Engineering Journal issn: 0125-8281

Article

Design of Control Systems with Multiple Backlash Nonlinearities Subject to Inputs Restricted in Magnitude and Slope

Tadchanon Chuman[a] **and Suchin Arunsawatwong**[b,*]

Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University, Bangkok 10330, Thailand
E-mail: [a]tadchanon01@hotmail.com, [b,*]suchin.a@chula.ac.th (Corresponding author)

Abstract. This paper develops a computational method for designing a control system that is an interconnection of transfer functions and multiple decoupled backlash nonlinearities where each backlash is modelled as an uncertain band containing multi-valued functions. The design objective is to ensure that the system outputs and the nonlinearity inputs always stay within their prescribed bounds in the presence of all inputs whose magnitude and whose slope are bounded by respective numbers. By using a known technique, each backlash is decomposed as a linear gain and a bounded disturbance. Essentially, the original design problem is replaced by a surrogate design problem that is related to a linear system and thereby can readily be solved by tools available in previous work. Moreover, as a result of using the convolution algebra $\mathcal{A}$, the method developed here is applicable to rational and nonrational transfer functions. To illustrate the usefulness of the method, linear decentralized controllers are designed for a binary distillation column where valve stiction characteristics are taken into account.

Keywords: Nonlinear control systems, backlash, computer-aided design, peak output, process control, valve stiction, method of inequalities.

ENGINEERING JOURNAL Volume 27 Issue 6
Received 29 January 2023
Accepted 12 June 2023
Published 30 June 2023
Online at https://engj.org/
DOI:10.4186/ej.2023.27.6.11

DOI:10.4186/ej.2023.27.6.11

1. Introduction

Backlash nonlinearities are found in many practical control systems as well as system components such as sensors and actuators (see, e.g., [1, 2, 3] and also the references therein). They, if ignored during the control design, can give rise to significant performance degradation or even instability in the closed-loop systems. A number of researchers have been prompted to develop methods for compensating (or mitigating) the adverse effects of backlash so that the control systems are ensured to be stable or to operate satisfactorily.

An approach for backlash compensation is to employ adaptive control schemes for ensuring desirable properties in the closed-loop systems. Because the literature on this subject is extensive and the space available is limited, readers are referred to [1, 2] and also the references therein. Another approach is to design a linear controller, where one tries to obtain conditions on the dynamical subsystems that yields a satisfactory controller. For example, Barreiro and Baños [4] and Tarbouriech, Queinnec, and Prieur [5] provide useful conditions for guaranteeing the stability of control systems that are an interconnection of linear time-invariant (LTI) subsystems and a backlash. In fact, our work is in this direction.

In this paper, we develop a computational method for designing the control system shown in Fig. 1 where $\mathbf{p} \in \mathbb{R}^N$ is a vector of design parameters, $\Psi \triangleq [\psi_1, \psi_2, \ldots, \psi_n]^T$ is the vector of backlash nonlinearities, $\mathbf{z} \triangleq [z_1, z_2, \ldots, z_m]^T$ is the vector of outputs of interest, $\mathbf{v} \triangleq [v_1, v_2, \ldots, v_n]^T$ is the vector of the nonlinearity outputs, $\mathbf{u} \triangleq [u_1, u_2, \ldots, u_n]^T$ is the vectors of the nonlinearity inputs, and f denotes an exogenous input.

Suppose that the exogeneous input f is known only to the extent that it belongs to the input set $\mathcal{F}$ described by

$$\mathcal{F} = \{f \in \mathbb{L}^\infty : \|f\|_\infty \leq M \text{ and } \|\dot{f}\|_\infty \leq D\} \quad (1)$$

where M and D are given bounds. As usual, for a function $x : \mathbb{R}_+ \to \mathbb{R}$, $\|x\|_\infty \triangleq \sup_{t\geq 0} |x(t)|$ and $\mathbb{L}^\infty \triangleq \{x : \|x\|_\infty < \infty\}$. A noteworthy feature of the set $\mathcal{F}$ is that it is suitable for characterizing inputs that vary persistently for all time, called *persistent inputs*. When all inputs are persistent and do not have stepwise discontinuities, using $\mathcal{F}$ makes the design formulation more realistic and more appropriate than using $\mathbb{L}^\infty$ ([6, 7, 8]). For different characterizations of the input set and their implications, readers are referred to, e.g., [6, 7, 8, 9].

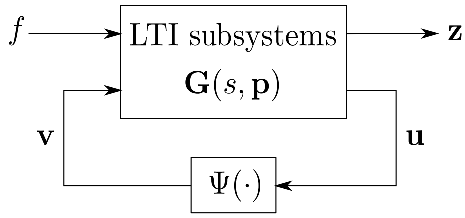


Fig. 1. General configuration for nonlinear control systems considered in the paper.

Suppose that each backlash ψ_j is described by an uncertain band model ([4])

$$\left.\begin{array}{rcl}\psi_j(x) & = & K_j x + \eta_j(x) \\ \eta_j(x) & = & [-a_j, a_j] \quad \forall x \in \mathbb{R}\end{array}\right\} \quad (2)$$

where K_j is a linear gain and $\eta_j(\cdot)$ is the multi-valued function mapping $\mathbb{R}$ to $2^{\mathbb{R}}$. Here, $2^{\mathbb{R}}$ denotes the set of all subsets of $\mathbb{R}$ ([10]). The model is depicted in Fig. 2. In spite of certain amount of conservatism, the uncertain band model has the following advantages: (i) it simplicity enables us to develop the design method presented here; (ii) the backlash width does not need to know exactly; and (iii) it is such a general model that contains some nonlinearities such as friction-driven hysteresis, backlash-like hysteresis, inertia-driven hysteresis and dead zone ([11, 2, 4, 5]).

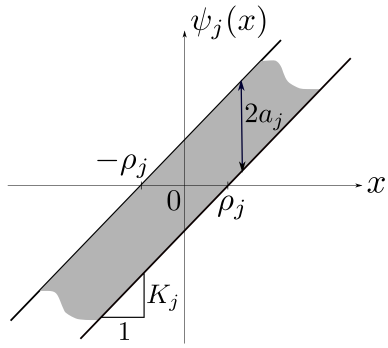


Fig. 2. Uncertain band model of backlash where $a_j = K_j \rho_j$.

The design problem considered in this work is to determine a value of $\mathbf{p} \in \mathbb{R}^N$ satisfying the following criteria:

$$\left.\begin{array}{rcll}\hat{z}_i(\mathbf{p}) & \leq & \varepsilon_i, & i = 1, 2, \ldots, m \\ \hat{u}_j(\mathbf{p}) & \leq & \sigma_j, & j = 1, 2, \ldots, n\end{array}\right\} \quad (3)$$

where $\hat{z}_i(\mathbf{p})$ and $\hat{u}_j(\mathbf{p})$ are the performance measures defined by

$$\hat{z}_i(\mathbf{p}) \triangleq \sup_{f\in\mathcal{F}} \|z_i\|_\infty \quad \text{and} \quad \hat{u}_j(\mathbf{p}) \triangleq \sup_{f\in\mathcal{F}} \|u_j\|_\infty \quad (4)$$

and ε_i and σ_j are specified bounds. The numbers $\hat{z}_i(\mathbf{p})$ and $\hat{u}_j(\mathbf{p})$ are called the peak outputs of z_i and u_j, respectively, for the input set $\mathcal{F}$ ([6, 8, 9]).

Regarding the criteria (3), it is worth noting that many researchers (see, e.g., [12, 13, 6, 8, 14, 9] and also the references therein) have been prompted to develop various methods for designing linear control systems to satisfy the criterion of the form

$$\hat{v} \le \varepsilon, \quad \hat{v} \triangleq \sup_{f \in \mathcal{F}} \|v\|_\infty \tag{5}$$

where $\hat{v}$ is the peak output of the system's response v for the set $\mathcal{F}$ and ε is the largest value of $\hat{v}$ that can be accepted or tolerated. The criterion (5) is particularly useful for designing *critical control systems* ([15, 16]), in which $v(t)$ is required to stay strictly within the interval $[-\varepsilon, \varepsilon]$ for all t, any violation resulting in unacceptable operation. Moreover, the criterion has been used in the design of various linear control systems (e.g., [17, 18, 19]).

It may be noted that prior to this work, preliminary investigations were carried out by Nguyen and Arunsawatwong [20, 21] for the case of single-loop unity-feedback control systems consisting of a linear controller, a backlash represented by the uncertain band model (2) and a linear plant where the design objective is to ensure that the error and the controller output stay within their prescribed bounds for all time and for all inputs in the set $\mathcal{F}$.

The main contributions of the present paper are twofold.

- First and foremost, we extend the results presented in [20] to a more general case in which the LTI subsystems are a multivariable system interconnecting with multiple decoupled backlash nonlinearities. By the decomposition technique used in [22], each backlash nonlinearity is replaced by a linear gain and a bounded disturbance, thus resulting in a linear system subject to $n+1$ inputs. Then by using Kakutani's fixed point theorem (see, e.g., [10]), sufficient conditions for the design criteria (3) are derived from the resultant linear system. Such conditions are used further to develop readily computable design inequalities (to be called surrogate design criteria) in place of the original criteria (3). Furthermore, a sufficient condition for the solvability of such design inequalities are given.
- Second, in order to illustrate the usefulness of the developed method, a numerical example is given in which linear decentralized controller are design for a binary distillation column and in which valve stiction characteristics are taken into account.

The structure of the paper is as follows: Section 2 presents the main theoretical result (Theorem 2.2), which provides sufficient conditions for the satisfaction of the design criteria (3). Based on the main result in Section 2, Section 3 further develops sufficient conditions for the criteria (3) in the form of inequalities that can be solved in practice by numerical methods and also provides a sufficient condition for the solvability of such inequalities. In Section 4, the usefulness of the developed method is illustrated by a controller design example of a binary distillation column with two valve stiction elements. Finally, conclusions and discussion are given in Section 5.

2. Theoretical Results

This section presents the main theoretical result of the article. The result is presented as Theorem 2.2, providing sufficient conditions for the satisfaction of the criteria (3).

Suppose that $\mathbf{G}(s, \mathbf{p})$ is a transfer matrix represented by

$$\mathbf{G}(s, \mathbf{p}) = \begin{bmatrix} \mathbf{G}_{zf}(s, \mathbf{p}) & \mathbf{G}_{zv}(s, \mathbf{p}) \\ \mathbf{G}_{uf}(s, \mathbf{p}) & \mathbf{G}_{uv}(s, \mathbf{p}) \end{bmatrix}$$

where $\mathbf{G}_{zf}(s, \mathbf{p}) \triangleq [G_{z_i f}(s, \mathbf{p})]_{m \times 1}$, $\mathbf{G}_{zv}(s, \mathbf{p}) \triangleq [G_{z_i v_k}(s, \mathbf{p})]_{m \times n}$, $\mathbf{G}_{uf}(s, \mathbf{p}) \triangleq [G_{u_j f}(s, \mathbf{p})]_{n \times 1}$, $\mathbf{G}_{uv}(s, \mathbf{p}) \triangleq [G_{u_j v_k}(s, \mathbf{p})]_{n \times n}$. Then the mathematical model of the system in Fig. 1 is described by

$$\begin{aligned} & z_i = g_{z_i f} * f + \sum_{k=1}^{n} (g_{z_i v_k} * v_k), \quad i = 1, 2, \ldots, m \\ & \left. \begin{aligned} u_j &= g_{u_j f} * f + \sum_{k=1}^{n} (g_{u_j v_k} * v_k) \\ v_j &= \psi_j(u_j) \end{aligned} \right\}, \quad j = 1, 2, \ldots, n \end{aligned} \tag{6}$$

where $g_{z_i f}$, $g_{z_i v_k}$, $g_{u_j f}$ and $g_{u_j v_k}$ denote the inverse Laplace transforms of $G_{z_i f}(s, \mathbf{p})$, $G_{z_i v_k}(s, \mathbf{p})$, $G_{u_j f}(s, \mathbf{p})$, and $G_{u_j v_k}(s, \mathbf{p})$, respectively. As usual the symbol $*$ denotes the convolution; i.e., for $x : \mathbb{R}_+ \to \mathbb{R}$ and $y : \mathbb{R}_+ \to \mathbb{R}$,

$$(x * y)(t) = \int_0^t x(t - \tau) y(\tau) d\tau, \quad t > 0.$$

Assumption 1. *The linear part of the system* (6) *is a time-invariant and non-anticipative system with zero initial conditions.*

Assumption 2. *The vector* Ψ *is described by* $\Psi(\mathbf{u}) = [\psi_1(u_1), \psi_2(u_2), \ldots, \psi_n(u_n)]^T$ *where each* ψ_j *is represented by the uncertain band model* (2).

Assumption 3. *There exists at least a solution* $\mathbf{z} : \mathbb{R}_+ \to \mathbb{R}^m$ *and* $\mathbf{u} : \mathbb{R}_+ \to \mathbb{R}^n$ *that satisfies* (6) *for every input* $f \in \mathcal{F}$.

DOI:10.4186/ej.2023.27.6.11

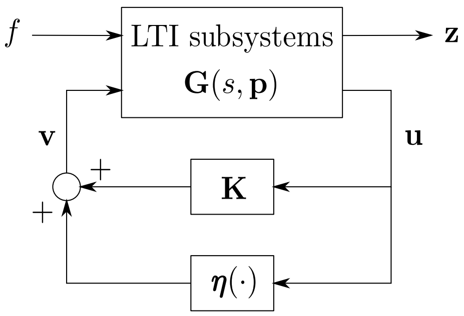


Fig. 3. Equivalent system of the system (6).

In connection with the decomposition technique used in [22], the backlash nonlinearities ψ_j are replaced by

$$\psi_j(u_j) = K_j u_j + \eta_j(u_j),\ j = 1, 2, \ldots, n. \quad (7)$$

Then the system (6) becomes equivalent to the system shown in Fig. 3 where $\mathbf{K} \triangleq \mathrm{diag}(K_1, K_2, \ldots, K_n)$ and $\boldsymbol{\eta}(\mathbf{u}) \triangleq [\eta_1(u_1), \eta_2(u_2), \ldots, \eta_n(u_n)]^T$. Note, by Assumption 2, that $||\eta_j(u_j)||_\infty \leq a_j$ for all j whenever $\mathbf{u}$ is bounded.

Now define

$$\begin{aligned} &\mathcal{U} \triangleq \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n, \\ &\mathcal{U}_j \triangleq \{x \in \mathbb{L}^\infty : \|x\|_\infty \leq \sigma_j\}, \quad j = 1, \ldots, n. \end{aligned} \quad (8)$$

Suppose that $\mathbf{w} \triangleq [w_1, w_2, \ldots, w_n]^T$ belongs to the set $\mathcal{U}$. Then, by replacing $\boldsymbol{\eta}(\mathbf{u})$ with $\boldsymbol{\eta}(\mathbf{w})$, we obtain the auxiliary system described by

$$\left.\begin{aligned} z_i' &= g_{z_i f} * f + \sum_{k=1}^{n} g_{z_i v_k} * (K_k u_k' + d_{w_k}), \\ &\quad i = 1, 2, \ldots, m \\ u_j' &= g_{u_j f} * f + \sum_{k=1}^{n} g_{u_j v_k} * (K_k u_k' + d_{w_k}) \\ d_{w_j} &= \eta_j(w_j) \\ &\quad j = 1, 2, \ldots, n \end{aligned}\right\}, \quad (9)$$

where $f \in \mathcal{F}$ and $\mathbf{w} \in \mathcal{U}$. The system (9) is depicted in Fig. 4 where $\mathbf{z}' \triangleq [z_1', z_2', \ldots, z_m']^T$, $\mathbf{u}' \triangleq [u_1', u_2', \ldots, u_n']^T$, $\mathbf{v}' \triangleq [v_1', v_2', \ldots, v_n']^T$, and $\mathbf{d}_w \triangleq [d_{w_1}, d_{w_2}, \ldots, d_{w_n}]^T$.

For the responses z_i' ($i = 1, 2, \ldots, m$) and u_j' ($j = 1, 2, \ldots, n$) of the system (9), define the peak outputs $\hat{z}_i'(\mathbf{p})$ and $\hat{u}_j'(\mathbf{p})$ as follows:

$$\begin{aligned} \hat{z}_i'(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{w} \in \mathcal{U}} \|z_i'\|_\infty, \\ \hat{u}_j'(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{w} \in \mathcal{U}} \|u_j'\|_\infty. \end{aligned}$$

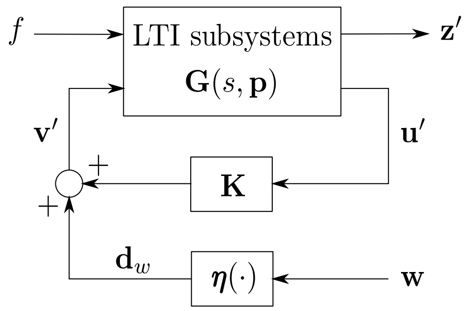


Fig. 4. Auxiliary system of the system (6).

In order to make this paper's contribution applicable to lumped- and distributed-parameter systems, the following notation is useful. Let $\mathcal{A}$ denote the convolution algebra whose elements take the form

$$g(t) = \begin{cases} g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (10)$$

where $\delta(\cdot)$ is the Dirac delta function, $0 = t_0 < t_1 < t_2 < \cdots$ are constants,

$$\int_0^\infty |g_a(t)| < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |g_i| < \infty.$$

For the details on the algebra $\mathcal{A}$, see, e.g., [23].

For the system (9), define the transfer matrix from $[f, \mathbf{d}_w^T]^T$ to $[\mathbf{z}'^T, \mathbf{u}'^T]^T$ as follows:

$$\mathbf{H}(s, \mathbf{p}) \triangleq \begin{bmatrix} \mathbf{H}_{zf}(s, \mathbf{p}) & \mathbf{H}_{zd}(s, \mathbf{p}) \\ \mathbf{H}_{uf}(s, \mathbf{p}) & \mathbf{H}_{ud}(s, \mathbf{p}) \end{bmatrix} \quad (11)$$

where

$$\begin{aligned} \mathbf{H}_{zf}(s, \mathbf{p}) &\triangleq [H_{z_i f}(s, \mathbf{p})]_{m \times 1}, \\ \mathbf{H}_{zd}(s, \mathbf{p}) &\triangleq [H_{z_i d_k}(s, \mathbf{p})]_{m \times n}, \\ \mathbf{H}_{uf}(s, \mathbf{p}) &\triangleq [H_{u_j f}(s, \mathbf{p})]_{n \times 1}, \\ \mathbf{H}_{ud}(s, \mathbf{p}) &\triangleq [H_{u_j d_k}(s, \mathbf{p})]_{n \times n}. \end{aligned} \quad (12)$$

Then one can verify that

$$\begin{aligned} \mathbf{H}_{zf}(s, \mathbf{p}) = {} & \mathbf{G}_{zv}(s, \mathbf{p})\mathbf{K}[I - \mathbf{G}_{uv}(s, \mathbf{p})\mathbf{K}]^{-1}\mathbf{G}_{uf}(s, \mathbf{p}) \\ & + \mathbf{G}_{zf}(s, \mathbf{p}), \end{aligned} \quad (13a)$$

$$\mathbf{H}_{zd}(s, \mathbf{p}) = \mathbf{G}_{zv}(s, \mathbf{p})[I - \mathbf{K}\mathbf{G}_{uv}(s, \mathbf{p})]^{-1}, \quad (13b)$$

$$\mathbf{H}_{uf}(s, \mathbf{p}) = [I - \mathbf{G}_{uv}(s, \mathbf{p})\mathbf{K}]^{-1}\mathbf{G}_{uf}(s, \mathbf{p}), \quad (13c)$$

$$\mathbf{H}_{ud}(s, \mathbf{p}) = [I - \mathbf{G}_{uv}(s, \mathbf{p})\mathbf{K}]^{-1}\mathbf{G}_{uv}(s, \mathbf{p}). \quad (13d)$$

Furthermore, let $h_{z_i f}$, $h_{z_i d_k}$, $h_{u_j f}$ and $h_{u_j d_k}$ denote the inverse Laplace transforms of $H_{z_i f}(s, \mathbf{p})$, $H_{z_i d_k}(s, \mathbf{p})$, $H_{u_j f}(s, \mathbf{p})$ and $H_{u_j d_k}(s, \mathbf{p})$, respectively.

The following notations will be used in the proof of the main result. Define the space $\mathbb{L}_n^\infty \triangleq \underbrace{\mathbb{L}^\infty \times \mathbb{L}^\infty \ldots \times \mathbb{L}^\infty}_{n}$.

For any $\mathbf{x} \triangleq [x_1, x_2, \ldots, x_n]^T \in \mathbb{L}_n^\infty$, let

$$||\mathbf{x}|| \triangleq \sum_{i=1}^{n} ||x_i||_\infty.$$

For any $\mathbf{x} \in \mathbb{L}_n^\infty$ and for any $T > 0$, define the truncated function $\mathbf{x}_T$ as follows:

$$\mathbf{x}_T(t) \triangleq \begin{cases} \mathbf{x}(t), & 0 \le t \le T \\ 0, & t > T \end{cases}.$$

For $X \subseteq \mathbb{L}_n^\infty$, define the truncated set X_T as $X_T \triangleq \{\mathbf{x}_T : \mathbf{x} \in X\}$. For $X, Y \subseteq \mathbb{L}_n^\infty$, let $S : M \subseteq X \to 2^Y$ be a multi-valued map where 2^Y denotes the set of all subsets of Y. Finally, let $S(X)$ denote the union of all sets $S(\mathbf{x})$ over $\mathbf{x} \in X$, i.e.,

$$S(X) = \bigcup_{\mathbf{x} \in X} S(\mathbf{x}).$$

Lemma 2.1. *Let $X \subset \mathbb{L}_n^\infty$. For every $T > 0$, let $\mathcal{H}$ denote the operator defined over X_T by*

$$\begin{aligned} \mathcal{H}\mathbf{x}(t) &\triangleq [\mathcal{H}_1\mathbf{x}(t), \mathcal{H}_2\mathbf{x}(t), \ldots, \mathcal{H}_n\mathbf{x}(t)]^T, \\ \mathcal{H}_j\mathbf{x}(t) &\triangleq \sum_{k=1}^{n} \int_0^T h_{jk}(t-\tau)x_k(\tau)d\tau + r_j(t), \\ &\quad \forall t \in [0, T] \end{aligned} \tag{14}$$

where $h_{jk} : \mathbb{R}_+ \to \mathbb{R}$ ($j, k = 1, 2, \ldots, n$) are given, $r_j : [0, T] \to \mathbb{R}$ ($j = 1, 2, \ldots, n$) are continuous and satisfy $\|r_j\|_\infty \le M$ for some $M > 0$. If both h_{jk} and $\dot{h}_{jk}$ belong to $\mathcal{A}$ for all j, k, then $\mathcal{H}$ is compact.

Proof. See Appendix A. □

Now it is ready to state the main result of this section, which is obtained by using the decomposition (7) and Kakutani's fixed point theorem (see, e.g., [10]). This technique was previously used by [4] and [20]. Note further that [4] investigates the stability properties of systems with backlash whereas our paper aims to develop design methods in connection with the criteria (3).

Theorem 2.2. *Consider the system* (6) *and let Assumptions 1–3 hold. Let $h_{u_jf} \in \mathcal{A}$ for $j = 1, 2, \ldots, n$ and let $h_{u_jd_k}, \dot{h}_{u_jd_k} \in \mathcal{A}$ for $j, k = 1, 2, \ldots, n$. The original design criteria* (3) *are satisfied if, for the auxiliary system* (9)*, the following conditions hold.*

$$\hat{z}_i'(\mathbf{p}) \le \varepsilon_i, \quad i = 1, 2, \ldots, m, \tag{15a}$$
$$\hat{u}_j'(\mathbf{p}) \le \sigma_j, \quad j = 1, 2, \ldots, n. \tag{15b}$$

Proof. Let $f \in \mathcal{F}$ be a fixed input and let conditions (15) hold. Consider the auxiliary system (9). From (13c) and (13d) it can be verified that for $j = 1, 2, \ldots, n$,

$$u_j' = \sum_{k=1}^{n} h_{u_jd_k} * \eta_k(w_k) + h_{u_jf} * f \triangleq \Phi_j(\mathbf{w}) \tag{16}$$

for any input $\mathbf{w}$. Let $T > 0$ be fixed. Truncating to both sides of (16) yields

$$\begin{aligned} u_{j,T}' &= \sum_{k=1}^{n} \Big(h_{u_jd_k} * \eta_k(w_{k,T})\Big)_T + \Big(h_{u_jf} * f\Big)_T \\ &\triangleq \Phi_{j,T}(\mathbf{w}_T). \end{aligned} \tag{17}$$

In connection with (17), we define a map Φ_T such that

$$\Phi_T(\mathbf{w}_T) \triangleq [\Phi_{1,T}(\mathbf{w}_T), \Phi_{2,T}(\mathbf{w}_T), \ldots, \Phi_{n,T}(\mathbf{w}_T)]^T. \tag{18}$$

Now suppose that $\mathbf{w} \in \mathcal{U}$. Since conditions (15b) hold, it follows that $\mathbf{u}' \subseteq \mathcal{U}$ and hence $\mathbf{u}_T' \subseteq \mathcal{U}_T$. Thus, $\Phi_T(\mathbf{w}_T) \subseteq \mathcal{U}_T$; i.e., Φ_T maps $\mathcal{U}_T$ into $2^{\mathcal{U}_T}$. In the following, we will show that Φ_T has a fixed point in $\mathcal{U}_T$. To this end, we write $\Phi_T = \mathcal{H}\mathcal{Q}$, where

$$\begin{aligned} \mathcal{H}\mathbf{x} &= [\mathcal{H}_1\mathbf{x}, \mathcal{H}_2\mathbf{x}, \ldots, \mathcal{H}_n\mathbf{x}]^T, \\ \mathcal{H}_j\mathbf{x} &= \sum_{k=1}^{n} (h_{u_jd_k} * x_k)_T + (h_{u_jf} * f)_T, \\ \mathcal{Q}\mathbf{x} &= \boldsymbol{\eta}(\mathbf{x}). \end{aligned}$$

Then, by using Lemma 2.1, the conditions $h_{u_jf} \in \mathcal{A}$ for all j and $h_{u_jd_k}, \dot{h}_{u_jd_k} \in \mathcal{A}$ for all j, k imply that $\mathcal{H}$ is compact over $\mathcal{U}_T$. Consequently, since $Q(\mathcal{U}_T)$ is bounded, we conclude by Definition A.4 that $\Phi(\mathcal{U}_T)$ is relatively compact. Furthermore, it is easy to verify that (i) Φ_T is upper semi-continuous; (ii) $\mathcal{U}_T$ is nonempty, closed and convex; (iii) $\Phi_T(\mathbf{x})$ is nonempty, closed and convex for all $\mathbf{x} \in \mathcal{U}_T$. Thus, by applying Kakutani's fixed point theorem (see, e.g., [10]), one can see that Φ_T always has a fixed point $\mathbf{u}_T^\dagger \triangleq [u_{1,T}^\dagger, u_{2,T}^\dagger, \ldots, u_{n,T}^\dagger]^T \in \mathcal{U}_T$ such that

$$\mathbf{u}_T^\dagger \in \Phi_T(\mathbf{u}_T^\dagger). \tag{19}$$

From (17)–(19), it follows immediately that for each j,

$$u_{j,T}^\dagger = \sum_{k=1}^{n} \Big(h_{u_jd_k} * \eta_k(u_{k,T}^\dagger)\Big)_T + \Big(h_{u_jf} * f\Big)_T.$$

Consequently, by using (13c) and (13d), one can verify that

$$u_{j,T}^\dagger = \sum_{k=1}^{n} \Big(g_{u_jv_k} * (K_k u_{k,T}^\dagger + \eta_k(u_{k,T}^\dagger))\Big)_T + \Big(g_{u_jf} * f\Big)_T. \tag{20}$$

By using the decomposition in (7), equation (20) turns out to be

$$u_{j,T}^\dagger = \sum_{k=1}^{n} \Big(g_{u_jv_k} * \psi_k(u_{k,T}^\dagger)\Big)_T + \Big(g_{u_jf} * f\Big)_T. \tag{21}$$

Next, let $\mathbf{z}^\dagger \triangleq [z_{1,T}^\dagger, z_{2,T}^\dagger, \ldots, z_{m,T}^\dagger]^T$ denote the vector of the associated outputs of interest of the auxiliary system (9) when $\mathbf{w} = \mathbf{u}_T^\dagger$. Clearly, conditions (15a) imply that $\|z_{i,T}^\dagger\|_\infty \le \varepsilon_i$ for all i. Moreover, since

$\mathbf{u}'_T = \mathbf{u}^\dagger_T$ when $\mathbf{w} = \mathbf{u}^\dagger_T$ (see above), it follows from (9) that for each i,

$$\begin{aligned} z^\dagger_{i,T} &= \sum_{k=1}^{n} \Big(g_{z_i v_k} * \big(K_k u^\dagger_{k,T} + \eta_k(u^\dagger_{k,T})\big)\Big)_T \\ &\quad + \Big(g_{z_i f} * f \Big)_T \\ &= \sum_{k=1}^{n} \Big(g_{z_i v_k} * \psi_k(u^\dagger_{k,T})\Big)_T + \Big(g_{z_i f} * f \Big)_T \end{aligned} \tag{22}$$

where we use the decomposition in (7) again in (22).

Now, it follows from (21) and (22) that $\mathbf{z}^\dagger_T$ and $\mathbf{u}^\dagger_T$ are also the responses $\mathbf{z}_T$ and $\mathbf{u}_T$ of the nonlinear system (6). Moreover, $||z^\dagger_{i,T}||_\infty \le \varepsilon_i$ for all i and $\|u^\dagger_{j,T}\| \le \sigma_j$ for all j. Finally, since the above arguments hold for any $f \in \mathcal{F}$ and any $T > 0$, we conclude that the criteria (3) are satisfied for the system (6). □

3. Surrogate Design Criteria

In contrast to the original system (6), the auxiliary system (9) is linear; therefore, inequalities (15) are easier to solve than the original design criteria (3). However, solving (15) is still not convenient because the input $\mathbf{d}_w$ depends on the characteristic of $\boldsymbol{\eta}$. In this regard, based on the main result in Theorem 2.2, this section provides practical sufficient conditions for (15), which are expressed as inequalities that are more computationally tractable.

3.1. Practical Design Inequalities

Consider the system (9). By defining

$$\mathcal{D}_w \triangleq \{\mathbf{d}_w : \mathbf{d}_w = \boldsymbol{\eta}(\mathbf{w}), \mathbf{w} \in \mathcal{U}\}, \tag{23}$$

one can easily see that

$$\begin{aligned} \hat{z}'_i(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{d}_w \in \mathcal{D}_w} \|z'_i\|_\infty, \\ \hat{u}'_j(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{d}_w \in \mathcal{D}_w} \|u'_j\|_\infty. \end{aligned}$$

Now define

$$\mathcal{D} \triangleq \{\mathbf{d} \in \mathbb{L}^\infty_n : ||d_j||_\infty \le a_j \ \forall j\} \tag{24}$$

where $\mathbf{d} \triangleq [d_1, d_1, \ldots, d_n]^T$. From (2) and (23), one can see that for any $\mathbf{d}_w \in \mathcal{D}_w$,

$$\sup_{w_j \in \mathcal{U}_j} \|d_{w_j}\|_\infty = a_j \ \ \forall j.$$

Consequently, $\mathcal{D}_w \subseteq \mathcal{D}$.

By replacing $\mathbf{d}_w$ in (9) with $\mathbf{d} \in \mathcal{D}$, we obtain the nominal system described by

$$\begin{aligned} & z^*_i = g_{z_i f} * f + \sum_{k=1}^{n} g_{z_i v_k} * (K_k u^*_k + d_k), \\ & i = 1, 2, \ldots, m \\ & u^*_j = g_{u_j f} * f + \sum_{k=1}^{n} g_{u_j v_k} * (K_k u^*_k + d_k), \\ & j = 1, 2, \ldots, n \end{aligned} \tag{25}$$

where $f \in \mathcal{F}$ and $\mathbf{d} \in \mathcal{D}$. The system (25) is depicted in Fig. 5 where $\mathbf{z}^* \triangleq [z^*_1, z^*_2, \ldots, z^*_m]^T$, $\mathbf{u}^* \triangleq [u^*_1, u^*_2, \ldots, u^*_n]^T$, and $\mathbf{v}^* \triangleq [v^*_1, v^*_2, \ldots, v^*_n]^T$. Additionally, the closed-loop transfer matrix of the system (25) from $[f, \mathbf{d}^T]^T$ to $[\mathbf{z}^{*T}, \mathbf{u}^{*T}]^T$ is identical to $\mathbf{H}(s, \mathbf{p})$ described in (11)–(13).

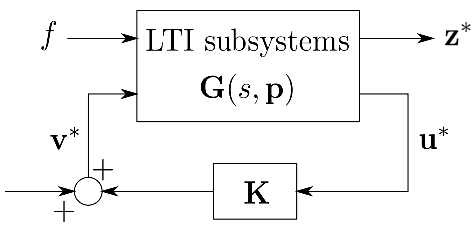


Fig. 5. Nominal system.

Since $\mathcal{D}_w \subseteq \mathcal{D}$, it follows that $\hat{z}'_i(\mathbf{p}) \le \hat{z}^*_i(\mathbf{p})$ for $i = 1, 2, \ldots, m$ and $\hat{u}'_j(\mathbf{p}) \le \hat{u}^*_j(\mathbf{p})$ for $j = 1, 2, \ldots, n$ where $\hat{z}^*_i(\mathbf{p})$ and $\hat{u}^*_j(\mathbf{p})$ are the peak outputs of the system (25) and are defined by

$$\begin{aligned} \hat{z}^*_i(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{d} \in \mathcal{D}} \|z^*_i\|_\infty, \\ \hat{u}^*_j(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{d} \in \mathcal{D}} \|u^*_j\|_\infty. \end{aligned}$$

By using linearity properties of the system as well as a well-known result in linear system theory ([24]), the peak outputs $\hat{z}^*_i(\mathbf{p})$ and $\hat{u}^*_j(\mathbf{p})$ can be given by

$$\begin{aligned} \hat{z}^*_i(\mathbf{p}) &= \hat{z}^*_{i,f}(\mathbf{p}) + \sum_{k=1}^{n} a_k \|h_{z_i d_k}\|_1, \ i = 1, \ldots, m \\ \hat{u}^*_j(\mathbf{p}) &= \hat{u}^*_{j,f}(\mathbf{p}) + \sum_{k=1}^{n} a_k \|h_{u_j d_k}\|_1, \ j = 1, \ldots, n \end{aligned} \tag{26}$$

where

$$\begin{aligned} \hat{z}^*_{i,f}(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{d} = \mathbf{0}} \|z^*_i\|_\infty, \\ \hat{u}^*_{j,f}(\mathbf{p}) &\triangleq \sup_{f \in \mathcal{F}, \mathbf{d} = \mathbf{0}} \|u^*_j\|_\infty. \end{aligned}$$

The peak outputs $\hat{z}^*_{i,f}(\mathbf{p})$ and $\hat{u}^*_{j,f}(\mathbf{p})$ can be computed by using efficient methods for computing peak outputs of linear time-invariant systems (e.g., [17, 14, 9] and the references therein). In this work, the method presented in [9] is employed because it is simple and efficient.

In connection with (26), we provide practical design inequalities as follows:

Theorem 3.1. *Consider the system* (6) *and let Assumptions 1–3 hold. Suppose that* $h_{u_j f} \in \mathcal{A}$ *for* $j = 1, 2, \ldots, n$ *and that* $h_{u_j d_k}, \dot{h}_{u_j d_k} \in \mathcal{A}$ *for* $j, k = 1, 2, \ldots, n$. *Then the original design criteria* (3) *are satisfied if, for the nominal system* (25), *the following conditions hold.*

$$\begin{array}{rcll} \hat{z}_i^*(\mathbf{p}) & \leq & \varepsilon_i, & i = 1, 2, \ldots, m \\ \hat{u}_j^*(\mathbf{p}) & \leq & \sigma_j, & j = 1, 2, \ldots, n \end{array} . \quad (27)$$

Proof. The proof readily follows from Theorem 2.2 and the above discussion. □

From Theorem 3.1, it readily follows that if a solution of inequalities (27) exists, then it is also a design solution of the original design criteria (3) for the system (6). Accordingly, one can obtain a design solution for the system (6) by solving inequalities (27). For this reason, inequalities (27) are appropriately called the *surrogate design criteria*.

Since each of inequalities (27) represents an individual design objective, it is evident that solving these inequalities is a multiobjective design problem, which can readily be treated by the method of inequalities ([25, 6, 7, 8]). According to this, the primary goal in solving inequalities (27) is to make sure that the inequalities has at least a solution (that is to say, the set of all solution of inequalities (27) is not empty). In this connection, the solvability of inequalities (27) is provided in Section 3.3.

3.2. Finiteness of $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$

Following previous work ([25, 6, 26, 7, 8, 9]), it is noted that in solving inequalities (27) by numerical methods, a search algorithm needs to start from a point $\mathbf{p} \in \mathbb{R}^N$ such that $\hat{z}_i^*(\mathbf{p}) < \infty$ for all i and $\hat{u}_j^*(\mathbf{p}) < \infty$ for all j. In this regard, the relationship between the bounded-input bounded-output (BIBO) stability of the nominal system (25) and the finiteness of peak outputs $\hat{z}_i^*(\mathbf{p})$, $\hat{u}_j^*(\mathbf{p})$ is stated as follows:

Proposition 3.2. *Consider the nominal system* (25). *Assume that* $f \in \mathbb{L}^\infty$ *and* $\mathbf{d} \in \mathbb{L}_n^\infty$. *Then* $\mathbf{z}^* \in \mathbb{L}_m^\infty$ *and* $\mathbf{u}^* \in \mathbb{L}_n^\infty$ *if the following two conditions hold.*

(a) The transfer functions $H_{z_i f}(s, \mathbf{p})$ *(*$i = 1, 2, \ldots, m$*) and* $H_{z_i d_k}(s, \mathbf{p})$ *(*$i = 1, 2, \ldots, m$ *and* $k = 1, 2, \ldots, n$*) are BIBO stable.*

(b) The transfer functions $H_{u_j f}(s, \mathbf{p})$ *(*$j = 1, 2, \ldots, n$*) and* $H_{u_j d_k}(s, \mathbf{p})$ *(*$j, k = 1, 2, \ldots, n$*) are BIBO stable.*

Proof. From (13) and (25), one can verify that

$$\begin{array}{l} z_i^* = h_{z_i f} * f + \displaystyle\sum_{k=1}^{n} h_{z_i d_k} * d_k, \ i = 1, \ldots, m \\ u_j^* = h_{u_j f} * f + \displaystyle\sum_{k=1}^{n} h_{u_j d_k} * d_k, \ j = 1, \ldots, n \end{array} . \quad (28)$$

Since $f \in \mathbb{L}^\infty$ and $\mathbf{d} \in \mathbb{L}_n^\infty$, conditions (a) and (b) imply, respectively, that $\mathbf{z} \in \mathbb{L}_m^\infty$ and $\mathbf{u} \in \mathbb{L}_n^\infty$. □

By Proposition 3.2 and by noting that $\mathcal{F} \subset \mathbb{L}^\infty$ and $\mathcal{D} \subset \mathbb{L}_n^\infty$, one can easily see that for a given $\mathbf{p}$, if all elements of $\mathbf{H}(s, \mathbf{p})$ are BIBO stable transfer functions, then the responses $\mathbf{z}^*$ and $\mathbf{u}^*$ are bounded; hence, all the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ are finite.

In the literature, necessary and sufficient conditions for BIBO stability are available for various classes of linear time-invariant systems. Some of the conditions are used in developing a numerical procedure (in conjunction with the method of inequalities) for determining a point $\mathbf{p}_0$ for which the systems are BIBO stable; such a procedure is sometimes called numerical stabilization. For retarded delay differential systems, a necessary and sufficient condition for BIBO stability and a useful inequality for numerical stabilization are mentioned in Section 4, where controller design for a time-delay control system is considered. For a wider class of linear systems called *retarded fractional delay differential systems*, a BIBO stability condition and an inequality for numerical stabilization are presented in [27]. It may be noted further that for numerical stabilization in connection with a class of nonlinear systems, readers are referred to [28].

3.3. Solvability of Design Inequalities (27)

It is noted ([25, 6, 7, 8]) that not every design problem that is cast as a set of inequalities has a solution and therefore the designer has to face the possibility that such a design problem has no solution. In solving inequalities (27) by numerical methods (which is usually a non-convex problem in the space $\mathbb{R}^N$), when a search algorithm fails to find a solution of the inequalities after a large number of iterations, a question often arising is whether or not the inequalities have a solution. When no solution exists, the designer has to reformulate the inequalities by simply increasing some bounds ε_i or σ_j or, in some cases, using a different controller structure with non-decreasing complexity so that the resultant inequalities have a solution.

From a computational viewpoint, it is useful to know under which condition, inequalities (27) are guaranteed to have (at least) a solution. To this end, a sufficient condition for the solvability of inequalities (27) is stated below.

Definition 3.3. Inequalities (27) are said to be solvable if they have (at least) a solution for sufficiently large bounds ε_i and σ_j.

Proposition 3.4. *Consider the nominal system* (25). *Inequalities* (27) *are solvable if there exists a vector* $\mathbf{p}^* \in \mathbb{R}^N$ *such that all elements of* $\mathbf{H}(s, \mathbf{p}^*)$ *are BIBO stable.*

Proof. Assume that there exists a $\mathbf{p}^* \in \mathbb{R}^N$ such that all elements of $\mathbf{H}(s, \mathbf{p}^*)$ are BIBO stable. Then it follows from Proposition 3.2 that all the peak outputs $\hat{z}_i^*(\mathbf{p}^*)$ and $\hat{u}_j^*(\mathbf{p}^*)$ are finite. Using (26) and the fact that $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ do not depend on ε_i and σ_j respectively, we conclude that inequalities (27) have a solution whenever $\varepsilon_i \geq \hat{z}_i^*(\mathbf{p}^*)$ and $\sigma_j \geq \hat{u}_j^*(\mathbf{p}^*)$. Thus, by Definition 3.3, inequalities (27) are solvable. □

Proposition 3.4 suggests that inequalities (27) are guaranteed to have a solution if the following two conditions are satisfied.

(I) There exists a $\mathbf{p}^* \in \mathbb{R}^N$ for which the nominal system (25) is BIBO stable.

(II) The bounds ε_i and σ_j are sufficiently large. More specifically, $\varepsilon_i \geq \hat{z}_i^*(\mathbf{p}^*)$ and $\sigma_j \geq \hat{u}_j^*(\mathbf{p}^*)$.

After such a point $\mathbf{p}^*$ is found, a better design can be obtained by decreasing some of the bounds ε_i and σ_j and then solving inequalities (27) again. This process of reformulating and solving inequalities (27) can be carried out successively until the inequalities have only one solution. It is known ([7, 8] and also the references therein) that this solution yields an optimal design in the Pareto sense, which is optimal in the sense that a reduction in any of the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ can be achieved only by increasing at least one of the others. It may be noted further that the Pareto optimal solution is not unique.

4. Numerical Examples

In this example, a controller design for a binary distillation column is given where two valve stiction elements are explicitly taken into account. Stiction is the most common valve problem in the process industry and can cause oscillations in the process output ([29]). For the details on valve stiction, see, e.g., [29] and the references therein.

The block diagram of the control system for a binary distillation column is shown in Fig. 6 where $\mathbf{e} = [e_1, e_2]^T$, $\mathbf{u} = [u_1, u_2]^T$, and $\mathbf{v} = [v_1, v_2]^T$. Following [30], the plant transfer matrices $\mathbf{G}_p(s)$ and $\mathbf{G}_d(s)$ are given by

$$\mathbf{G}_p(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-2s}}{14.4s+1} \end{bmatrix}$$

and

$$\mathbf{G}_d(s) = \begin{bmatrix} \dfrac{3.8e^{-8s}}{14.9s+1} \\ \dfrac{4.9e^{-3s}}{13.2s+1} \end{bmatrix}.$$

Let the nonlinearity $\mathbf{\Psi}$ be described by

$$\Psi(\mathbf{u}) = [\psi_1(u_1),\ \psi_2(u_2)]^T$$

where ψ_1 and ψ_2 represent the valve stiction characteristics shown in Fig. 7 with the parameters $S_1 = S_2 = 0.05$, $J_1 = J_2 = 0.02$, and $m_1 = m_2 = 1$. In order to apply the design method developed in Sections 2 and 3, we will use the uncertain band model (2) instead of the valve stiction where $K_j = m_j = 1$. Therefore, this results in the uncertain band with $a_j = \rho_j = (S_j + J_j)/2 = 0.035$.

Let the controller transfer matrix $\mathbf{C}(s, \mathbf{p})$ have the form of PID structure as follows:

$$\begin{aligned} \mathbf{C}(s,\mathbf{p}) &= \begin{bmatrix} C_1(s,\mathbf{p}) & 0 \\ 0 & C_2(s,\mathbf{p}) \end{bmatrix}, \\ C_1(s,\mathbf{p}) &= p_1\left(1 + \frac{1}{p_2 s} + \frac{p_3 s}{1+p_4 s}\right), \\ C_2(s,\mathbf{p}) &= -p_5\left(1 + \frac{1}{p_6 s} + \frac{p_7 s}{1+p_8 s}\right) \end{aligned} \tag{29}$$

where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8]^T$ satisfies the constraints

$$\begin{aligned} &p_1 > 0,\ p_2(p_3+p_4) > 0,\ p_2+p_4 > 0,\ p_4 > 0, \\ &p_5 > 0,\ p_6(p_7+p_8) > 0,\ p_6+p_8 > 0,\ p_8 > 0. \end{aligned} \tag{30}$$

The constraints (30) are imposed so as to ensure that $C_1(s, \mathbf{p})$ and $C_2(s, \mathbf{p})$ are minimum-phase transfer functions.

Following [18], assume that the deviation of feed rate f belongs to the set $\mathcal{F}$ where

$$M = 0.2 \text{ lb/min} \ \text{ and } \ D = 0.2 \text{ lb/min}^2. \tag{31}$$

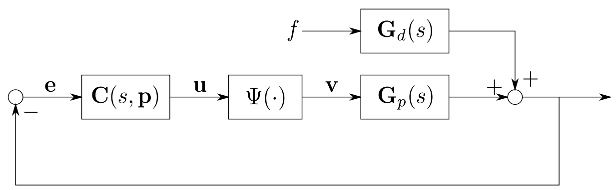


Fig. 6. Control system of a binary distillation column.

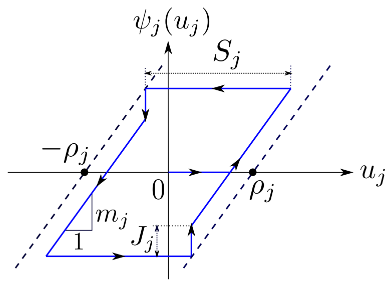


Fig. 7. Valve stiction model ([29]).

The main design objective is to ensure that for any $f \in \mathcal{F}$,

1) the top product deviation e_1 stays within ± 0.30 mol%,
2) the bottom product deviation e_2 stays within ± 0.50 mol%,
3) the deviation of the reflux rate u_1 stays within ± 0.50 lb/min,
4) the deviation of the reboiler rate u_2 stays within ± 0.50 lb/min.

It is easy to see that the system in Fig. 6 can be represented as the one in Fig. 1 where $\mathbf{z} = \mathbf{e}$,

$$\begin{aligned}&\mathbf{G}_{zf}(s) = -\mathbf{G}_d(s), \quad \mathbf{G}_{zv}(s) = -\mathbf{G}_p(s),\\ &\mathbf{G}_{uf}(s,\mathbf{p}) = -\mathbf{C}(s,\mathbf{p})\mathbf{G}_d(s),\\ &\mathbf{G}_{uv}(s,\mathbf{p}) = -\mathbf{C}(s,\mathbf{p})\mathbf{G}_p(s).\end{aligned}$$

Accordingly, the design objective stated above are expressed as the following design criteria:

$$\left.\begin{array}{rcl}\hat{z}_1(\mathbf{p}) & \leq & 0.30 \text{ mol\%}\\ \hat{z}_2(\mathbf{p}) & \leq & 0.50 \text{ mol\%}\\ \hat{u}_1(\mathbf{p}) & \leq & 0.50 \text{ lb/min}\\ \hat{u}_2(\mathbf{p}) & \leq & 0.50 \text{ lb/min}\end{array}\right\}. \tag{32}$$

It is known (see, e.g., [31]) that a retarded delay differential system is BIBO stable if and only if all the characteristic roots have negative real parts. Let $f(s)$ be the characteristic function of the nominal system and let $\alpha(\mathbf{p})$ denote the abscissa of stability of $f(s)$ defined by

$$\alpha(\mathbf{p}) \triangleq \max\{\mathrm{Re}\, s : f(s) = 0\}. \tag{33}$$

Then conditions (a) and (b) in Proposition 3.2 are satisfied if

$$\alpha(\mathbf{p}) \leq -\varepsilon$$

where $0 < \varepsilon \ll 1$. In this work, the abscissa of stability $\alpha(\mathbf{p})$ is computed by the method developed by [31].

From Theorem 3.1 and Proposition 3.2, it follows that a design solution $\mathbf{p}$ is obtained by solving the constraints (30) and the following inequalities:

$$\alpha(\mathbf{p}) \leq -10^{-6}, \tag{34}$$

$$\left.\begin{array}{rcl}\hat{z}_1^*(\mathbf{p}) & \leq & 0.30 \text{ mol\%}\\ \hat{z}_2^*(\mathbf{p}) & \leq & 0.50 \text{ mol\%}\\ \hat{u}_1^*(\mathbf{p}) & \leq & 0.50 \text{ lb/min}\\ \hat{u}_2^*(\mathbf{p}) & \leq & 0.50 \text{ lb/min}\end{array}\right\} \tag{35}$$

where inequality (34) ensures the finiteness of the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ of the nominal system, and inequalities (35) are the surrogate design criteria, which ensures for the satisfaction of (32).

In this work, a solution $\mathbf{p}$ of inequalities (30), (34) and (35) is determined by using a search algorithm called the moving-boundaries-process (MBP). The detail of the MBP algorithm can be found in [25, 8]. Alternatively, other algorithms for solving inequalities may also be used (see, e.g., Chapters 7 and 8 of [8] and the references therein).

At this point, it is worth mentioning that since searching for a solution of inequalities (35) in the space $\mathbb{R}^N$ is in general a non-convex problem, the search algorithm could sometimes be hindered by a computational trap. However, as long as a solution exists, this can be overcome in practice, for example, by choosing a new starting point which is sufficiently far away from the trap or by temporarily relaxing some of the bounds ε_i and σ_j so that the algorithm can escape from the trap. Detailed discussion on this can be found in [8].

4.1. Case I

In this subsection, the controller transfer functions $C_1(s,\mathbf{p})$ and $C_2(s,\mathbf{p})$ will be designed with $a_1 = a_2 = 0.035$ so that inequalities (35), and hence inequalities (32), are satisfied.

After a number of iterations, the MBP algorithm locates a design solution $\mathbf{p}^{\mathrm{I}}$ resulting in the following transfer functions:

$$\begin{aligned}C_1(s,\mathbf{p}^{\mathrm{I}}) &= 2.70\left(1 + \frac{1}{4,824s} - \frac{9.39s}{1+10.9s}\right),\\ C_2(s,\mathbf{p}^{\mathrm{I}}) &= -0.384\left(1 + \frac{1}{7,005s} - \frac{0.700s}{1+2.11s}\right)\end{aligned} \tag{36}$$

where the corresponding performance measures are

$$\left.\begin{array}{rcl}\alpha(\mathbf{p}^{\mathrm{I}}) & = & -6.104\times 10^{-5}\\ \hat{z}_1^*(\mathbf{p}^{\mathrm{I}}) & = & 0.2943 \text{ mol\%}\\ \hat{z}_2^*(\mathbf{p}^{\mathrm{I}}) & = & 0.5000 \text{ mol\%}\\ \hat{u}_1^*(\mathbf{p}^{\mathrm{I}}) & = & 0.1033 \text{ lb/min}\\ \hat{u}_2^*(\mathbf{p}^{\mathrm{I}}) & = & 0.1022 \text{ lb/min}\end{array}\right\}. \tag{37}$$

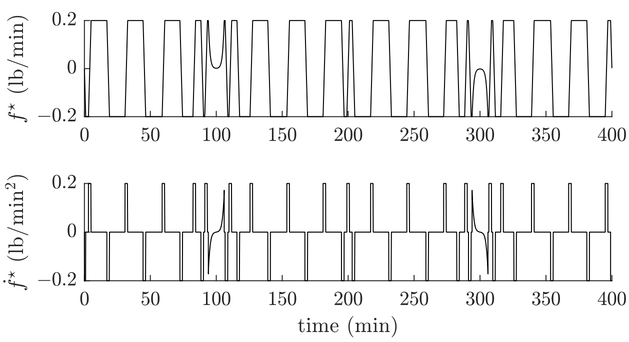


Fig. 8. Waveforms of the test input $f^\star$ and its derivative.

DOI:10.4186/ej.2023.27.6.11

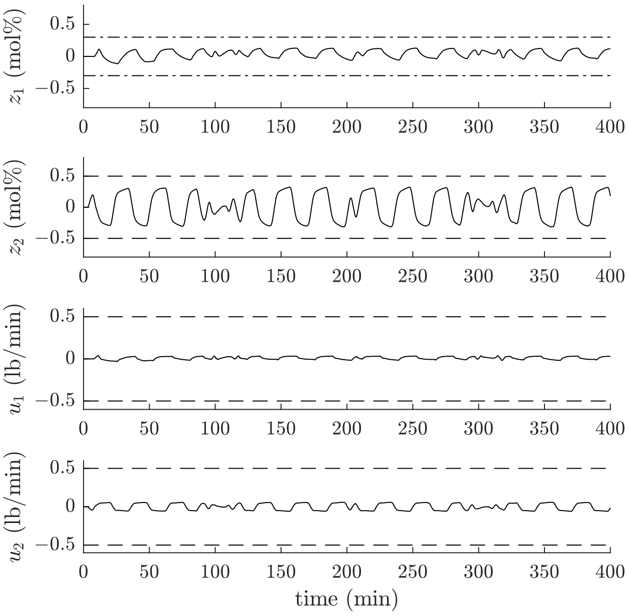


Fig. 9. Responses of the nonlinear system subject to the test input $f^\star$ with the controller (36).

To verify the design, a simulation is carried out with the original nonlinear system subject to a test input $f^\star \in \mathcal{F}$, which is a concatenation of the maximal inputs that induce the peak outputs $\hat{z}^*_{1,f}(\mathbf{p}^{\mathrm{I}})$, $\hat{z}^*_{2,f}(\mathbf{p}^{\mathrm{I}})$, $\hat{u}^*_{1,f}(\mathbf{p}^{\mathrm{I}})$ and $\hat{u}^*_{2,f}(\mathbf{p}^{\mathrm{I}})$ (cf. (26)). Such maximal inputs are obtained by using the method in [9]. The waveform of $f^\star$ and the corresponding system responses are shown in Figs. 8 and 9. Clearly, the design objective (32) are satisfied.

4.2. Case II

In this subsection, we attempt to find the maximum width of the uncertain band model such that inequalities (35) has a solution for the controller structure defined in (29). For simplicity, we assume that $a_1 = a_2 = a$. Then we successively solve inequalities (35) for a design solution with a specified value of a that is gradually increased, until no design solution is found. The so-obtained value of a results in the maximum width of the uncertain band, which is equal to $2a$, for each valve stiction characteristics.

After extensive computation, we find the maximum value of a as follows:

$$a = 0.0385. \tag{38}$$

In this case, the MBP algorithm locates a design solution $\mathbf{p}^{\mathrm{II}}$ resulting in the following transfer functions:

$$\begin{aligned} C_1(s, \mathbf{p}^{\mathrm{II}}) &= 34.1 \left(1 + \frac{1}{4,987s} - \frac{126s}{1 + 128s}\right), \\ C_2(s, \mathbf{p}^{\mathrm{II}}) &= -0.433 \left(1 + \frac{1}{4,548s} - \frac{0.737s}{1 + 2.02s}\right) \end{aligned} \tag{39}$$

where the corresponding performance measures are

$$\left.\begin{aligned} \alpha(\mathbf{p}^{\mathrm{II}}) &= -1.376 \times 10^{-3} \\ \hat{z}^*_1(\mathbf{p}^{\mathrm{II}}) &= 0.2977 \text{ mol\%} \\ \hat{z}^*_2(\mathbf{p}^{\mathrm{II}}) &= 0.5000 \text{ mol\%} \\ \hat{u}^*_1(\mathbf{p}^{\mathrm{II}}) &= 0.1978 \text{ lb/min} \\ \hat{u}^*_2(\mathbf{p}^{\mathrm{II}}) &= 0.1278 \text{ lb/min} \end{aligned}\right\}. \tag{40}$$

To verify the design of Case II, a numerical simulation is carried out for the system with the controller (39) for the test input $f^\star$ as before. For the maximum value of a in (38), the parameters of both valve stiction models are modified so that $S_j = 0.055$, $J_j = 0.022$ and $m_j = 1$, which yields $(S_j + J_j)/2 = 0.0385 = a_j$. The responses of the corresponding system due to $f^\star$ is displayed in Fig. 10.

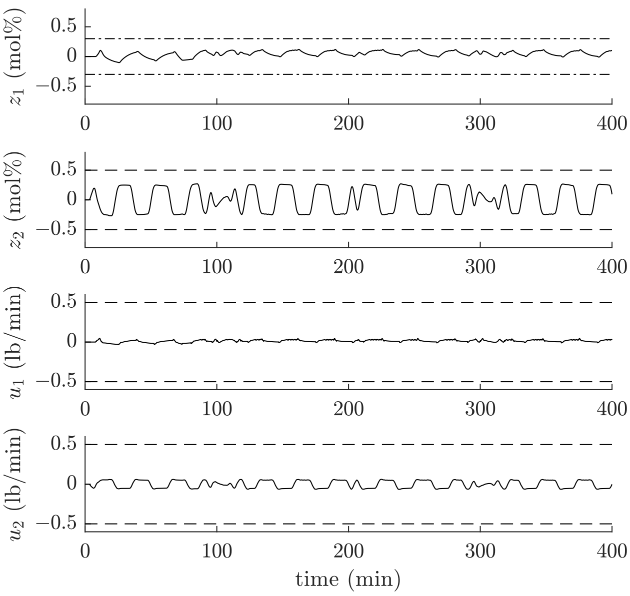


Fig. 10. Responses of the nonlinear system subject to the test input $f^\star$ with the controller (39).

From (37) and (40), it is clear that both of the design results for Cases I and II satisfy the design criteria (32). Notice, however, that the peak output $\hat{u}^*_1(\mathbf{p}^{\mathrm{II}})$ is 1.9 times of $\hat{u}^*_1(\mathbf{p}^{\mathrm{I}})$ and $\hat{u}^*_2(\mathbf{p}^{\mathrm{II}})$ is 1.2 times of $\hat{u}^*_2(\mathbf{p}^{\mathrm{I}})$, whereas there are no significant differences between $\hat{z}^*_i(\mathbf{p}^{\mathrm{II}})$ and $\hat{z}^*_i(\mathbf{p}^{\mathrm{I}})$ for $i = 1, 2$.

5. Conclusions and Discussion

This article has developed a practical and systematic method for designing the nonlinear system (6) so as to ensure that the outputs z_i and the nonlinearity inputs u_j stay within the prescribed ranges $\pm\varepsilon_i$ and $\pm\sigma_j$, respectively, for all time and for all inputs $f \in \mathcal{F}$. The input set $\mathcal{F}$ can be regarded as the set of all inputs that happen or are likely to happen; for this reason, it has been called the possible set ([7, 8]). The method developed in this work can be seen as an adjunct to Zakian's framework, which is a control design framework comprising the principle of matching ([7, 8]) and the method of inequalities ([25]).

For simplicity, we focus our attention in this work only to the input set characterized by (1). It should be noted that the methodology used in the paper can also be applied to input sets with different characterizations in a straightforward manner.

Being obtained by using the decomposition (7) and Kakutani's fixed point theorem (the fixed point theorem for multi-valued functions), Theorem 2.2 provides an essential basis for developing the surrogate design criteria (27), which are associated with the nominal linear system subject to the input $f \in \mathcal{F}$ and an additional disturbance $\mathbf{d} \in \mathcal{D}$. As a consequence, a solution of the original design problem can be obtained by solving the surrogate problem with the computational tools developed previously for linear systems. Inequalities (27) are used in conjunction with conditions (a) and (b) in Proposition 3.2 to obtain a solution of the original design problem (3) by numerical methods. From a computational viewpoint, we show that inequalities (27) have a solution for sufficiently large bounds ε_i and σ_j whenever the nominal system (25) can be stabilized.

As a result of using the convolution algebra $\mathcal{A}$ in Section 2, the method developed here is applicable to control systems whose LTI subsystems consisting of lumped- and/or distributed-parameter components as long as the conditions $h_{u_j f} \in \mathcal{A}$ for all j and $h_{u_j d_k}, \dot{h}_{u_j d_k} \in \mathcal{A}$ for all j, k hold.

Following previous work ([25,6,26,7,8,9]), it is interesting to note that in solving inequalities (27) by numerical methods, one has to deal with two computational problems as follows.

- It is required that for a given $\mathbf{p}$, the evaluation of the peak outputs $\hat{z}_i^*(\mathbf{p})$ and $\hat{u}_j^*(\mathbf{p})$ should be carried out reliably and sufficiently fast, because the process of searching for a solution of inequalities (27) involves the evaluation of the peak outputs for a long sequence of $\mathbf{p}$.
- It is necessary to obtain a point $\mathbf{p}$ such that the nominal system (25) is BIBO stable, because numerical algorithms in general are able to search for a solution of inequalities (27) only if they start from such a point.

With the computational tools that are available (see [9],[27],[31] for details), the method is applicable to a class of retarded fractional delay differential systems, which can be found in many practical applications. However, there are still systems that do not belong to this class; in order to make the method applicable to such systems, we need to resolve the two problems mentioned above. And this can be topics for future investigation.

In the numerical example, linear decentralized controllers are designed for a binary distillation column where the uncertain band models of backlash are used during the design procedure instead of the actual valve stiction characteristics. The numerical results clearly demonstrate the usefulness of the contribution of this work.

Acknowledgement

T. Chuman gratefully acknowledges the support from the 100th Anniversary Chulalongkorn University Fund for Doctoral Scholarship and the 90th Anniversary Chulalongkorn University Fund (Ratchadaphiseksomphot Endowment Fund).

References

[1] G. Tao and P. V. Kokotovic, *Adaptive Control of Systems with Actuator and Sensor Nonlinearities*. John Wiley & Sons, Inc., 1996.

[2] M. Nordin and P. O. Gutman, "Controlling mechanical systems with backlash—a survey," *Automatica*, vol. 38, no. 10, pp. 1633–1649, 2002.

[3] J. Vörös, "Modeling and identification of systems with backlash," *Automatica*, vol. 46, no. 2, pp. 369–374, 2010.

[4] A. Barreiro and A. Baños, "Input–output stability of systems with backlash," *Automatica*, vol. 42, no. 6, pp. 1017–1024, 2006.

[5] S. Tarbouriech, I. Queinnec, and C. Prieur, "Stability analysis and stabilization of systems with input backlash," *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 488–494, 2013.

[6] V. Zakian, "New formulation for the method of inequalities," *Proceedings of the Institution of Electrical Engineers*, vol. 126, no. 6, pp. 579–584, 1979.

[7] ——, "Perspectives on the principle of matching and the method of inequalities," *International Journal of Control*, vol. 65, no. 1, pp. 147–175, 1996.

[8] ——, *Control Systems Design: A New Framework*. London: Springer-Verlag, 2005.

[9] W. Silpsrikul and S. Arunsawatwong, "Computation of peak output for inputs satisfying many bounding conditions on magnitude and slope," *International Journal of Control*, vol. 83, no. 1, pp. 49–65, 2010.

[10] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, Part I - Fixed-Point Theorems*. New York: Springer-Verlag, 1986.

[11] C.-Y. Su, Y. Stepanenko, J. Svoboda, and T. P. Leung, "Robust adaptive control of a class of nonlinear systems with unknown backlash-like hysteresis," *IEEE Transactions on Automatic Control*, vol. 45, no. 12, pp. 2427–2432, 2000.

[12] B. J. Birch and R. Jackson, "The behaviour of linear systems with inputs satisfying certain bounding conditions," *Journal of Electronics and Control*, vol. 6, no. 4, pp. 366–375, 1959.

[13] I. M. Horowitz, "Analysis and synthesis of linear systems with inputs satisfying certain bounding conditions," *Journal of Electronics and Control*, vol. 12, no. 3, pp. 195–208, 1962.

[14] W. Reinelt, "Maximum output amplitude of linear systems for certain input constraints," in *Proceedings of the 39th IEEE Conference on Decision and Control*, 2000, pp. 1075–1080.

[15] V. Zakian, "Critical systems and tolerable inputs," *International Journal of Control*, vol. 49, no. 4, pp. 1285–1289, 1989.

[16] J. F. Whidborne and G. P. Liu, *Critical Control Systems: Theory Design and Applications*. Taunton: Research Studies Press, 1993.

[17] P. G. Lane, "Design of control systems with inputs and outputs satisfying certain bounding conditions," PhD Thesis, University of Manchester Institute of Science and Technology, Manchester, UK, 1992.

[18] W. Silpsrikul and S. Arunsawatwong, "Design of fractional PI controllers for a binary distillation column with disturbances restricted in magnitude and slope," in *Proceedings of the 18th IFAC World Congress*, 2011, pp. 7702–7707.

[19] P. Kalvibool and S. Arunsawatwong, "Design of load frequency regulator for power systems subject to bounded persistent disturbance considering generation rate constraint," in *Proceedings of SICE Annual Conference*, 2015, pp. 916–921.

[20] H. H. Nguyen and S. Arunsawatwong, "Design of feedback systems with backlash for inputs restricted in magnitude and slope," in *Proceedings of SICE Annual Conference*, 2013, pp. 229–234.

[21] ——, "Robust controller design for feedback systems with uncertain backlash and plant uncertainties subject to inputs satisfying bounding conditions," in *Proceedings of the 19th IFAC World Congress*, 2014, pp. 1643–1648.

[22] S. Oldak, C. Baril, and P. O. Gutman, "Quantitative design of a class of nonlinear systems with parameter uncertainty," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 1, pp. 101–117, 1994.

[23] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic Press, 1975.

[24] E. P. Pfeiffer, "The maximum response ratio of linear systems," *Transactions of the American Institute of Electrical Engineers*, vol. 73, no. 6, pp. 480–484, 1955.

[25] V. Zakian and U. Al-Naib, "Design of dynamical and control systems by the method of inequalities," *Proceedings of the Institution of Electrical Engineers*, vol. 120, no. 11, pp. 1421–1427, 1973.

[26] V. Zakian, "Design formulations," *International Journal of Control*, vol. 46, no. 2, pp. 403–408, 1987.

[27] S. Arunsawatwong and V. Q. Nguyen, "Design of retarded fractional delay differential systems using the method of inequalities," *International Journal of Automation and Computing*, vol. 6, no. 1, pp. 22–28, 2009.

[28] S. Arunsawatwong and T. Chuman, "Stability of control systems with multiple sector-bounded nonlinearities for inputs having bounded magnitude and bounded slope," *Engineering Journal*, vol. 27, no. 5, pp. 69–81, 2023.

[29] M. A. A. S. Choudhury, N. F. Thornhill, and S. L. Shah, "Modelling valve stiction," *Control Engineering Practice*, vol. 13, no. 5, pp. 641–658, 2005.

[30] R. K. Wood and M. W. Berry, "Terminal composition control of a binary distillation column," *Chemical Engineering Science*, vol. 28, no. 9, pp. 1707–1717, 1973.

[31] S. Arunsawatwong, "Stability of retarded delay differential systems," *International Journal of Control*, vol. 65, no. 2, pp. 347–364, 1996.

[32] E. J. McShane, *Integration*. Princeton, NJ: Princeton University Press, 1944.

[33] V. S. Mai, "Design of feedback control systems with a sector-bounded nonlinearity using Zakian's framework," Master's thesis, Chulalongkorn University, Bangkok, Thailand, 2010.

[34] R. G. Bartle, *The Elements of Integration and Lebesgue Measure*. New York: Wiley, 1995.

Appendix A. Proof of Lemma 2.1

Definition A.1 ([32]). Let (E, ρ) be a metric space. Let $\mathcal{G}$ denote a set of functions that are defined and finite-valued on E. The set $\mathcal{G}$ is said to be equicontinuous if, for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that for all $f \in \mathcal{G}$,

$$|f(x_1) - f(x_2)| < \varepsilon \quad (41)$$

whenever $x_1, x_2 \in E$ and $\rho(x_1, x_2) < \delta$. The set $\mathcal{G}$ is said to be uniformly bounded if there is an $M < \infty$ such that

$$|f(x)| \leq M, \ \forall\, x \in E, \ \forall\, f \in \mathcal{G}. \quad (42)$$

Theorem A.2 (Ascoli's Theorem [32]). *Let $\mathcal{G}$ denote a set of functions that are defined on a bounded and closed set. If $\mathcal{G}$ is equicontinuous and uniformly bounded, then it is possible to select a uniformly convergent subsequence from every sequence $\{f_n\}$ of functions of $\mathcal{G}$.*

Proposition A.3 ([10]). *In a Banach space, a subset K is relatively compact if and only if every sequence in K contains a subsequence.*

Definition A.4 ([10]). Let X and Y be Banach spaces, and $\mathcal{H} : D \subset X \to Y$ be an operator. Then $\mathcal{H}$ is called compact if and only if (i) $\mathcal{H}$ is continuous; and (ii) $\mathcal{H}$ maps bounded sets into relatively compact sets.

Proof of Lemma 2.1. The lemma is proved with the technique used in [33]. Note that this lemma considers the case in which $\mathcal{H}$ is an affine operator (see (14)), whereas [33] considers the case in which $\mathcal{H}$ is a linear operator. For the sake of brevity, the sketch of the proof is given here.

Since $h_{jk} \in \mathcal{A}$ for all j, k, it follows from (14) that there exists $C_0 < \infty$ such that

$$||\mathcal{H}\mathbf{x} - \mathcal{H}\mathbf{y}|| \leq nC_0 \sum_{j=1}^{n} ||x_j - y_j||_\infty = nC_0||\mathbf{x} - \mathbf{y}||.$$

Thus, we conclude that $\mathcal{H}$ is continuous.

Let $\{\mathbf{x}^{(l)}\}$ be any sequence in X_T and let $\mathbf{y}^{(l)}$ be defined by

$$\mathbf{y}^{(l)}(t) = \big(\mathcal{H}\mathbf{x}^{(l)}\big)(t), \quad \forall\, t \in [0, T].$$

Then we can write

$$y^{(l)}(t) = \big(\mathcal{H}_j\mathbf{x}^{(l)}\big)(t), \quad j = 1, 2, \ldots, n.$$

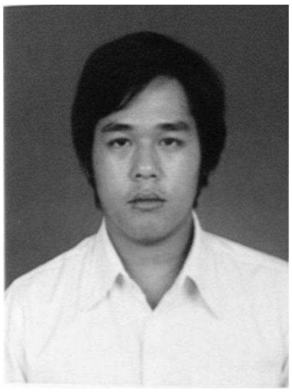
Since $X_T \subset \mathbb{L}^\infty_{n,T}$ and since $h_{jk} \in \mathcal{A}$ for all j, k, it follows from (14) that $\{y_j^{(l)}\}$ are uniformly bounded on $[0, T]$ for any fixed $T > 0$ and for all j. Furthermore, by using conditions $\dot{h}_{jk} \in \mathcal{A}$ for all j, k and by the dominated convergence theorem (see, e.g., [34]), one can verify that for any $t_1, t_2 \in [0, T]$, any $l > 0$ and any $\varepsilon > 0$, there always exists $\delta > 0$ such that

$$|y_j^{(l)}(t_1) - |y_j^{(l)}(t_2)| \leq \varepsilon \ \forall\, j \text{ whenever } |t_1 - t_2| \leq \delta.$$

Consequently, we conclude by Definition A.1 that $\{y_j^{(l)}\}$ $(j = 1, 2, \ldots, n)$ are equicontinuous. Hence, in view of Theorem A.2, $\{y_j^{(l)}\}$ contain a convergent subsequence for $j = 1, 2, \ldots, n$.

Now, by using Proposition A.3, the sets $\mathcal{H}_1(X_T)$, $\mathcal{H}_2(X_T)$, $\ldots$, $\mathcal{H}_n(X_T)$ are relatively compact. Thus, by virtue of Tychonoff's theorem (see, e.g., [10]), the set $\mathcal{H}(X_T) = \mathcal{H}_1(X_T) \times \mathcal{H}_2(X_T) \times \cdots \times \mathcal{H}_n(X_T)$ is relatively compact and so we conclude that the operator $\mathcal{H}$ maps bounded sets into relatively compact sets. Hence, the compactness of $\mathcal{H}$ readily follows from Definition A.4 since H is continuous. □

DOI:10.4186/ej.2023.27.6.11



Tadchanon Chuman received his B.Eng. and M.Eng. degrees in electrical engineering from Chulalongkorn University, Thailand in 2012 and 2014, respectively. He is currently studying for a Ph.D. at Control Systems Research Laboratory, the Department of Electrical Engineering, Chulalongkorn University. His main research interests include control systems design using the method of inequalities and the principle of matching, control systems with nonlinearities, and load frequency control.

Suchin Arunsawatwong received his B.Eng. and M.Eng. degrees in electrical engineering from Chulalongkorn University, Thailand, in 1985 and 1988, respectively, and Ph.D. degree in control engineering from the Control Systems Centre, University of Manchester Institute of Science and Technology, U.K., in 1995. He is currently an assistant professor at the Department of Electrical Engineering, Chulalongkorn University. His main research interests include control systems design by the method of inequalities and the principle of matching, delay differential systems, fractional-order control systems, control systems with nonlinearities, and power systems control applications.