_{G_1}(u_i)^2+d_{G_1}(u_j)^2+2d_{G_1}(u_i)d_{G_1}(u_j))+M_2(G_1)M_2(G_2)+|E_1|M_2(G_2)$$

$$=|E_2|F(G_1)+2|E_2|M_2(G_1)+M_2(G_1)M_2(G_2)+|E_1|M_2(G_2).$$

Now,

$$\sum_{v\in V_2}\sum_{ux\in E(Q(G_1))}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(x,v)\big)=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,x\in V_1^*}}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(x,v)\big)$$

$$+\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(x,v)\big).$$

Now we find each sum separately

$$\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,\,x\in V_1^*}}\big(d_{(G_1*_QG_2)}(u,v)d_{(G_1*_QG_2)}(x,v)\big)=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,\,x\in V_1^*}}d_{Q(G_1)}(u)\left(d_{Q(G_1)}(x)+d_{G_2}(v)\right)$$

$$=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,\,x\in V_1^*}}d_{Q(G_1)}(u)d_{Q(G_1)}(x)+d_{G_2}(v)d_{Q(G_1)}(u)$$

$$=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,\,x\in V_1^*}}d_{G_1}(u)d_{Q(G_1)}(x)+\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u\in V_1,\,x\in V_1^*}}d_{G_1}(u)d_{G_2}(v)$$

$$=|V_2|(F(G_1)+2M_2(G_1))+2|E_2|M_1(G_1).$$

The second part is

$$\sum_{v\in V_2}\left(\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}\left(d_{Q(G_1)}(u)+d_{G_2}(v)\right)\left(d_{Q(G_1)}(x)+d_{G_2}(v)\right)\right)$$

$$=\sum_{v\in V_2}\sum_{\substack{ux\in E(Q(G_1)),\\ u,x\in V_1^*}}\left(d_{Q(G_1)}(u)d_{Q(G_1)}(x)+d_{G_2}(v)\left(d_{Q(G_1)}(u)+d_{Q(G_1)}(x)\right)+d_{G_2}(v)^2\right)$$

$$=\sum_{v\in V_2}\left(\sum_{\substack{u_iu_j\in E_1,\\ u_ju_k\in E_1}}\left(d_{G_1}(u_i)+d_{G_1}(u_j)\right)\left(d_{G_1}(u_j)+d_{G_1}(u_k)\right)\right)$$

$$+\sum_{v\in V_2}d_{G_2}(v)\left(\sum_{\substack{u_iu_j\in E_1,\\ u_ju_k\in E_1}}\left(d_{G_1}(u_i)+d_{G_1}(u_j)+d_{G_1}(u_j)+d_{G_1}(u_k)\right)\right)+(|E(Q(G_1))|-2|E_1|)M_1(G_2)$$

$$= |V_2|\Big(\sum_{u_j\in V_1} C^2_{d_{G_1}(u_j)} d_{G_1}(u_j)^2 + \sum_{u_i,u_j\in V_1} r_{ij} d_{G_1}(u_i) d_{G_1}(u_j) + \sum_{u_j\in V_1}(d_{G_1}(u_j)-1)d_{G_1}(u_j)\sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}} d_{G_1}(u_i)\Big)$$

$$+2|E_2|\Big(2\sum_{u_j\in V_1} C^2_{d_{G_1}(u_j)} d_{G_1}(u_j) + \sum_{u_j\in V_1}(d_{G_1}(u_j)-1)\sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}} d_{G_1}(u_i)\Big) + \big(|E(Q(G_1))| - 2|E_1|\big)M_1(G_2)$$

$$= |V_2|\Big(\frac{1}{2}\sum_{u_j\in V_1}(d_{G_1}(u_j)^4 - d_{G_1}(u_j)^3) + \sum_{u_i,u_j\in V_1} r_{ij} d_{G_1}(u_i) d_{G_1}(u_j)\Big)$$

$$+|V_2|\Big(\sum_{u_j\in V_1} d_{G_1}(u_j)^2 \sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}} d_{G_1}(u_i) - 2M_2(G_1)\Big)$$

$$+2|E_2|\Big(\sum_{u_j\in V_1}(d_{G_1}(u_j)^3 - d_{G_1}(u_j)^2) + \sum_{u_j\in V_1}(d_{G_1}(u_j)-1)\sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}} d_{G_1}(u_i)\Big) + \big(|E(Q(G_1))| - 2|E_1|\big)M_1(G_2)$$

$$= |V_2|\Big(\frac{1}{2}M_4(G_1) - \frac{1}{2}F(G_1) + \sum_{u_i,u_j\in V_1} r_{ij} d_{G_1}(u_i) d_{G_1}(u_j) + \sum_{u_j\in V_1} d_{G_1}(u_j)^2 \sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}} d_{G_1}(u_i) - 2M_2(G_1)\Big)$$

$$+2|E_2|\left(F(G_1) + 2M_2(G_1) - 2M_1(G_1)\right) + (|E(Q(G_1))| - 2|E_1|)M_1(G_2).$$

Here u_iu_j is the edge corresponding to the vertex u and u_ju_k is the edge corresponding to the vertex x, r_{ij} denotes the number of common vertices adjacent to both u_i and u_j. Thus we obtain

$$M_2(G_1 *_Q G_2) = |E_2|M_2(G_1) + |E_1|M_2(G_2) + M_2(G_1)M_2(G_2) + 2|E_2|M_1(G_1)$$
$$+\frac{1}{2}\left[|V_2|M_4(G_1) + (2|E_2| + |V_2|F(G_1))\right]$$
$$+|V_2|\Big(\sum_{u_i,u_j\in V_1} r_{ij} d_{G_1}(u_i) d_{G_1}(u_j) + \sum_{u_j\in V_1} d_{G_1}(u_j)^2 \sum_{\substack{u_i\in V_1,\\ u_iu_j\in E_1}} d_{G_1}(u_i)\Big).$$

□

Theorem 4. *Let G_1 and G_2 be two connected graphs, then*

(*a*) $M_1(G_1 *_T G_2) = 2|E_2|M_1(G_1) + |E_1|M_1(G_2) + 2|V_2|M_2(G_1)$
$+ |V_2|F(G_1) + 4\big(|E(T(G_1))| - 3|E_1|\big)|E_2|,$

(*b*) $M_2(G_1 *_T G_2) = 5|E_2|M_2(G_1) + \big(4|V_2| + |E_1|\big)M_2(G_2) + M_2(G_1)M_2(G_2) - 2|E_2|M_1(G_1)$
$+ \big(|E(T(G_1))| - 3|E_1|\big)M_1(G_2) + \frac{1}{2}\big[|V_2|M_4(G_1) + (2|E_2| + |V_2|)F(G_1)\big]$
$+ |V_2|\Big(\sum_{u_i,u_j\in V_1} r_{ij} d_{G_1}(u_i) d_{G_1}(u_j) + \sum_{u_j\in V_1} d_{G_1}(u_j)^2 \sum_{u_i\in V_1, u_iu_j\in E_1} d_{G_1}(u_i)\Big).$

where r_{ij} denotes the number of common vertices adjacent to both u_i, u_j.

P r o o f. We prove this theorem using Theorem 2 and Theorem 3. When $u \in V_1^*$ and $vy \in E_2$

$$\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1 *_T G_2)}(u,v) + d_{(G_1 *_T G_2)}(u,y)\big) = \sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1 *_Q G_2)}(u,v) + d_{(G_1 *_Q G_2)}(u,y)\big).$$

From Theorem 3

$$\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1*_TG_2)}(u,v)+d_{(G_1*_TG_2)}(u,y)\big)=2|E_2|M_1(G_1)+|E_1|M_1(G_2).$$

Also

$$\begin{aligned}&\sum_{v\in V_2}\sum_{ux\in E(T(G_1))}\big(d_{(G_1*_TG_2)}(u,v)+d_{(G_1*_TG_2)}(x,v)\big)\\&=\sum_{v\in V_2}\sum_{\substack{ux\in E(T(G_1)),\\ u\in V_1,\,x\in V_1^*}}\big(d_{(G_1*_TG_2)}(u,v)+d_{(G_1*_TG_2)}(x,v)\big)\\&+\sum_{v\in V_2}\sum_{\substack{ux\in E(T(G_1)),\\ u,x\in V_1^*}}\big(d_{(G_1*_TG_2)}(u,v)+d_{(G_1*_TG_2)}(x,v)\big)\\&+\sum_{v\in V_2}\sum_{\substack{ux\in E(T(G_1)),\\ u,x\in V_1}}\big(d_{(G_1*_TG_2)}(u,v)+d_{(G_1*_TG_2)}(x,v)\big).\end{aligned}$$

Also from Theorem 2 and Theorem 3

$$\begin{aligned}&\sum_{v\in V_2}\sum_{ux\in E(T(G_1))}\big(d_{(G_1*_TG_2)}(u,v)+d_{(G_1*_TG_2)}(x,v)\big)\\&=4\big(|E(T(G_1))|-3|E_1|\big)|E_2|+|V_2|\big(F(G_1)+2M_2(G_1)-2M_1(G_1)\big)+2|V_2|M_1(G_1).\end{aligned}$$

Thus,

$$\begin{aligned}M_1(G_1*_TG_2)&=2|E_2|M_1(G_1)+|E_1|M_1(G_2)+2|V_2|M_2(G_1)\\&+|V_2|F(G_1)+4\big(|E(T(G_1))|-3|E_1|\big)|E_2|.\end{aligned}$$

Similarly for M_2, from Theorem 3

$$\begin{aligned}&\sum_{u\in V_1^*}\sum_{vy\in E_2}\big(d_{(G_1*_TG_2)}(u,v)d_{(G_1*_TG_2)}(u,y)\big)\\&=|E_2|F(G_1)+|E_2|M_2(G_1)+M_2(G_1)M_2(G_2)+|E_1|M_2(G_2).\end{aligned}$$

The second part of the sum is

$$\begin{aligned}\sum_{v\in V_2}\sum_{ux\in E(T(G_1))}\big(d_{(G_1*_TG_2)}(u,v)d_{(G_1*_TG_2)}(x,v)\big)&=\sum_{v\in V_2}\sum_{\substack{ux\in E(T(G_1)),\\ u\in V_1,\,x\in V_1^*}}\big(d_{(G_1*_TG_2)}(u,v)d_{(G_1*_TG_2)}(x,v)\big)\\&+\sum_{v\in V_2}\sum_{\substack{ux\in E(T(G_1)),\\ u,x\in V_1^*}}\big(d_{(G_1*_TG_2)}(u,v)d_{(G_1*_TG_2)}(x,v)\big)\\&+\sum_{v\in V_2}\sum_{ux\in E(T(G_1)),u,x\in V_1}\big(d_{(G_1*_TG_2)}(u,v)d_{(G_1*_TG_2)}(x,v)\big).\end{aligned}$$

From Theorem 2 and Theorem 3 we get

$$\begin{aligned}&\sum_{v\in V_2}\sum_{ux\in E(T(G_1))}\big(d_{(G_1*_TG_2)}(u,v)d_{(G_1*_TG_2)}(x,v)\big)=|V_2|\big(F(G_1)+2M_2(G_1)\big)+2|E_2|M_1(G_1)\\&+|V_2|\Bigg(\frac{1}{2}M_4(G_1)-\frac{1}{2}F(G_1)+\sum_{u_i,u_j\in V_1}r_{ij}d_{G_1}(u_i)d_{G_1}(u_j)+\sum_{u_j\in V_1}d_{G_1}(u_j)^2\sum_{\substack{u_i\in V_1\\ u_iu_j\in E_1}}d_{G_1}(u_i)-2M_2(G_1)\Bigg)\\&+2|E_2|\big(F(G_1)+2M_2(G_1)-2M_1(G_1)\big)+\big(|E(Q(G_1))|-2|E_1|\big)M_1(G_2)+4|V_2|M_2(G_2),\end{aligned}$$

here r_{ij} denotes the number of common vertices adjacent to both u_i, u_j. Thus we obtain

$$M_2(G_1 *_T G_2) = 5|E_2|M_2(G_1) + \big(4|V_2| + |E_1|\big)M_2(G_2) + M_2(G_1)M_2(G_2) - 2|E_2|M_1(G_1)$$
$$+\big(|E(T(G_1))| - 3|E_1|\big)M_1(G_2) + \frac{1}{2}\big[|V_2|M_4(G_1) + (2|E_2| + |V_2|)F(G_1)\big]$$
$$+|V_2|\bigg(\sum_{u_i,u_j\in V_1} r_{ij}d_{G_1}(u_i)d_{G_1}(u_j) + \sum_{u_j\in V_1} d_{G_1}(u_j)^2 \sum_{u_i\in V_1, u_iu_j\in E_1} d_{G_1}(u_i)\bigg).$$

□

4. Applications with illustration

The above computational procedure can be used to find the respective indices for many classes of graphs very easily. As an illustration we provide the following.

Example 1. When $G_1 = P_n$, $G_2 = P_m$, $n, m > 3$, using the theorem, we easily obtain the following results

1. $M_1(P_n *_S P_m) = 20mn - 22m - 14n + 14$,
   $M_2(P_n *_S P_m) = 32mn - 40m - 24n + 38$;
2. $M_1(P_n *_R P_m) = 32mn - 40m - 14n + 14$,
   $M_2(P_n *_R P_m) = 64mn - 48m + 24n - 80$;
3. $M_1(P_n *_Q P_m) = 40mn - 64m - 22n + 30$,
   $M_2(P_n *_Q P_m) = 96mn - 184m + 18n + 134$;
4. $M_1(P_n *_T P_m) = 48mn - 82m - 22n + 30$,
   $M_2(P_n *_T P_m) = 136mn - 258m - 86n + 146$.

Let $\mathcal{T}_{n,m}$ denote the torus grid graph obtained from the cycle C_n and C_m. Using F^* sums, we can compute the Zagreb indices of torus grid graph $\mathcal{T}_{2n,m}$ since $\mathcal{T}_{2n,m} = C_n *_S C_m$.

Example 2. When $G_1 = C_n$, $G_2 = C_m$, $n, m > 3$, using the theorem, we easily obtain the following results

1. $M_1(C_n *_S C_m) = 20mn$,
   $M_2(C_n *_S P_m) = 32mn$;
2. $M_1(C_n *_R C_m) = 32mn$,
   $M_2(C_n *_R C_m) = 48mn$;
3. $M_1(C_n *_Q C_m) = 40mn$,
   $M_2(C_n *_Q C_m) = 96mn$;
4. $M_1(C_n *_T C_m) = 52mn$,
   $M_2(C_n *_T C_m) = 136mn$.

We can also find the Zagreb indices of some chemical structures using the expressions of F^* sums.

Example 3. Let $n \geq 3$ be an integer, then Zagreb indices of the the zigzag polyhex nanotube $TUHC6[2n, 2]$

$M_1(TUHC6[2n,2]) = 26n$,
$M_2(TUHC6[2n,2]) = 33n$.
Since $TUHC6[2n,2] = C_n *_S P_2$, then by Theorem 1.

Using F^* sums, we can also find the Zagreb indices of some classes of bridge graphs. Let $v_1, v_2, \ldots, v_n$ be vertices of graphs $G_1, G_2, \ldots, G_n$ respectively. The bridge graph using $v_1, v_2, \ldots, v_n$ is secured by joining the vertices v_i of G_i to v_{i+1} of G_{i+1} for $i = 1, 2, \ldots n-1$ and it is denoted by $B(G_1, G_2, \ldots, G_n; v_1, v_2, \ldots, v_n)$. If $G_i \cong G_{i+1} \cong G$ and $v_i = v_{i+1} = v$ for all $i = 1, 2, \ldots n$, then $B(G, G, \ldots, G; v, v \ldots, v) = G_n(G, v)$. Let $B_n = G_n(P_3, v)$ where the degree $d(v) = 2$ and $T_{n,3} = G_n(C_3, v)$ [14] be two class of bridge graphs.

Example 4. Let $n \geq 2$ be an integer, then
$M_1(B_n) = 18n - 14$,
$M_2(B_n) = 24n - 28$;

$M_1(T_{n,3}) = 24n - 14$,
$M_2(T_{n,3}) = 36n - 32$.
Since $B_n = P_2 *_S P_n$ and $T_{n,3} = P_2 *_R P_n$ and by Theorem 1 and Theorem 2.

5. Summary and Conclusion

The F sum of graphs was a new sum defined by M. Eliasi, B. Taeri in [6], a lot of research has been done on this to compute various topological indices of this F sum. In this paper we have defined a similar new operation and computed the first and second Zagreb index of this sum. Computing other topological indices on these sums is an area which researchers may find helpful.

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REFERENCES

1. Alex L., Indulal G. Some degree based topological indices of a generalised F sums of graphs. *Electron. J. Math. Anal. Appl.*, 2021. Vol. 9, No. 1. P. 91–111.
2. Alex L., Indulal G. On the Wiener index of F_H sums of graphs. *J. Comput. Sci. Appl. Math.*, 2021. Vol. 3, No. 2. P. 37–57. DOI: 10.37418/jcsam.3.2.1
3. Balaban A. T., Motoc I., Bonchev D., Mekenyan O. Topological indices for structure-activity correlations. In: *Topics Curr. Chem., vol 114: Steric Effects in Drug Design.* Berlin, Heidelberg: Springer, 1983. P. 21–55. DOI: 10.1007/BFb0111212
4. Cvetković D. M., Doob M., Sachs H. *Spectra of Graphs: Theory and Application.* New York: Academic Press, 1980. 368 p.
5. Deng H., Sarala D., Ayyaswamy S. K., Balachandran S. The Zagreb indices of four operations on graphs. *Appl. Math. Comput.*, 2016. Vol. 275. P. 422–431. DOI: 10.1016/j.amc.2015.11.058
6. Eliasi M., Taeri B. Four new sums of graphs and their Wiener indices. *Discrete Appl. Math.,* 2009. Vol. 157, No. 4. P. 794–803. DOI: 10.1016/j.dam.2008.07.001
7. Furtula B., Gutman I. A forgotten topological index. *J. Math. Chem.,* 2015. Vol. 53. P. 1184-1190. DOI: 10.1007/s10910-015-0480-z
8. Gutman I. On the origin of two degree-based topological indices. *Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur.,* 2014, Vol. 146. P. 39–52.
9. Gutman I., Das K. C. The first Zagreb index 30 years after. *MATCH Commun. Math. Comput. Chem.,* 2004. Vol. 50. P. 83–92.

10. Gutman I., Milovanović E., Milovanović I. Beyond the Zagreb indices. *AKCE Int. J. Graphs and Combinatorics*, 2018. DOI: 10.1016/j.akcej.2018.05.002
11. Gutman I., Trinajstić N. Graph theory and molecular orbitals. Total φ-electron energy of alternant hydrocarbons. *Chem. Phys. Lett.,* 1972, Vol. 17, No. 4. P. 535–538. DOI: 10.1016/0009-2614(72)85099-1
12. Gutman I., Ruščić B., N. Trinajstić, Wilcox C.F. Graph theory and molecular orbitals. XII. Acyclic polyenes. *J. Chem. Phys.*, 1975. Vol. 62, No. 9. P. 3399–3405.
13. Gutman I. An exceptional property of the first Zagreb index. *MATCH Commun. Math. Comput. Chem.,* 2014. Vol. 72. P. 733–740.
14. Imran M., Akhter S., Iqbal Z. Edge Mostar index of chemical structures and nanostructures using graph operations. *Int. J. Quantum Chem.,* 2020. Vol. 120, No. 15. Art. no. e26259. DOI: 10.1002/qua.26259
15. Indulal G., Alex L., Gutman I. On graphs preserving PI index upon edge removal. *J. Math. Chem.*, 2021. Vol. 59. P. 1603–1609. DOI: 10.1007/s10910-021-01255-1
16. Khalifeh M.H., Yousefi-Azari H., Ashrafi A.R. The first and second Zagreb indices of some graph operations. *Discrete Appl. Math.*, 2009. Vol. 157. P. 804–811. DOI: 10.1016/j.dam.2008.06.015
17. Li X., Zhao H. Trees with the first three smallest and largest generalized topological indices. *MATCH Commun. Math. Comput. Chem.,* 2004. Vol. 50. P. 57–62.
18. Li X., Zheng J. A unified approach to the extremal trees for different indices. *MATCH Commun. Math. Comput. Chem.,* 2005. Vol. 54, P. 195–208.
19. Nikolić S., Kovačević G., Miličević A., Trinajstić N. The Zagreb indices 30 years after. *Croat. Chem. Acta.*, 2003. Vol. 76, No. 2. P. 113–124.
20. Stevanović D. *Mathematical Properties of Zagreb Indices.* Beograd: Akademska misao, 2014. (in Serbian)
21. Wiener H. Structural determination of paraffin boiling points. *J. Am. Chem. Soc.,* 1947. Vol. 69. P. 17–20. DOI: 10.1021/ja01193a005