Stability of domination in graphs

Reeja Kuriakose*
Parvathy K S†

Abstract

The stability of dominating sets in Graphs is introduced and studied, in this paper. Here D is a dominating set of Graph G. In this paper the vertices of D and vertices of $V-D$ are called donors and acceptors respectively. For a vertex u in D, let $\psi_D(u)$ denote the number $|N(u)\cap(V-D)|$. The donor instability or simply d-instability $d^D_{inst}(e)$ of an edge e connecting two donor vertices v and u is $|\psi_D(u)-\psi_D(v)|$. The d-instability of D, $\psi_d(D)$ is the sum of d-instabilities of all edges connecting vertices in D. For a vertex u not in D, let $\phi_D(u)$ denote the number $|N(u)\cap D|$. The Acceptor Instability or simply a-instability $a^D_{inst}(e)$ of an edge e connecting two acceptor vertices u and v is $|\phi_D(u)-\phi_D(v)|$. The a-instability of D, $\phi_a(D)$ is the sum of a-instabilities of all edges connecting vertices in $V-D$. The dominating set D is d-stable if $\psi_d(D)=0$ and a-stable if $\phi_a(D)=0$. D is stable, if $\psi_d(D)=0$ and $\phi_a(D)=0$. Given a non negative integer α, D is α-d-stable, if $d^D_{inst}(e)\leq\alpha$ for any edge e connecting two donor vertices and D is α-a-stable, if $a^D_{inst}(e)\leq\alpha$ for any edge e connecting two acceptor vertices. Here we study α-stability number of graphs for non negative integer α.

Keywords: Domination number; stable domination
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*Department of Mathematics, Govt. Polytechnic College, Koratty, Thrissur, Kerala, India; reejaiykulambil@gmail.com, reeja.kuriakose@smctsr.ac.in

†Department of Mathematics, St. Mary's College, Thrissur, Kerala, India; parvathy.ks@smctsr.ac.in (Corresponding author).

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1 Introduction

Graphs considered here are the simple undirected graphs. The study of domination in graphs started in 1850 with a problem of finding the minimum number of queens that are needed to place on a Chess board such that each field not occupied by queen can be attacked by atleast one. In 1962, Ore was the first in publishing about domination in graphs. Many researchers including Ernie Cockayne, Michael Henning have contributed much to this field. A beautiful survey of results in domination is given in fundamentals of domination in graphs by W.Haynes et al. [1998]. The theory of domination has applications in the study of facility location problems, social networking and so on. Several types of domination number such as perfect domination number, connected domination number, have been defined and studied.
In this paper the elements of a dominating set are called donors and the other vertices are called acceptors. A subset D of vertices in a social network graph with the condition that each member in D dominates almost equally many members in $V - D$, or that each member in $V - D$ is dominated by almost equally many members in D, or both, plays a key role. This concept of equitable domination in graphs was defined and studied by A. Anitha, S. Arumugam and Mustapha Chellali. It is referred in the paper Anitha et al. [2011]. In social network problems related to marketing, banking and others the instability affects the system when adjacent acceptors are dominated by unequal number of donors or adjacent donors dominates unequal number of acceptors. This situation become worse when the instability is large. Motivated from this idea the concept of α-stability of dominating sets is being introduced.

2 α-d stable domination

Definition 2.1. *Let D be a dominating set. For a vertex u in D let $\psi_D(u) = |N(u) \cap (V - D)|$. The donor instability or d-instability of an edge e connecting two donor vertices u and v, $d^D_{inst}(e)$= $|\psi_D(u) - \psi_D(v)|$. Let $D \subset V$, the d-instability of D, is the sum of d-instabilities of all edges connecting vertices in D, $\psi_d(D) = \sum_{e \in G[D]} d^D_{inst}(e)$.*

Definition 2.2. *Let D be a dominating set. Given a non negative integer α, D is an α-d-stable dominating set, if $d^D_{inst}(e) \leq \alpha$ for any edge e connecting two donor vertices. Cardinality of a minimum α-d-stable dominating set is α-d-stable domination number and it is denoted by $\gamma^\alpha_d(G)$.*

Definition 2.3. *A dominating set D is d-stable if $\psi_d(D) = 0$. Cardinality of a minimum d-stable dominating set is d-stable domination number and it is denoted* *by* $\gamma_d^0(G)$.

Observation 2.1. *If* $\alpha \geq \beta$, *then* $\gamma(G) \leq \gamma_d^{\alpha}(G) \leq \gamma_d^{\beta}(G)$.

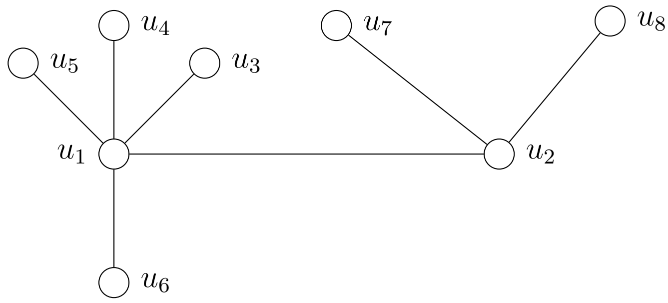

Figure 1: Graph with $\gamma(G) \leq \gamma_d^{\alpha}(G) \leq \gamma_d^{\beta}(G)$

Example 2.1. *In Figure 1,* $D = \{u_1, u_2\}$ *is the minimum dominating set.* $\psi_D(u_1) = 4$ *and* $\psi_D(u_2) = 2$. *And* $d_{inst}^D(u_1u_2) = 2$. *Hence* D *is a minimum 2-*d*-stable dominating set. And for* $\alpha \geq 2$, $\gamma_d^{\alpha}(G) = 2$.
If $S = \{u_1, u_7, u_8\}$, $\psi_D(u_1) = 3$, $\psi_D(u_7) = 0$ *and* $\psi_D(u_8) = 0$. *None of them are adjacent. Hence* $\gamma_d^1(G) = \gamma_d^0(G) = 3$.

Observation 2.2. *Property of being* α*-*d*-stable dominating set is neither super-hereditary nor hereditary.*

Theorem 2.1. *An* α*-*d*- stable dominating set* D *is a minimal* α*-*d*- stable dominating set if and only if for each vertex* v *in* D *one of the following conditions holds*

1. v *is an isolate of* D.
2. v *has a private neighbour* u *in* $V - D$.
3. *There exist two adjacent vertices* u_1 *and* u_2 *different from* v *in* D, u_1 *adjacent to v,* u_2 *not adjacent to v and* $\psi_D(u_1) = \psi_D(u_2) + \alpha$.

Proof. If an α-d- stable dominating set D is minimal, then D is an α-d- stable dominating set and for each vertex v in D, $D - \{v\}$ is not an α-d- stable dominating set. This means that some vertex u in $(V - D) \cup \{v\}$ is not dominated by $D - \{v\}$ or there exist two adjacent vertices u_1 and u_2 different from v in D with $|\psi_D(u_1) - \psi_D(u_2)| \leq \alpha$ but $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| > \alpha$.

Now if some vertex u in $(V - D) \cup \{v\}$ is not dominated by any vertex in $D - \{v\}$, either $u = v$, means v is an isolate of D or $u \in V - D$. If u is not dominated by $D - \{v\}$, then u is adjacent only to vertex v in D. ie, v has a private neighbour u in $V - D$.

If $|\psi_D(u_1) - \psi_D(u_2)| \leq \alpha$ and $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| > \alpha$, let $\alpha = 0$, then $\psi_D(u_1) = \psi_D(u_2)$ and $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| = \alpha + 1$. Assume $\psi_{D-\{v\}}(u_1) > \psi_{D-\{v\}}(u_2)$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\psi_D(u_1) = \psi_D(u_2) + \alpha$. If $\alpha > 0$, then assume $\psi_D(u_1) > \psi_D(u_2)$. Then $\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2) = \alpha + 1$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\psi_D(u_1) = \psi_D(u_2) + \alpha$.

Conversely, suppose that D is an α-d-stable dominating set and for each vertex $v \in D$, one of the three statements holds. We show that D is a minimal α-d-stable dominating set. If D is not a minimal α-d-stable dominating set, then there exists a vertex $v \in D$ such that $D - \{v\}$ is an α-d-stable dominating set. Then each vertex u in $(V - D) \cup \{v\}$ is adjacent with atleast one vertex in $D - \{v\}$. Then v is not an isolate of D and condition 1 does not hold. And v has no private neighbour in $V - D$ and condition 2 does not hold. $D - \{v\}$ is an α-d-stable dominating set implies for any two adjacent vertices u_1 and u_2 in $D - \{v\}$, $\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2) \leq \alpha$. Hence condition 3 does not hold. Hence D is a minimal α-d-stable dominating set. □

Observation 2.3. *For non negative integer α, $\gamma_d^\alpha(G) = 1 \iff \gamma(G) = 1$.*

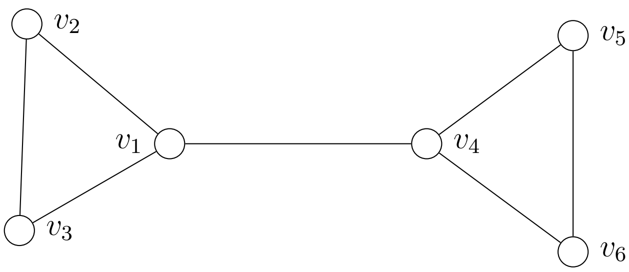

Figure 2: Graph with $\gamma_d^\alpha(G) = 1$

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Theorem 2.2. *For a graph G and non negative integer α, $\beta_o(G) \geq \gamma_d^\alpha(G)$.*

Proof. Let S be a maximum independent set. Then, every vertex in $V - S$ is adjacent with atleast one vertex in S. Thus S is a dominating set. S is an independent set. It follows that S is a d-stable dominating set. Hence, $\beta_o(G) \geq \gamma_d^\alpha(G)$.For the graph in figure 2, $\gamma_d^\alpha(G) = 2 = \beta_o(G)$. Hence the bound is sharp. □

Theorem 2.3. *If D is an α-d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) = d(u) + k + \alpha$, $k \in Z^+$, then D contains atleast k elements from $(N[v] - N[u])$.*

Proof. If D is an α-d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) = d(u) + k + \alpha$, $k \in Z^+$, then $|\psi_D(v) - \psi_D(u)| \leq \alpha$.

Hence $|N[v] \cap (V - D)| \leq |N[u] \cap (V - D)| + \alpha$. Thus, $d(v) - d(u) \leq |(N[v] - N[u]) \cap D| + \alpha$.

Hence D contains atleast k elements from $(N[v] - N[u])$. □

Corolary 2.1. *If D is a d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) > d(u)$, then D contains atleast $d(v) - d(u)$ elements from $(N[v] - N[u])$.*

Corolary 2.2. *If u is a pendant vertex adjacent to v, D is a d-stable dominating set and $u, v \in D$, then $N[v] \subset D$.*

Theorem 2.4. *For any non negative integer β, there exist graph G with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)...... > \gamma_d^\beta(G)$*

Proof. Take $k = \beta + 1$
Construct G as follows,
Step 1:- Let H be the complete graph with vertex set $\{a_1, a_2, ..., a_k\}$.
Step 2:- Let $A_i = \{a_{i,1}, a_{i,2}, ..., a_{i,i+1}\}$ for $i = 1, 2, ..., k$.
Form G by joining each vertices in A_i with a_i in H for $i = 1, 2..., k$.
Let D be a d-stable dominating set, $B_i = \{a_i, a_{i,1}, a_{i,2}, ..., a_{i,i+1}\}$ and $C_i = B_i \cap D$.

If $a_r, a_s \in D$ with $r < s$ then by corollary 2.1, D contains $s - r$ elements from A_s and hence by corollary 2.2, $N[a_s] \subset D$. Thus $B_s \subset D$. Thus $C_i = B_i$ for all $i = 1, 2, ..., k$.

Hence $D = V(G)$ or $D = A_1 \cup A_2 \cup A_3 \cup ...... \cup A_{s-1} \cup A_{s+1} \cup ..... \cup A_{k-1} \cup A_k \cup \{a_s\}$. Hence we take $D = A_1 \cup A_2 \cup A_3 \cup ...... \cup A_{s-1} \cup A_{s+1} \cup ..... \cup A_{k-1} \cup A_k \cup \{a_s\}$.

Thus $|C_i| = i + 1$ for $i \neq s$ and $|C_s| = 1$. Then,

$$\begin{aligned} |D| &= 2 + 3 + ... + (s) + 1 + (s + 2) + ... + (k + 1) \\ &= \tfrac{(k+1)(k+2)}{2} - (s + 1). \end{aligned}$$

Hence $|D|$ is minimum when $\frac{(k+2)(k+1)}{2} - (s+1)$ is minimum. That is when $s = k$.
And $D = A_1 \cup A_2 \cup A_3 \cup ...... \cup A_{k-1} \cup \{a_k\}$ will form a d- stable dominating set with $|D| = \frac{(k+2)(k+1)}{2} - (k + 1)$.
Hence $\gamma_d^0(G) = \frac{(k+2)(k+1)}{2} - (k + 1) = \frac{(k)(k+1)}{2}$.
Similarly,

$$\begin{aligned} \gamma_d^0(G) &= \tfrac{(k)(k+1)}{2} \\ \gamma_d^1(G) &= \tfrac{k(k-1)}{2} + 1 \\ &\vdots \\ \gamma_d^\beta(G) &= \tfrac{(k-\beta+1)(k-\beta)}{2} + \beta. \end{aligned}$$

Figure 3 illustrates the graph with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)$. □

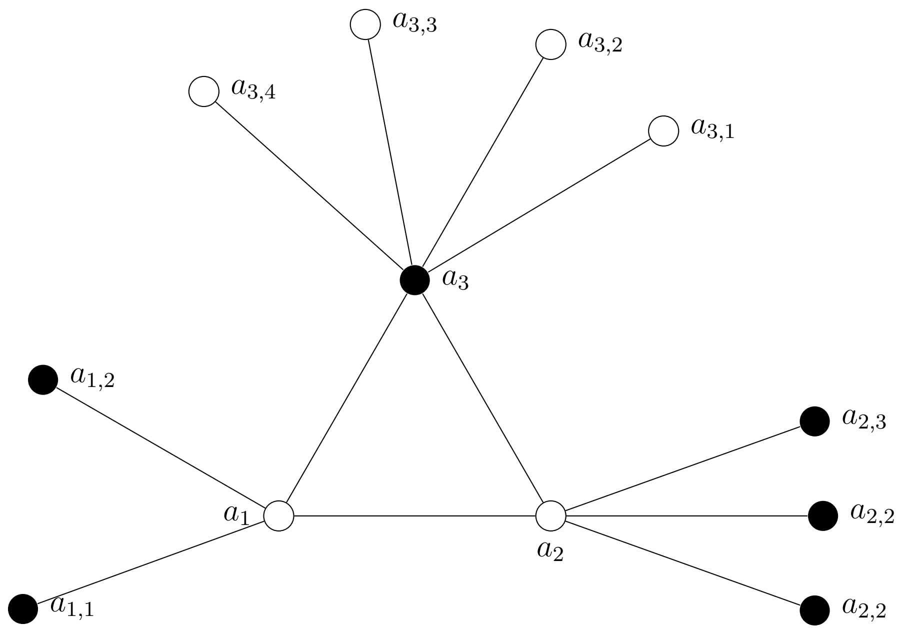

Figure 3: Graph with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)$

3 α-a-stable domination

Definition 3.1. *Let D be a dominating set. For a vertex u not in D, let $\phi_D(u)$ = $|N(u) \cap D|$. The Acceptor Instability or a-instability of an edge e connecting two acceptor vertices u and v is, $a_{inst}^D(e)$ = $|\phi_D(u) - \phi_D(v)|$. The a-instability of D, $\phi_a(D)$ is the sum of a-instabilities of all edges connecting vertices in $V - D$, $\phi_a(D) = \sum_{e \in G[V-D]} a_{inst}^D(e)$.*

Definition 3.2. *Let D be a dominating set. Given a non negative integer α, D is an α-a-stable dominating set, if $a_{inst}^D(e) \leq \alpha$ for any edge e connecting two acceptor vertices. Cardinality of a minimum α-a-stable dominating set is α-a-stable domination number and denoted by $\gamma_a^\alpha(G)$.*

Definition 3.3. *The dominating set D is a-stable if $\phi_a(D) = 0$. Minimum cardinality of an a-stable dominating set is a-stable domination number and denoted by $\gamma_a^0(G)$.*

Observation 3.1. *If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma_a^\alpha(G) \leq \gamma_a^\beta(G)$*

Example 3.1. *In Figure 1, $D = \{u_1, u_2\}$ is the minimum dominating set. $\phi_D(u_3) = \phi_D(u_4) = \phi_D(u_5) = \phi_D(u_6) = \phi_D(u_7) = \phi_D(u_8) = 1$. Hence D is a minimum a-stable dominating set and $\gamma_a^\alpha(G) = 2$ for all non negative integer α.*

Observation 3.2. *Property of being α-a-stable dominating set is neither super-hereditary nor hereditary.*

Theorem 3.1. *An α-a- stable dominating set D is a minimal α-a- stable dominating set if and only if for each vertex v in D one of the following conditions holds*

1. *v is an isolate of D.*
2. *v has a private neighbour u in $V - D$.*
3. *There exist two adjacent vertices u_1 and u_2 in V-D, u_1 adjacent to v, u_2 not adjacent to v and $\phi_D(u_2) = \phi_D(u_1) + \alpha$.*

Proof. If an α-a- stable dominating set D is minimal then D is an α-a- stable dominating set and for each vertex v in D, $D - \{v\}$ is not an α-a- stable dominating set. This means that some vertex u in $(V - D) \cup \{v\}$ is not dominated by $D - \{v\}$ or there exist two adjacent vertices u_1 and u_2 in $V - D$ with $|\phi_D(u_1) - \phi_D(u_2)| \leq \alpha$ but $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| > \alpha$.
Now if some vertex u in $(V - D) \cup \{v\}$ is not dominated by any vertex in $D - \{v\}$, either $u = v$, means v is an isolate of D or $u \in V - D$. If u is not dominated by $D - \{v\}$, then u is adjacent only to vertex v in D. ie, v has a private neighbour u in $V - D$.
If $|\phi_D(u_1) - \phi_D(u_2)| \leq \alpha$ and $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| > \alpha$, let $\alpha = 0$, then $\phi_D(u_1) = \phi_D(u_2)$ and $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| = \alpha + 1$. Assume $\phi_{D-\{v\}}(u_2) > \phi_{D-\{v\}}(u_1)$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\phi_D(u_2) = \phi_D(u_1) + \alpha$. If $\alpha > 0$, then assume $\phi_D(u_2) > \phi_D(u_1)$. Then $\phi_D(u_2) - \phi_D(u_1) = \alpha$ and $\phi_{D-\{v\}}(u_2) - \phi_{D-\{v\}}(u_1) = \alpha + 1$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\phi_D(u_2) = \phi_D(u_1) + \alpha$.
Conversely, suppose that D is an α-a-stable dominating set and for each vertex $v \in D$, one of the three statements holds. We show that D is a minimal α-a-stable dominating set. If D is not a minimal α-a-stable dominating set,then there exists a vertex $v \in D$ such that $D - \{v\}$ is an α-a-stable dominating set. Then each vertex u in $(V - D) \cup \{v\}$ is adjacent with atleast one vertex in $D - \{v\}$. Then v is not an isolate of D and condition 1 does not hold. And v has no private neighbour in $V - D$ and condition 2 does not hold. If $D - \{v\}$ is an α-a-stable dominating set then for any adjacent vertices u_1 and u_2 in $(V - D) \cup \{v\}$, $\phi_{D-\{v\}}(u_2) - \phi_{D-\{v\}}(u_1) \leq \alpha$. Hence condition 3 does not hold. Hence D is a minimal α-a-stable dominating set. □

Observation 3.3. *For non negative integer α, $\gamma_a^\alpha(G) = 1 \iff \gamma(G) = 1$*

Theorem 3.2. *For $\alpha \geq 1$, $\gamma_a^\alpha(G) = 2 \iff \gamma(G) = 2$*

Proof. If $\gamma(G) = 2$, then for a minimum dominating set D, $|D| = 2$

$$|D| = 2 \Rightarrow \phi_D(v) = 1 \quad or \quad \phi_D(v) = 2 \quad \forall v \in V - D$$

$$\Rightarrow |\phi_D(v_1) - \phi_D(v_2)| \leq 1, \qquad \forall v_1, v_2 \in V - D$$

$$\Rightarrow \gamma_a^\alpha(G) = 2$$

Conversely, if $\gamma_a^\alpha(G) = 2$, then $\gamma(G) \neq 1$. If D is a minimum α-a- stable dominating set , then $|D| = 2$, and D is a dominating set. Thus, $\gamma(G) = 2$ □

Theorem 3.3. *For any graph G and non negative integer α, $\gamma_a^\alpha(G) \leq \gamma_p(G)$. And this bound is sharp.*

Proof. If D is a γ_P- set, then $|N[v] \cap D| = 1, \forall v \in (V - D)$. Hence $\phi_D(v) = 1$, for all $v \in (V - D)$. And so D is an a-stable dominating set . Thus every perfect dominating set is an a-stable dominating set . Hence, $\gamma_a^\alpha(G) \leq \gamma_p(G)$. For $G = P_{3n}$, $\gamma_p(G) = n = \gamma_a^\alpha(G)$. So we can see that the bound is sharp. □

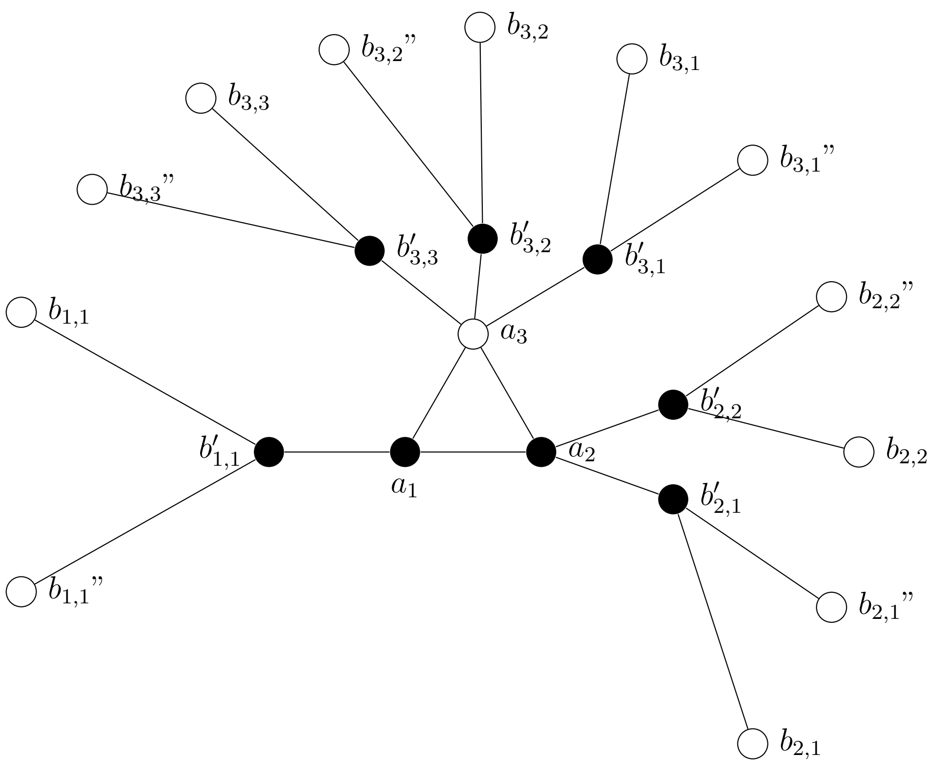

Figure 4: Graph with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)$

Theorem 3.4. *For any positive integer β, there exist graph G with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)...... > \gamma_a^\beta(G)$.*

Proof. Let $k = \beta + 1$. Construct G as follows
Step 1:Let H be the complete graph with vertex set $\{a_1, a_2, ..., a_k\}$
Step 2: For each i take i copies of P_3 with vertex set $\{b_{i,j}, b'_{i,j}, b_{i,j}"\}$ for $j = 1, 2, ..., i$ and join $b'_{i,j}$ with a_i for each $j = 1, 2, ..., i$.
Let $A_i^j = \{b_{ij}, b'_{ij}, b_{ij}"\}$ for $j = 1, 2, ...i$ and $A_i = \{a_i\} \cup \cup_{j=1}^{i} \{b_{ij}, b'_{ij}, b_{ij}"\}$ for all $i \in \{1, 2, ..., k\}$.
Let D be an a-stable dominating set. If $a_i \in D$ then $|A_i \cap D| \geq i + 1$.
Let r be the smallest integer such that $a_r \notin D$. Then $|A_r \cap D| \geq r$.
If $s > r$ and $a_s \notin D$, since $\gamma_{inst}^a(a_r, a_s) = 0$, $b'_{sj} \in D$ for atmost r values of j.
And if there exist j for which $b'_{sj} \notin D$ then $b_{sj}, b_{sj}" \in D$.

$$\implies \begin{array}{rcl} |A_s \cap D| & \geq & r + 2(s - r) \\ & = & 2s - r \geq s + 1 \end{array}$$

$$\begin{array}{rcl} Hence, |D| & \geq & |A_1 \cap D| + |A_2 \cap D| + |A_3 \cap D| + ... + |A_{r-1} \cap D| + \\ & & |A_r \cap D| + |A_{r+1} \cap D| + ... + |A_k \cap D| \\ & \geq & 2 + 3 + ... + (r - 1 + 1) + r + (r + 2) + ... + (k + 1) \ . \\ & = & 1 + 2 + ... + k + k - 1 \\ & = & \frac{k(k+1)}{2} + (k - 1) \end{array}$$

Thus, $\gamma_d^0(G) \geq \frac{k(k+1)}{2} + (k - 1)$.
And $D' = \{a_1, ..., a_{k-1}\} \cup \cup_{i=1}^{k} \{b'_{i1}, b'_{i2}, ...b'_{ii}\}$ is an a-stable dominating set with $|D'| = \frac{k(k+1)}{2} + (k - 1)$.
Hence, $\gamma_a^0(G) \leq \frac{k(k+1)}{2} + (k - 1)$
Thus, $\gamma_a^0(G) = \frac{k(k+1)}{2} + (k - 1)$
Similarly,

$$\begin{array}{rcl} \gamma_a^1(G) & = & \frac{k(k+1)}{2} + (k - 2) \\ \gamma_a^2(G) & = & \frac{k(k+1)}{2} + (k - 3) \\ \gamma_a^3(G) & = & \frac{k(k+1)}{2} + (k - 4) \\ & \vdots & \\ \gamma_a^{\beta-1}(G) & = & \frac{k(k+1)}{2} + 1 \\ \gamma_a^\beta(G) & = & \frac{k(k+1)}{2} \end{array}$$

Figure 4 illustrates the graph with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)$. □

4 α-stable domination

Definition 4.1. *A dominating set D is stable, if $\psi_d(D) = 0$ and $\phi_a(D) = 0$. Minimum cardinality of a stable dominating set is called stable domination number and denoted by $\gamma^0(G)$.*

Definition 4.2. *If a dominating set D is an α-d-stable dominating set and α-a - stable dominating set, then D is called an α-stable dominating set and cardinality of a minimum α-stable dominating set is defined as α-stable domination number and denoted by $\gamma^{\alpha}(G)$*

Observation 4.1. *If a minimum α -a-stable dominating set is an α-d-stable dominating set, then $\gamma^{\alpha}(G) = \gamma^{\alpha}_{a}(G)$. And if a minimum α-d- stable dominating set is an α-a- stable dominating set, then $\gamma^{\alpha}(G) = \gamma^{\alpha}_{d}(G)$.*

Observation 4.2. *If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma^{\alpha}(G) \leq \gamma^{\beta}(G)$.*

Definition 4.3. *Minimum α so that $\gamma^{\alpha}(G) = \gamma(G)$ is called stable dominating index and denoted by $I_{sd}(G)$.*

Example 4.1. *In figure 1, the minimum d-stable dominating set $\{u_1, u_7, u_8\}$ is an a-stable dominating set. Hence $\{u_1, u_7, u_8\}$ is a minimum stable dominating set and $\gamma^{0}(G) = 3$.*
A minimum 1-d-stable dominating set $\{u_1, u_7, u_8\}$ form a 1-a- stable dominating set and hence $\{u_1, u_7, u_8\}$ is a minimum 1-stable dominating set and $\gamma^{1}(G) = 3$.
And minimum dominating set $\{u_1, u_2\}$ is a 2-a- stable dominating set and a 2-d- stable dominating set. $\{u_1, u_2\}$ form a minimum 2-stable dominating set. Hence, $\gamma^{2}(G) = 2$.
And $\forall \alpha \geq 2$, $\gamma^{\alpha}(G) = 2 = \gamma(G)$. Hence, $I_{sd}(G) = 2$.

Compliment of a minimum α-stable dominating set need not be an α-stable dominating set. In graph figure 1, $\{u_1, u_2, u_3\}$ is a minimum 1-stable dominating set but its compliment is not a 1-stable dominating set.

Observation 4.3. *Property of being α-stable dominating set is neither super-hereditary nor hereditary.*

Theorem 4.1. *For any graph G and for any non-negative integer α, $\gamma(G) = 1 \iff \gamma^{\alpha}(G) = 1$.*

Proof. If $\gamma(G) = 1$, then the single vertex set $\{v\}$ which dominates all vertices of G, is an α-d-stable dominating set and an α-a-stable dominating set. Then $\gamma^{\alpha}(G) = 1$. Also any α-stable dominating set is a dominating set. So, if $\gamma^{\alpha}(G) = 1$ then $\gamma(G) = 1$. □

Lemma 4.1. *For any graph G and for any non negative integer α, $\gamma^{\alpha}(G) = n \iff G = \overline{K_n}$.*

Proof. If $G \neq \overline{K_n}$, there is atleast one vertex v with $d(v) \geq 1$. Then $V - \{v\}$ is an α-stable dominating set. This means that $\gamma^{\alpha}(G) \leq n - 1$. Hence if $\gamma^{\alpha}(G) = n$, then $G = \overline{K_n}$. If $G = \overline{K_n}$, then $\gamma^{\alpha}(G) = n$ trivially. □

Theorem 4.2. *For a graph G with $\delta(G) \geq 1$, $\gamma^{\alpha}(G) \leq n-1$.*

Proof. From lemma 3.9 it is clear that $\gamma^{\alpha}(G) \leq n-1$. □

Theorem 4.3. *For any non negative integer α, $\gamma^{\alpha}(G) = \gamma(G)$ for the following Graphs*

- *Complete graph K_n*
- *Path P_n*
- *Cycle C_n*
- *Wheel graph W_n*
- *Helm graph H_n*

Proof. In these graphs minimum dominating set D, form an α-stable dominating set. Hence α-stable domination number is same as its domination number. □

Theorem 4.4. *Let G and H be two graphs of order n_1 and n_2 , then for any non-negative integer α ,*

- $\gamma_a^{\alpha}(G\Box H) \leq min\{n_1\gamma_a^{\alpha}(H), n_2\gamma_a^{\alpha}(G)\}$
- $\gamma_d^{\alpha}(G\Box H) \leq min\{n_1\gamma_d^{\alpha}(H), n_2\gamma_d^{\alpha}(G)\}$
- $\gamma^{\alpha}(G\Box H) \leq min\{n_1\gamma^{\alpha}(H), n_2\gamma^{\alpha}(G)\}$.

Proof. Let S_H be a minimum α-a-stable dominating set of H. Let us see that $S = V(G) \times S_H$ is an α-a-stable dominating set of $G\Box H$. If $(u,v) \in (V(G) \times V(H)) - S$. Then (u,v) is adjacent to atleast one vertex in S. And if $(u_1,v_1) \in (V(G) \times V(H)) - S$ and $(u_2,v_2) \in (V(G) \times V(H)) - S$ and (u_1,v_1) adjacent to (u_2,v_2). Then,

$$\begin{aligned} \phi_S(u_1,v_1) &= |\{(u,v) \in S : (u_1,v_1) \ adjacent \ to \ (u,v)\}| \\ &= |\{(u,v) \in S : u_1 = u \ \ and \ \ v_1 \ adjacent \ \ to \ \ v\} \cup \\ &\quad \{(u,v) \in S : u_1 \ \ adjacent \ to \ u \ \ and \ \ v_1 = v\}| \\ &= |\{(u,v) \in S : u_1 = uand \ \ v_1 \ adjacent \ \ tov\}| \\ &= \phi_{S_H}(v_1). \end{aligned}$$

$\phi_S(u_2,v_2) = \phi_{S_H}(v_2)$. And

$$\begin{aligned} |\phi_S(u_1,v_1) - \phi_S(u_2,v_2)| &= |\phi_{S_H}(v_1) - \phi_{S_H}(v_2) \\ &\leq \alpha. \end{aligned}$$

Hence S is an α-a-stable dominating set of $G\Box H$.

Similarly, if S_G is a minimum α-a-stable dominating set of G, then $S_G \times V(H)$ is an α-a-stable dominating set of $G\Box H$.
Thus, $\gamma_a^\alpha(G\Box H) \leq min\{n_1\gamma_a^\alpha(H), n_2\gamma_a^\alpha(G)\}$.

Let S_H be a minimum α-d-stable dominating set of H. Let $S = V(G) \times S_H$. If $(u, v) \in (V(G) \times V(H)) - S$, then (u, v) is adjacent to atleast one vertex in S. And if $(u_1, v_1) \in S$ and $(u_2, v_2) \in S$ and (u_1, v_1) adjacent to (u_2, v_2). Then,

$$\begin{aligned}
&\psi_S(u_1, v_1)\\
=\ & |\{(u, v) \in (V(G) \times V(H)) - S : (u_1, v_1) \ \ adjacent \ \ to \ \ (u, v)\}|\\
=\ & |\{(u, v) \in (V(G) \times V(H)) - S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\} \cup\\
& \{(u, v) \in (V(G) \times V(H)) - S : u_1 \ \ adjacent \ \ to \ \ u \ \ and \ \ v_1 = v\}|\\
=\ & |\{(u, v) \in (V(G) \times V(H)) - S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\}|\\
=\ & \psi_{S_H}(v_1).
\end{aligned}$$

Similarly $\psi_S(u_2, v_2) = \psi_{S_H}(v_2)$.

Thus

$$\begin{aligned}
|\psi_S(u_1, v_1) - \psi_S(u_2, v_2)| &= |\psi_{S_H}(v_1) - \psi_{S_H}(v_2)\\
&\leq \alpha
\end{aligned}$$

Thus S is an α-d-stable dominating set of $G\Box H$.

Similarly, if S_G is a minimum α-d-stable dominating set of G, $S_G \times V(H)$ is an α-d-stable dominating set of $G\Box H$. Thus,
$\gamma_d^\alpha(G\Box H) \leq min\{n_1\gamma_d^\alpha(H), n_2\gamma_d^\alpha(G)\}$.

Hence, $\gamma^\alpha(G\Box H) \leq min\{n_1\gamma^\alpha(H), n_2\gamma^\alpha(G)\}$. □

Remark 4.1. *The bound in theorem 3.12 is attained if* $G = K_n$ *and* $H = K_2$*; because* $\gamma^\alpha(K_n\Box K_2) = 2 = min\{2\gamma^\alpha(K_n), n\gamma^\alpha(K_2)\}$.

Theorem 4.5. *For any two graphs G and* K *and non negative integer* α*,* α*-stable domination number of its corona,* $\gamma^\alpha(GoK) = |V(G)|$.

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of G and $\{u_1, u_2, ...u_m\}$ be the vertices of K. K_i be the i^{th} copy K in GoK. To make sure that each vertex of K_i is dominated, we need atleast one vertex of K_i or v_i. Thus the dominating set contains atleast n vertices. Let $D = \{v_1, v_2, ..., v_n\}$. Then each vertex of $V - D$ is adjacent with exactly one vertex of D and each vertex of D dominates exactly m vertices in $V - D$. And so $\psi_D(v) = m$, $\forall v \in D$ and $\phi_D(v) = 1, \forall v \in V - D$. Therefore, $|\phi_D(v_1) - \phi_D(v_2)| = 0, \forall v_1, v_2 \in V - D$ and $|\psi_D(v_1) - \psi_D(v_2)| = 0, \forall v_1, v_2 \in D$. And so D is a minimum α-stable dominating set. Thus $\gamma^\alpha(GoK) = n = |V(G)|$. □

5 Conclusions

Motivated from this idea the concept of α-stability of dominating sets is being introduced. The stability of dominating sets in Graphs is introduced and studied, in this paper. In social network problems related to marketing, banking and others the instability affects the system when adjacent acceptors are dominated by unequal number of donors or adjacent donors dominates unequal number of acceptors. Several types of domination number have been defined and studied. This situation become worse when the instability is large. Here we study α- stability number of graphs for non negative integer α. Instability of domination in Graphs the concept of stability of domination in Graphs are defined. The open problems are Characterize the graphs with $I_{sd}(G) = 0$. Characterize the graphs for which the compliment of minimum α-stable dominating set is an α-stable dominating set. Develop an algorithm to find the α-stable domination number of a graph. In future we extended this topic.

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