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Journal Pre-proof

Computing the μ -bases of algebraic monoid curves and surfaces

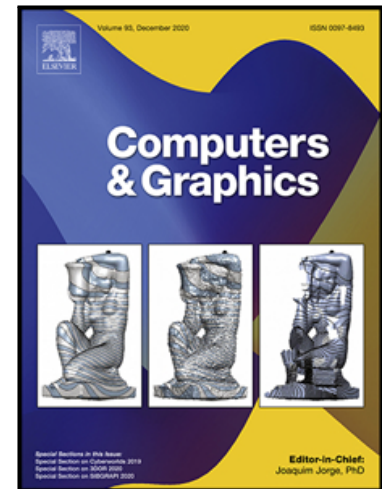
Sonia Pérez-Díaz, Li-Yong Shen

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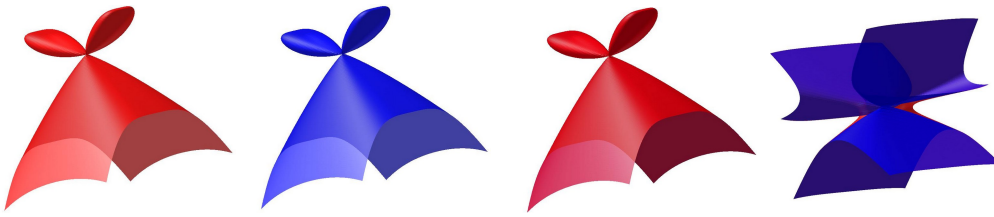
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Highlights

- We propose the μ -basis formulas for the **implicit** monoid curves/surfaces;
- New μ -basis formulas for the parametric monoid curves/surfaces in **general expression**;
- **Approximate μ -bases** are computed for the monoid curves/surfaces as well as the error estimations

Journal Pre-proof



Finding the approximate parametrization of algebraic surface by μ -basis

Input surface (red color with 25% transparency), output surface (blue color with 25% transparency), both surfaces in same figure and enlarged surfaces near the origin

Journal Pre-proof

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Author names and affiliations

Sonia Pérez-Díaz

{Dpto. de Física y Matemáticas, Universidad de Alcalá, 28871-Alcalá de Henares,
Madrid, Spain

Email: sonia.perez@uah.es

Li-Yong Shen

School of Mathematical Sciences, University of Chinese Academy of Sciences, 100049,
Beijing, China

Email: lyshen@ucas.ac.cn

Corresponding authors:

Li-Yong Shen

Email: lyshen@ucas.ac.cn

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Computing the μ -bases of algebraic monoid curves and surfaces

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ABSTRACT

The μ -basis is a developing algebraic tool to study the expressions of rational curves and surfaces. It can play a bridge role between the parametric forms and implicit forms and show some advantages in implicitization, inversion formulas and singularity computation. However, it is difficult and there are few works to compute the μ -basis from an implicit form. In this paper, we derive the explicit forms of μ -basis for implicit monoid curves and surfaces, including the conics and quadrics which are particular cases of these entities. Additionally, we also provide the explicit form of μ -basis for monoid curves and surfaces defined by any rational parametrization (not necessarily in standard proper form). Our technique is simply based on the linear coordinate transformation and standard forms of these curves and surfaces. As a practical application in numerical situation, if an exact multiple point can not be computed, we can consider the problem of computing “approximate μ -basis” as well as the error estimation.

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1. Introduction

The μ -basis was first introduced in [7] to provide a compact representation for the implicit equation of a rational parametric curve. The μ -basis of a rational planar curve is a basis for the syzygy module with respect to homogeneous parameterizations, the degrees of their elements are unique and sum to the degree of the parametrization, their cross product retrieves the homogeneous parametrization, and their resultant generates the implicit equation of the curve [3]. The μ -basis can be used not only to recover the parametric equation of a rational curve but also to derive its implicit equation. **Additionally the μ -bases are useful used to compute and to analyze all the singular points of low degree rational planar and space curves (see [27] and [28]).** There are several efficient methods to compute the μ -bases for rational curves by computing two moving lines which satisfy the required properties [7] or based on the vector elimination [3].

The μ -basis has also been generalized to rational surfaces [5] and the situation for rational surfaces is quite different: even the degrees of μ -basis elements can be different. Currently, the only known algorithm to compute a weak μ -basis of a rational surface is designed based on the polynomial matrix factorization [8]. However, for certain rational surfaces with special ge-

ometry, the μ -basis can be defined well and there are some explorations on the μ -bases of such special surfaces, Steiner surfaces, surfaces of revolution, rational surfaces, ruled surfaces, cyclides as well as canal surfaces etc [14].

In all above discussions, the μ -basis is derived from the parametric expression. For an implicit curve or surface, it is difficult to compute a μ -basis, since finding a rational parametric expression from an implicit equation is nontrivial, which is known as the parametrization problem. To benefit from the bridge role of the μ -basis, there are still few papers trying to find the μ -basis from an implicit equation. For quadratic surfaces with two base points or cubic surfaces with six base points, the minimal μ -basis was proved to be all linear in the parametric variables and the minimal μ -basis can be computed either from the parametric equation or the implicit equation [4]. The situation was generalized to the quadratic surface with one simple base point [29]. Another work was to compute μ -bases from algebraic ruled surfaces [25].

A monoid hypersurface is an irreducible algebraic hypersurface which has a singularity of multiplicity one less than the degree of the hypersurface [15]. Note that conics and quadrics are monoid curves and surfaces of degree 2 that have a point of maximum multiplicity (in this case, a simple point) and they

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are very popularly used in geometric modeling and computer graphics [12, 9, 13]. In the Bernstein-Bézier formulation, a monoid curve has a simple representation if we choose one vertex of the reference triangle to be the multiple point. This property can be easily extended to the surface case using barycentre coordinates in a tetrahedron. Hence the monoid curves and surfaces are used to approximate implicitization [26] and approximate parametrization [17, 18]. In [2], the authors discussed the minimal generators of the defining ideal of the Rees Algebra associated to monoid parameterizations, on the other words, they found the generators of the syzygy model associated to the standard parameterizations. If the parameterization corresponds to a curve or a surface then the minimal generators form the μ -basis of the curve or the surface.

In this paper, we attempt to compute μ -basis from another direction, i.e., we give a uniform and explicit way to get μ -basis from the standard implicit form. Our method is simply based on the linear algebra computations and the fact that all partial derivatives of the implicit equation of a monoid curve or surface of degree d , till order $d - 2$, vanish at the singular point. The parametrization is used only as the auxiliary role in our discussion. Once we get the formula form of the μ -basis, people can find a μ -basis from the implicit equation without parametrization any more. Notice that by our lemmas, we can also compute the μ -basis from a rational parametrization not necessary in standard form in [2].

We note that for checking the existence and actual computation of a singularity of multiplicity $d - 1$ for a surface (similarly for a plane curve), one has to solve the system of algebraic equations

$$\mathcal{A} = \left\{ \frac{\partial^{i+j+k} f}{\partial^i x \partial^j y \partial^k z} (x, y, z) = 0 \right\}_{i+j+k=0, \dots, d-2}.$$

The system \mathcal{A} may be simplified by reducing the number of equations and their degrees. More precisely, first, we choose three triples (i_ℓ, j_ℓ, k_ℓ) , with $\ell = 1, 2, 3$, such that $i_\ell + j_\ell + k_\ell = d - 2$, and we consider the new subsystem

$$\mathcal{B} = \left\{ \frac{\partial^{d-2} f}{\partial^{i_\ell} x \partial^{j_\ell} y \partial^{k_\ell} z} (x, y, z) = 0 \right\}, \ell = 1, 2, 3.$$

that only involves quadratic equations. After computing the solutions of \mathcal{B} , one chooses the one satisfying the system \mathcal{A} (if we have a monoid surface this solution exists and it is unique). So, the symbolic method requires the computation of a simple point on these quadrics; once the point is determined, the remaining steps can be executed symbolically without further difficulties (similarly for the case of plane curves). The computation of this point can be performed either symbolically, for instance introducing algebraic numbers, or numerically by root finding methods (see [1], [10], [11]). In this case, we should consider the problem of computing approximate μ -bases.

In this paper, we present two methods. The first one (see Theorems 3, 8, and Corollaries 1, 3) uses well-known linear algebra techniques to transform a conics or quadrics into the standard form (i.e. defined by a polynomial of the form $a_1 x_1^2 + \dots + a_n x_n^2$, $a_i \in \mathbb{K}$, $n = 2$ or 3 , and \mathbb{K} and algebraically

closed field of zero characteristic). Note that this transformation is very efficient since it only has to do with linear algebra operations. Afterwards, we give a uniform and explicit way to get μ -basis from the standard implicit form defining a conic or a quadric. The second method (see Theorems 4, 5, 9 and 10) assumes that the origin is a point of maximum multiplicity on the variety and therefore it can be applied not only to conics, quadrics, and but also in general to monoid curves and surfaces of any degree. The second method only has to do with the computation of some derivatives of order one of the implicit polynomial defining the monoid curve or surface.

In both methods, we show the relation of the μ -basis with the coefficients defining the implicit equation of the given variety and also with the rational parametrization defining the variety.

We can find that the second method can be applied to more curves and surfaces. Moreover, if an exact point can not be computed, we can not deal in an exact way with the implicit equation defining the variety. This leads us to consider the problem of computing "approximate μ -basis". Some comments concerning this problem are included at the end of each section. Precisely, using [17] and [18], we show how to compute μ -basis for an input variety not necessarily rational. This is the first process to find the approximate μ -bases from implicit curves and surfaces, since there was only one paper considering the approximate μ -bases of the rational parametric equations [24] without error estimation.

The paper is organized as follows. In Section 2, we recall the definition, properties and an algorithm as well as two new lemmas for the μ -basis of the rational curve. In Section 3, we propose the explicit μ -bases for the conics of the implicit form and the second method can be generalized to plane curves having a point of maximum multiplicity. In Section 4, the minimal μ -basis for the quadrics of the implicit form is discussed and the numerical consideration is also introduced for the curves and surfaces in Sections 3 and 4 respectively. We present algorithms and examples in Section 5. Finally, we conclude our paper in Section 6 with a brief summary of our work.

2. μ -Bases for rational planar curves

Here we review the definition of the μ -basis and propose some necessary properties. For this purpose, \mathbb{K} denotes and algebraically closed field of zero characteristic, and $\mathbb{K}(\cdot)$ the field of rational functions in the variables (\cdot) . We also will use $\mathbb{K}[\cdot]$ that denotes the polynomials in the variables (\cdot) .

The μ -basis of a rational planar curve $\mathcal{P}(t) = (\wp_1(t) : \wp_2(t) : \wp_3(t))$ is defined as a special basis of the moving line ideal of the rational curve in [7].

For a better understanding of the concept of μ -basis, syzygies can be used. A moving line $A(t)x_1 + B(t)x_2 + C(t)x_3 = 0$ is a line corresponding to a three dimensional vector $(A(t), B(t), C(t)) \in \mathbb{K}[t]^3$ with a parameter t , and we call a moving line follows $\mathcal{P}(t)$ if $A(t)\wp_1(t) + B(t)\wp_2(t) + C(t)\wp_3(t) \equiv 0$. In algebraic view, the $(A(t), B(t), C(t))$ exactly corresponds to a syzygy of $\mathcal{P}(t)$. Thus the set

$$M_{\mathcal{P}} := \{(A(t), B(t), C(t)) \in \mathbb{K}[t]^3 \mid \wp_1 A(t) + \wp_2 B(t) + \wp_3 C(t) \equiv 0\}$$

corresponds to all the moving lines following the rational curve $\mathcal{P}(t)$ defines a syzygy module over $\mathbb{K}[t]$. The module $M_{\mathcal{P}}$ is free of rank two, and a μ -basis for the rational curve $\mathcal{P}(t)$ is just a basis of the syzygy module $M_{\mathcal{P}}$ with the lowest possible degree. More precisely, one has the following formal definition (see e.g. [7]).

Definition 1. Two moving lines $p^{\mathcal{P}}(\bar{x}, t) = p_1(t)x_1 + p_2(t)x_2 + p_3(t)x_3 = 0$ and $q^{\mathcal{P}}(\bar{x}, t) = q_1(t)x_1 + q_2(t)x_2 + q_3(t)x_3 = 0$, or equivalently, two polynomial vectors $\mathbf{p}(t) = (p_1(t), p_2(t), p_3(t)) \in \mathbb{K}[t]^3$ and $\mathbf{q}(t) = (q_1(t), q_2(t), q_3(t)) \in \mathbb{K}[t]^3$ are a μ -basis of the curve defined by $\mathcal{P}(t)$ (or the syzygy module $M_{\mathcal{P}}$), if

1. $\mathbf{p}(t)$ and $\mathbf{q}(t)$ form a basis for the syzygy module $M_{\mathcal{P}}$, i.e., any moving line $L(t) \in M_{\mathcal{P}}$ can be expressed by $L = h_1\mathbf{p} + h_2\mathbf{q}$ with $h_1, h_2 \in \mathbb{K}[t]$; and
2. $\mathbf{p}(t)$ and $\mathbf{q}(t)$ have the lowest degree among all the bases of $M_{\mathcal{P}}$, i.e., assuming that $\deg(\mathbf{p}) \leq \deg(\mathbf{q})$, then there does not exist another basis $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ of $M_{\mathcal{P}}$ with $\deg(\bar{\mathbf{p}}) \leq \deg(\bar{\mathbf{q}})$ such that $\deg(\bar{\mathbf{p}}) < \deg(\mathbf{p})$ or $\deg(\bar{\mathbf{q}}) < \deg(\mathbf{q})$.

Based on the definitions, Chen and Wang in [3] derived the equivalent definitions of μ -basis, we review a necessary one below.

Theorem 1. Let $\mathbf{p}(t)$, $\mathbf{q}(t)$ be two moving lines following the curve defined by $\mathcal{P}(t)$ with $\deg(\mathbf{p}) \leq \deg(\mathbf{q})$. Then $\mathbf{p}(t)$, $\mathbf{q}(t)$ form a μ -basis for $\mathcal{P}(t)$ if and only if one of the following conditions holds:

1. $\mathbf{p} \times \mathbf{q} = k\mathcal{P}(t)$ for some nonzero constant k , and $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathcal{P})$.
2. $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathcal{P})$, and $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are $\mathbb{K}[t]$ -linearly independent.

The following properties of μ -basis can be easily obtained from the above definitions (see [3] and [20]).

Theorem 2. Let $\mathbf{p}(t)$, $\mathbf{q}(t)$ be a μ -basis for $\mathcal{P}(t)$ with $\deg(\mathbf{p}) \leq \deg(\mathbf{q})$. Then,

1. $\mathbf{p} \times \mathbf{q} = k\mathcal{P}(t)$ for some nonzero constant k .
2. If $\mathcal{P}(t)$ is a parametrization with fibre degree $\deg(\phi_{\mathcal{P}})$, then $\text{resultant}_t(p^{\mathcal{P}}, q^{\mathcal{P}})^{\deg(\phi_{\mathcal{P}})}$ is the implicit equation of the curve defined by $\mathcal{P}(t)$, where $p^{\mathcal{P}}, q^{\mathcal{P}}$ are introduced in Definition 1.

In the following, we prove some technical lemmas that analyze the behavior of the μ -basis under change of variables and change of coordinates and that will play an important role in Sections 3 and 4. The first lemma was proved in [20].

Lemma 1. Let $\bar{\mathbf{p}}(t)$, $\bar{\mathbf{q}}(t)$ be a μ -basis for a parametrization $\bar{Q}(t)$ with $\deg(\bar{\mathbf{p}}) \leq \deg(\bar{\mathbf{q}})$. Let $R(t) \in \mathbb{K}(t) \setminus \mathbb{K}$. It holds that $\mathbf{p}(t) = \bar{\mathbf{p}}(R(t))$ and $\mathbf{q}(t) = \bar{\mathbf{q}}(R(t))$ form a μ -basis for the reparametrization $\mathcal{P}(t) = \bar{Q}(R(t))$ with $\deg(\mathbf{p}) \leq \deg(\mathbf{q})$.

Note that we consider $Q(R(t))$, with $R(t) = r_1(t)/r_2(t) \in \mathbb{K}(t) \setminus \mathbb{K}$, in homogenous form. Hence, in this paper, $\mathcal{P}(t) = Q(R(t))$

means $\mathcal{P}(t) = Q(R(t))r_2(t)^{\deg(\mathcal{P})}$ which is a polynomial vector in homogenous form.

In the next lemma, we investigate the μ -bases of a rational curve $\mathcal{P}(t)$ and the curve $\bar{\mathcal{P}}(t)$ after a linear coordinate transformation from $\mathcal{P}(t)$.

Lemma 2. Let $\mathcal{P}(t) = (\varphi_1(t) : \varphi_2(t) : \varphi_3(t))$, and $T : \mathbb{K}^3 \rightarrow \mathbb{K}^3$, $T(\bar{x}) = (a_1x_1 + b_1x_2 + c_1x_3, a_2x_1 + b_2x_2 + c_2x_3, a_3x_1 + b_3x_2 + c_3x_3)$, (with $\bar{x} = (x_1, x_2, x_3)$) invertible such that $\bar{\mathcal{P}}(t) = T(\mathcal{P}(t))$ is a parametrization. Let $\bar{\mathbf{p}}(t)$, $\bar{\mathbf{q}}(t)$ be a μ -basis for $\bar{\mathcal{P}}(t)$, with $\deg(\bar{\mathbf{p}}) \leq \deg(\bar{\mathbf{q}})$. It holds that $S(\bar{\mathbf{p}})$ and $S(\bar{\mathbf{q}})$ form a μ -basis for $\mathcal{P}(t)$, where $S(\bar{x}) = (a_1x_1 + a_2x_2 + a_3x_3, b_1x_1 + b_2x_2 + b_3x_3, c_1x_1 + c_2x_2 + c_3x_3)$.

Proof. We first note that T is invertible and then $D \neq 0$ (where D is the determinant of the coefficients of the components of T). Then, $\bar{\mathcal{P}}(t) = T(\mathcal{P}(t)) = (a_1\varphi_1(t) + b_1\varphi_2(t) + c_1\varphi_3(t), a_2\varphi_1(t) + b_2\varphi_2(t) + c_2\varphi_3(t), a_3\varphi_1(t) + b_3\varphi_2(t) + c_3\varphi_3(t))$. Since $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ form a μ -basis for $\bar{\mathcal{P}}(t)$, $\bar{\mathbf{p}} \times \bar{\mathbf{q}} = k\bar{\mathcal{P}}$ for some non-zero constant k and $\deg(\bar{\mathbf{p}}) + \deg(\bar{\mathbf{q}}) = \deg(\bar{\mathcal{P}})$. Thus, it holds that $S(\bar{\mathbf{p}})(t)$ and $S(\bar{\mathbf{q}})(t)$ form a μ -basis for $\mathcal{P}(t)$. Indeed, we note that $\deg(S(\bar{\mathbf{p}})) \leq \deg(S(\bar{\mathbf{q}}))$ (note that $\deg(\bar{\mathbf{p}}) \leq \deg(\bar{\mathbf{q}})$ and S is invertible because $D \neq 0$). In addition, since $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ form a μ -basis for $\bar{\mathcal{P}}(t)$, we have that $\deg(\bar{\mathbf{p}}) + \deg(\bar{\mathbf{q}}) = \deg(\bar{\mathcal{P}})$, and then $\deg(S(\bar{\mathbf{p}})) + \deg(S(\bar{\mathbf{q}})) = \deg(\mathcal{P})$ (note that $\deg(\mathcal{P}) = \deg(\bar{\mathcal{P}})$, and $\deg(S(\bar{\mathbf{p}})) = \deg(\bar{\mathbf{p}})$, $\deg(S(\bar{\mathbf{q}})) = \deg(\bar{\mathbf{q}})$). Finally, we have that $0 = \bar{\mathcal{P}}(t) \cdot \bar{\mathbf{p}}(t) = \varphi_1(a_1\bar{\mathbf{p}}_1(t) + a_2\bar{\mathbf{p}}_2(t) + a_3\bar{\mathbf{p}}_3(t)) + \varphi_2(b_1\bar{\mathbf{p}}_1(t) + b_2\bar{\mathbf{p}}_2(t) + b_3\bar{\mathbf{p}}_3(t)) + \varphi_3(c_1\bar{\mathbf{p}}_1(t) + c_2\bar{\mathbf{p}}_2(t) + c_3\bar{\mathbf{p}}_3(t)) = \mathcal{P}(t) \cdot S(\bar{\mathbf{p}})$. We reason similarly for $S(\bar{\mathbf{q}})$. Since $S(\bar{\mathbf{p}})$ and $S(\bar{\mathbf{q}})$ are linearly independent we get that $S(\bar{\mathbf{p}}) \times S(\bar{\mathbf{q}}) = k\mathcal{P}$ for some non-zero constant k . From Theorem 1, we conclude that $S(\bar{\mathbf{p}})$ and $S(\bar{\mathbf{q}})$ is a μ -basis for $\mathcal{P}(t)$. \square

We observe that clearly the reciprocal of Lemma 2 also holds.

3. μ -Bases of implicit monoid curves

In this section, efficient methods are developed to compute the μ -basis either from the parametric equation or the implicit equation of a given monoid curve with a point P of high multiplicity (that is, all partial derivatives of the defining polynomial of the curve, till order $d - 2$, vanish at P). More precisely, we show how to compute μ -basis from a given proper parametrization of a conic and in general of a monoid curve (see Theorems 3, 4 and 5). From the expression of the μ -basis computed, we deduce that this μ -basis can be obtained directly from the implicit equation. We observe that, from Lemma 1, the properness of the parametrization can be assumed w.l.o.g (see also [20]).

We remind that the implicit equation and the parametric equation can be derived straightforward from the μ -basis (see Theorem 2). So we can represent the plane curve using its μ -basis instead.

Finally, we briefly consider the problem of computing "approximate μ -basis". More precisely, using [17], we show how to compute μ -basis for an input curve not necessarily rational and having an ϵ -point of maximum multiplicity.

To start with, let C be a projective conic with ground field \mathbb{K} (an algebraically closed field), implicitly defined by an homogeneous polynomial $F(\bar{x})$ of degree 2, where $\bar{x} = (x_1, x_2, x_3)$. By well-known techniques in linear algebra, we can find a linear change of coordinates over \mathbb{K} transforming the conic C to a conic of the form $F(\bar{x}) = a_1x_1^2 + a_2x_2^2 - a_3x_3^2$, where $a_i \in \mathbb{K}$. Furthermore, we may assume w.l.o.g that $a_1 \cdot a_2 \neq 0$. Let us denote this new conic also as C . Under these conditions, we consider a rational proper parametrization of C ,

$$\begin{aligned} \mathcal{P} : \mathbb{K} &\dashrightarrow C \subset \mathbb{P}^2(\mathbb{K}) \\ t &\longmapsto (\varphi_1(t) : \varphi_2(t) : \varphi_3(t)), \end{aligned} \quad (1)$$

where $\gcd(\varphi_1, \varphi_2, \varphi_3) = 1$. Note that any proper parametrization has $\deg(\mathcal{P}) = 2$ (see Chapter 4 in [23]).

Theorem 3. *It holds that*

$$\mathbf{p}(t) = (a_1\varphi_1'(t), a_2\varphi_2'(t), -a_3\varphi_3'(t))$$

$$\mathbf{q}(t) = (a_1(2\varphi_1(t) - t\varphi_1'(t)), a_2(2\varphi_2(t) - t\varphi_2'(t)), -a_3(2\varphi_3(t) - t\varphi_3'(t)))$$

form a μ -basis for any conic parametrization $\mathcal{P}(t) = (\varphi_1(t) : \varphi_2(t) : \varphi_3(t))$.

Proof. Since $F(\mathcal{P}) = 0$ and $\nabla(F(\mathcal{P}))\mathcal{P}'(t) = 0$, we get that $(a_1\varphi_1)\varphi_1 + (a_2\varphi_2)\varphi_2 + (-a_3\varphi_3)\varphi_3 = (a_1\varphi_1')\varphi_1 + (a_2\varphi_2')\varphi_2 + (-a_3\varphi_3')\varphi_3 = 0$. From these equalities we get that $a_1(2\varphi_1(t) - t\varphi_1'(t))\varphi_1 + a_2(2\varphi_2(t) - t\varphi_2'(t))\varphi_2 - a_3(2\varphi_3(t) - t\varphi_3'(t))\varphi_3 = 0$. Therefore, $\mathcal{P} \cdot \mathbf{p} = \mathcal{P} \cdot \mathbf{q} = 0$. In addition, since $\deg(\varphi_i) \leq 2, i = 1, 2, 3$ and $\deg(\varphi_j) = 2$ for some $j = 1, 2, 3$, we have $\deg(2\varphi_i - t\varphi_i') \leq 1$ and $\deg(2\varphi_j - t\varphi_j') = 1$ for some $j = 1, 2, 3$. Thus, $\deg(\mathbf{p}) = \deg(\mathbf{q}) = 1$ and this implies that $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathcal{P}) = 2$. Hence statement 1 in Theorem 1 holds and we get that $\mathbf{p}(t) = (a_1\varphi_1'(t), a_2\varphi_2'(t), -a_3\varphi_3'(t))$, $\mathbf{q}(t) = (a_1(2\varphi_1(t) - t\varphi_1'(t)), a_2(2\varphi_2(t) - t\varphi_2'(t)), -a_3(2\varphi_3(t) - t\varphi_3'(t)))$ form a μ -basis for any parametrization $\mathcal{P}(t)$. \square

Remark 1. *This theorem can also be obtained using theory of Rees Algebra by computing certain minimal set of generators (see Theorem 2.10 in [2]) when we focus the same monoid curve case. Here, we prefer to give a straightforward proof. Notice that here, by the formula one can find a μ -basis from the implicit equation without parameterizing.*

It is easy to prove that

$$\mathcal{P}(t) = (\varphi_1(t) : \varphi_2(t) : \varphi_3(t)) = \left(2t \sqrt{\frac{a_3}{a_1}} : (t^2 - 1) \sqrt{\frac{a_3}{a_2}} : t^2 + 1 \right)$$

is a rational proper parametrization of C . Remind that any proper parametrization of C is expressed as $\mathcal{P}(R(t))$, where $R(t) \in \mathbb{K}(t) \setminus \mathbb{K}$, $\deg(R) = 1$. In this case, we get the following corollary.

Corollary 1. *It holds that*

$$\mathbf{p}(t) = (\sqrt{a_1a_3}, \sqrt{a_2a_3}t, -a_3t), \mathbf{q}(t) = (\sqrt{a_1a_3}t, -\sqrt{a_2a_3}, -a_3)$$

form a μ -basis for the parametrization

$$\mathcal{P}(t) = (2t\sqrt{a_3/a_1} : (t^2 - 1)\sqrt{a_3/a_2} : t^2 + 1).$$

Since $\bar{x} \cdot \mathbf{p} = \bar{x} \cdot \mathbf{q} = 0$, for $\bar{x} \in C$ and where $\mathbf{p}(t), \mathbf{q}(t)$ form the μ -basis of Corollary 1, we also can easily compute an inverse of the input proper parametrization (see the results concerning the computation of the inverse presented in [20]). We remind that the inverse is unique modulo the implicit equation defining the algebraic plane curve. More precisely, we get the following corollary.

Corollary 2. *It holds that $\frac{-\sqrt{a_1a_3x_1}}{\sqrt{a_2a_3x_2 - a_3x_3}}$ is an inverse of $\mathcal{P}(t)$. In addition, $\frac{\sqrt{a_2a_3x_2 + a_3x_3}}{\sqrt{a_1a_3x_1}}$ is another inverse of $\mathcal{P}(t)$.*

Remark 2. *Using Lemmas 1 and 2 and Corollary 1, one gets a μ -basis for any proper parametrization $\mathcal{P}(t)$ of any conic C generally defined by an irreducible homogeneous polynomial of the form $F(\bar{x}) = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 + a_4x_1x_3 + a_5x_2x_3 + a_6x_3^2$, $a_i \in \mathbb{K}$.*

Example 1. *We consider the conic defined by the implicit homogeneous polynomial $-x_1^2 + 4x_1x_2 - x_2^2 + 2x_3^2$. By well-known techniques in linear algebra, we can find a linear change of coordinates transformation $T(\bar{x}) = (x_1\sqrt{2}/2 - x_3\sqrt{2}/2, x_1\sqrt{2}/2 + x_3\sqrt{2}/2, x_2)$, that transforms the input conic to the conic C defined by the polynomial $F(\bar{x}) = x_1^2 + 2x_2^2 - 3x_3^2$. From Corollary 1, we get that $\mathbf{p}(t) = (\sqrt{3}, t\sqrt{6}, -3t)$ and $\mathbf{q}(t) = (t\sqrt{3}, -\sqrt{6}, -3)$ form a μ -basis for the parametrization $\mathcal{P}(t) = (2t\sqrt{3} : \sqrt{6}/2(t^2 - 1) : t^2 + 1)$ of the curve C . Using Lemma 2 with the linear transformation $S(\bar{x}) = (x_1\sqrt{2}/2 + x_2\sqrt{2}/2, x_3, -x_1\sqrt{2}/2 + x_2\sqrt{2}/2)$ that can be easily derived from $S(\bar{x})$, one gets the μ -basis*

$$\begin{aligned} &(\sqrt{2}/2(\sqrt{3} + 3t), \sqrt{2}/2(\sqrt{3} - 3t), t\sqrt{6}), \\ &(\sqrt{6}/2(t + \sqrt{3}), -\sqrt{6}/2(-t + \sqrt{3}), -\sqrt{6}) \end{aligned}$$

for the input conic defined by the irreducible homogeneous polynomial $-x_1^2 + 4x_1x_2 - x_2^2 + 2x_3^2$. In addition (from Theorem 2), for the input conic we get the parametrization

$$\begin{aligned} &((-\sqrt{2}/2(-t + \sqrt{3} - \sqrt{2})(-t + \sqrt{3} + \sqrt{2}) : \\ &\sqrt{2}/2(t + \sqrt{3} + \sqrt{2})(t + \sqrt{3} - \sqrt{2}) : \sqrt{6}/2(t - 1)(t + 1)). \end{aligned}$$

It is well-known that we can find a linear change of coordinates over \mathbb{K} transforming the irreducible conic C to a conic of the form $F(\bar{x}) = f_2(x_1, x_2) + x_3f_1(x_1, x_2)$, where $f_i(x_1, x_2)$ are homogeneous polynomials of degree i . Note that we have assumed that C passes through the origin. Under these conditions, we have that

$$\mathcal{P}(t) = (\varphi_1(t) : \varphi_2(t) : \varphi_3(t)) = (-tf_1(t, 1), -f_1(t, 1), f_2(t, 1))$$

is a proper parametrization of C (see Section 4.6 in [23]).

In the following, we denote by g^z the derivative of a certain polynomial g with respect to the variable z . If g is a univariate polynomial, we will use the notation g' to represent the first derivative of the polynomial g w.r.t the variable.

Theorem 4. For a conic curve defined by $f_2(x_1, x_2) + x_3 f_1(x_1, x_2) = 0$, it holds that

$$\mathbf{p}(t) = (1, -t, 0)$$

$$\mathbf{q}(t) = (f_2^{x_1}(t, 1), f_2^{x_2}(t, 1), t f_1^{x_1}(t, 1) + f_1(t, 1) + f_1^{x_2}(t, 1))$$

form a μ -basis for $\mathcal{P}(t) = (-t f_1(t, 1), -f_1(t, 1), f_2(t, 1))$.

Proof. Let us use statement 1 in Theorem 1 to prove the theorem. For this purpose, we first note that $\deg(\mathbf{p}) = \deg(\mathbf{q}) = 1$ and thus $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathcal{P})$. In addition, $\mathcal{P} \cdot \mathbf{p} = 0$. Then, it suffices to prove that $\mathcal{P} \cdot \mathbf{q} = 0$.

Since $F(\mathcal{P}) = 0$, we get that $\nabla(F(\mathcal{P}))\mathcal{P}'(t) = 0$ which implies that

$$\begin{aligned} 0 &= (f_2^{x_1}(-t f_1(t, 1), -f_1(t, 1)) + \\ & f_2(t, 1) f_1^{x_1}(-t f_1(t, 1), -f_1(t, 1)))(-f_1(t, 1) - \\ & t f_1'(t, 1)) + (f_2^{x_2}(-t f_1(t, 1), -f_1(t, 1)) + \\ & f_2(t, 1) f_1^{x_2}(-t f_1(t, 1), -f_1(t, 1)))(-f_1'(t, 1)) + \\ & f_1(-t f_1(t, 1), -f_1(t, 1)) f_2'(t, 1). \end{aligned}$$

Then $0 = (-f_1(t, 1) f_2^{x_1}(t, 1) + f_2(t, 1) f_1^{x_1}(t, 1))(-f_1(t, 1) - t f_1'(t, 1)) + (-f_1(t, 1) f_2^{x_2}(t, 1) + f_2(t, 1) f_1^{x_2}(t, 1))(-f_1'(t, 1)) - f_1(t, 1)^2 f_2'(t, 1)$ (note that $f_1^{x_1}, f_1^{x_2} \in \mathbb{K}$ since $\deg(f_1) = 1$). Hence, using that $f_j^{x_i}(t, 1) = f_j'(t, 1)$, $j = 1, 2$, and dividing the equality by $f_1'(t, 1)$, we get that $0 = f_2^{x_1}(t, 1) t f_1(t, 1) + f_2^{x_2}(t, 1) f_1(t, 1) - f_2(t, 1)(f_1(t, 1) + t f_1'(t, 1) + f_1^{x_2}(t, 1))$ which implies that $\mathcal{P} \cdot \mathbf{q} = 0$. \square

Remark 3.

- Using Lemmas 1 and 2 and Theorem 4, one gets a μ -basis for any proper parametrization $\mathcal{P}(t)$ of any conic C generally defined by $F(\bar{x}) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1 x_3 + a_5 x_2 x_3 + a_6 x_3^2$.
- Reasoning as in Corollary 2, we can easily compute the inverse of the input parametrization from the μ -basis obtained in Theorem 4.
- We observe that although the μ -basis is defined for parametric equations, we can also design μ -basis for implicit equations since the expressions obtained in Theorem 4 can be directly obtained from the implicit equation defining the conic.

Theorem 4 can be generalized to plane curves having a point of maximum multiplicity. More precisely, in the following we assume we have a plane irreducible curve C of degree d where $(0 : 0 : 1)$ is a point of multiplicity $d - 1$ (if $(0 : 0 : 1)$ is not the point of maximum multiplicity, we consider a linear change of coordinates). Thus, we get that C is implicitly defined as $F(\bar{x}) = f_d(x_1, x_2) + x_3 f_{d-1}(x_1, x_2)$. Under these conditions, we have that

$$\mathcal{P}(t) = (\wp_1(t) : \wp_2(t) : \wp_3(t)) = (-t f_{d-1}(t, 1), -f_{d-1}(t, 1), f_d(t, 1))$$

is a proper parametrization of C (see Section 4.6 in [23]).

Theorem 5. For a planar curve defined by $f_d(x_1, x_2) + x_3 f_{d-1}(x_1, x_2) = 0$, it holds that

$$\mathbf{p}(t) = (1, -t, 0),$$

$$\mathbf{q}(t) = (f_d^{x_1}(t, 1), f_d^{x_2}(t, 1), t f_{d-1}^{x_1}(t, 1) + f_{d-1}(t, 1) + f_{d-1}^{x_2}(t, 1))$$

form a μ -basis for $\mathcal{P}(t) = (-t f_{d-1}(t, 1), -f_{d-1}(t, 1), f_d(t, 1))$.

Proof. Let us use statement 1 in Theorem 1 to prove the theorem. For this purpose, we first note that $\deg(\mathbf{p}) = 1$ and $\deg(\mathbf{q}) = d - 1$ and thus $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathcal{P}) = d$. In addition, $\mathcal{P} \cdot \mathbf{p} = 0$. Then, we only have to prove that $\mathcal{P} \cdot \mathbf{q} = 0$.

Since $F(\mathcal{P}) = 0$, we get that $\nabla(F(\mathcal{P}))\mathcal{P}'(t) = 0$ which implies that

$$\begin{aligned} 0 &= (f_d^{x_1}(-t f_{d-1}(t, 1), -f_{d-1}(t, 1)) + \\ & f_d(t, 1) f_{d-1}^{x_1}(-t f_{d-1}(t, 1), -f_{d-1}(t, 1)))(-f_{d-1}(t, 1) - \\ & t f_{d-1}'(t, 1)) + (f_d^{x_2}(-t f_{d-1}(t, 1), -f_{d-1}(t, 1)) + \\ & f_d(t, 1) f_{d-1}^{x_2}(-t f_{d-1}(t, 1), -f_{d-1}(t, 1)))(-f_{d-1}'(t, 1)) + \\ & f_{d-1}(-t f_{d-1}(t, 1), -f_{d-1}(t, 1)) f_d'(t, 1). \end{aligned}$$

Then, $0 = (-f_{d-1}(t, 1) f_d^{x_1}(t, 1) + f_d(t, 1) f_{d-1}^{x_1}(t, 1))(-f_{d-1}(t, 1) - t f_{d-1}'(t, 1)) + (-f_{d-1}(t, 1) f_d^{x_2}(t, 1) + f_d(t, 1) f_{d-1}^{x_2}(t, 1))(-f_{d-1}'(t, 1)) - f_{d-1}(t, 1)^2 f_d'(t, 1)$. Hence, using that $f_j^{x_i}(t, 1) = f_j'(t, 1)$, $j = d - 1, d$, and dividing the equality by $f_{d-1}'(t, 1)$, we get that $0 = f_d^{x_1}(t, 1) t f_{d-1}(t, 1) + f_d^{x_2}(t, 1) f_{d-1}(t, 1) - f_d(t, 1)(f_{d-1}(t, 1) + t f_{d-1}'(t, 1) + f_{d-1}^{x_2}(t, 1))$ which implies that $\mathcal{P} \cdot \mathbf{q} = 0$. \square

Note that we have the parallel discussions in Remark 3 for the general case but with Theorem 5 instead. More precisely and what is more important, using Lemmas 1 and 2 and Theorem 5, one gets a μ -basis for any curve C having a point of maximum multiplicity. Note that we can compute the μ -basis from the implicit or from the parametric equations (see Section 5).

3.1. Numerical μ -bases for implicit monoid curves

For the numerical μ -basis, there is only one initial ideal was tested but no error estimation is given [24]. It is expected that a different choice for the approximation criteria will lead to a different specification for the approximate μ -basis. We here introduce a different criteria. Namely, we could say that given a curve C implicitly defined by $F(\bar{x})$ with an approximate point of maximum multiplicity (an ϵ -point of maximum multiplicity) and thus not necessarily rational, we could find a μ -basis, $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$, such that $\bar{\mathbf{p}}(t) \times \bar{\mathbf{q}}(t)$ defines a rational parametrization $\bar{\mathcal{P}}(t)$ describing a new rational curve \bar{C} (defined by a new polynomial $\bar{F}(\bar{x})$) such that \bar{C} and C are "closed" enough in the sense given in [17]. Summarizing, we could compute a μ -basis, $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$, for an input curve, C , not necessarily rational.

To start with, we first summarize some notions and results presented in [17]. To be more precise, we first introduce the notion of ϵ -singularity. We assume that $\bar{P} = (\bar{a}, \bar{b}, 1)$ although we reason similarly for points of the form $\bar{P} = (\bar{a}, 1, \bar{b})$ or $\bar{P} = (1, \bar{a}, \bar{b})$. In addition, we use $\|\cdot\|$ that denotes the ∞ -norm. More precisely, as the implicit equation defines univocally a plane curve up to a non-zero constant, we need to normalize it by considering $\|F(x_1, x_2, 1)\|$, were $F(\bar{x})$ is the defined polynomial of the curve.

Definition 2. We say that $\bar{P} = (\bar{a}, \bar{b}, 1)$ is an ϵ -singularity of multiplicity r of an algebraic plane curve defined by a polynomial

$F(\bar{x})$ if it holds that $\frac{\|\partial^{i+j} F(\bar{P})\|}{\|F(x_1, x_2, 1)\|} \leq \epsilon$ for $0 \leq i + j \leq r - 1$,

and $\frac{\|\partial^{i_0} F(\bar{P})\|}{\|F(x_1, x_2, 1)\|} > \epsilon$ for some $i_0, j_0 \in \mathbb{N}$ with $i_0 + j_0 = r$.

The following theorem shows that the implicit equation of the rational curve approximating an input not necessarily rational curve with an ϵ -singularity of maximum multiplicity can be obtained by Taylor expansions at the ϵ -singularity (see results in [17]). In fact, the theorem includes the result for conics as a particular case.

Theorem 6. Let $F(\bar{x})$ be the implicit equation of a curve C of degree d with an ϵ -singularity of maximum multiplicity at \bar{P} . Let \bar{C} be the curve defined by the polynomial $\bar{F}(\bar{x}) = F(\bar{x}) - T_{\bar{P}}(\bar{x})$, where $T_{\bar{P}}(\bar{x})$ is the Taylor expansion up to order $d - 1$ of $F(\bar{x})$ at \bar{P} . Then, it holds that \bar{C} is a rational curve having a singularity at \bar{P} of maximum multiplicity. Furthermore, it holds that

1. For almost all point $\bar{Q} \in \bar{C}$ there exists a point $Q \in C$ (and reciprocally) such that $\|\bar{Q} - Q\|_2 \leq \sqrt{2}\epsilon^{\frac{1}{2d}} \exp(2)$.
2. C is contained in the offset region of \bar{C} (and reciprocally) at distance $2\sqrt{2}\epsilon^{\frac{1}{2d}} \exp(2)$.

Definition 3. Let $F(\bar{x})$ be the implicit equation of a curve C of degree d . We say that two polynomial vectors $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ form an ϵ - μ -basis of the curve C , if $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ form a μ -basis of a rational curve \bar{C} that is contained in the offset region of C (and reciprocally) at distance ϵ .

From Theorems 5 and 6 and using Definition 3, we get the following theorem.

Theorem 7. Let $F(\bar{x})$ be the implicit equation of a curve C of degree d . Let $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ be the μ -basis computed in Theorem 5 for the curve \bar{C} constructed in Theorem 6. It holds that $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ form a $2\sqrt{2}\epsilon^{\frac{1}{2d}} \exp(2)$ - μ -basis of the curve C .

In the following example, we illustrate the above results with a curve C of degree $d = 4$ that has an ϵ -singularity of multiplicity $d - 1 = 3$.

Example 2. Let

$$F(\bar{x}) = 4445x_1^3 + 321x_2^4 - 2234x_1x_2^2 + 0.0005x_2 - 0.001x_1x_2 - 0.0002x_1^2 - 0.001$$

be the implicit equation of a curve C of degree $d = 4$ (see Figure 1). Observe that C is not a rational curve but it has an ϵ -singularity of maximum multiplicity at the origin $\bar{P} = (0 : 0 : 1)$, with $\epsilon = 2 \cdot 10^{-7}$. Let \bar{C} be the curve defined by the polynomial $\bar{F}(\bar{x}) = F(\bar{x}) - T_{\bar{P}}(\bar{x}) = 4445x_1^3 + 321x_2^4 - 2234x_1x_2^2$, where $T_{\bar{P}}(\bar{x})$ is the Taylor expansion up to order 3 of $F(\bar{x})$ at \bar{P} . Then, it holds that \bar{C} is a rational curve having a singularity at \bar{P} of maximum multiplicity.

Let $\bar{\mathbf{p}}(t) = (1, -t, 0)$ and $\bar{\mathbf{q}}(t) = (0, 1284, 17780t^3 - 8936t)$ be the μ -basis computed in Theorem 5 for the curve \bar{C} . It holds that $\bar{\mathbf{p}}(t)$ and $\bar{\mathbf{q}}(t)$ form an 3.08 - μ -basis of the curve C (see Theorem 7).

By the statement 1 in Theorem 2, we conclude that $\bar{\mathbf{p}}(t) \times \bar{\mathbf{q}}(t) = (-4445t^4 + 2234t^2 : -4445t^3 + 2234t : 321)$ is a parametrization of $\bar{F}(\bar{x})$ and so an approximate parametrization of $F(\bar{x})$.

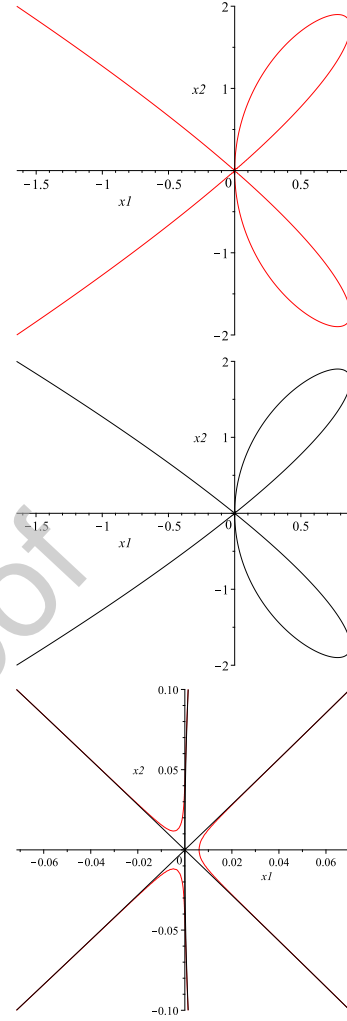


Fig. 1. Input curve (red color), output curve (black color) and both curves around the origin

Remark 4. We observe that the space curve could be included in this section. More precisely, for the univariate case, since the syzygy is always free, the μ -basis exists and we can generalize these results into the cases of dimension greater than two.

4. Minimal μ -bases of implicit monoid surfaces

In this section, we move to compute the minimal μ -basis either from the parametric equation or the implicit equation of a monoid surface. We first show how to compute μ -basis from a given proper parametrization of a quadric and in general of a monoid surface. From the expression of the μ -basis, a μ -basis can be obtained directly from the implicit equation. Using [18], we show how to compute "approximate μ -basis" for an input surface not necessarily rational and having an ϵ -point of maximum multiplicity.

To start with, we need to introduce some preliminaries of μ -basis for surfaces. Let

$$\mathcal{P}(\bar{t}) = (\varphi_1(\bar{t}) : \varphi_2(\bar{t}) : \varphi_3(\bar{t}) : \varphi_4(\bar{t})), \bar{t} = (t_1, t_2) \quad (2)$$

where $\gcd(\wp_1, \wp_2, \wp_3, \wp_4) = 1$, be a rational parametrization of a projective surface \mathcal{V} with the ground field \mathbb{K} (an algebraically closed field). A moving plane is a family of planes with parameter pair $\bar{t} = (t_1, t_2)$, defined by

$$L(\bar{t}) := L(x_1, x_2, x_3; \bar{t}) := A_1(\bar{t})x_1 + A_2(\bar{t})x_2 + A_3(\bar{t})x_3 + A_4(\bar{t})x_4 = 0,$$

or, in vector form $\mathbf{L}(\bar{t}) = (A_1(\bar{t}), A_2(\bar{t}), A_3(\bar{t}), A_4(\bar{t})) \in \mathbb{K}[\bar{t}]^4$. A moving plane $\mathbf{L}(\bar{t})$ is said to follow the rational surface $\mathcal{P}(\bar{t})$ if $\mathbf{L}(\bar{t}) \cdot \mathcal{P}(\bar{t}) = 0$. Thus the set $M_{\mathcal{P}} := \{\mathbf{L}(\bar{t}) \mid \mathbf{L}(\bar{t}) \cdot \mathcal{P}(\bar{t}) = 0\}$ is exactly the syzygy module $\text{syz}(\wp_1, \wp_2, \wp_3, \wp_4)$ and is a free module of rank 3 (see [5]).

Definition 4. Let $\mathbf{p}(\bar{t}), \mathbf{q}(\bar{t}), \mathbf{r}(\bar{t}) \in M_{\mathcal{P}}$ be three moving planes following the surface defined by $\mathcal{P}(\bar{t})$ such that $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\mathcal{P}(\bar{t})$ for some nonzero constant k , where $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ is the outer product of \mathbf{p}, \mathbf{q} and \mathbf{r} . Then $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is a μ -basis of the rational surface $\mathcal{P}(\bar{t})$.

If, in addition, among all the triples of \mathbf{p}, \mathbf{q} and \mathbf{r} satisfying that $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\mathcal{P}(\bar{t})$, the total degree $\deg(\mathbf{p}) + \deg(\mathbf{q}) + \deg(\mathbf{r})$ is smallest, then $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are called to form a minimal μ -basis.

The existence of μ -basis was proved in [5] and an algorithm was developed to compute a weak μ -basis in [8]. The μ -basis of improper parametrization is discussed in [21]. However, it is an unsolved problem to compute a minimal μ -basis for a general rational surface.

Some properties stated for curves also hold for surfaces. In particular, Lemmas 1 and 2.

Remark 5. Lemma 1 can be similarly proved for a given rational parametrization of a surface $Q(\bar{t}) = (\wp_1(\bar{t}), \dots, \wp_4(\bar{t}))$, $\bar{t} = (t_1, t_2)$ with $n \geq 2$. More precisely, if $\{\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \tilde{\mathbf{p}}_3\}$ is a μ -basis for a parametrization $Q(\bar{t})$ and $R(\bar{t}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$, it holds that $\mathbf{p}_i(\bar{t}) = \tilde{\mathbf{p}}_i(R(\bar{t}))$, $i = 1, 2, 3$, is a μ -basis for the reparametrization $\mathcal{P}(\bar{t}) = Q(R(\bar{t}))$ (see Definitions 4).

Remark 6. We easily get that Lemma 2 can be similarly proved for a parametrization of a surface $\mathcal{P}(\bar{t}) = (\wp_1(\bar{t}), \dots, \wp_4(\bar{t}))$, $\bar{t} = (t_1, t_2)$, and $T : \mathbb{K}^4 \rightarrow \mathbb{K}^4$, $T(\bar{x}) = (a_{11}x_1 + \dots + a_{14}x_4, \dots, a_{41}x_1 + \dots + a_{44}x_4)$, (with $\bar{x} = (x_1, \dots, x_4)$) invertible (see Definitions 4).

In the following, we first show that the minimal μ -basis of a quadric surface are linear in the variables and then we present a very simple algorithm to compute the minimal μ -basis. The conversion between the parametric form and the implicit form of a quadric is thus derived.

Similar to the curves, we consider \mathcal{V} be a projective irreducible quadric with ground field \mathbb{K} (an algebraically closed field), implicitly defined by an homogeneous polynomial, $F(x_1, x_2, x_3, x_4)$, of degree 2 (see [18]). There has a linear change of coordinates over \mathbb{K} transforming the quadric \mathcal{V} onto a quadric of the form $F(\bar{x}) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - a_4x_4^2$, where $\bar{x} = (x_1, x_2, x_3, x_4)$, $a_i \in \mathbb{K}$ and w.l.o.g. $a_1 \cdot a_2 \neq 0$. Let us denote this new quadric also as \mathcal{V} . There exists a rational proper parametrization of the form (2) such that $\deg(\mathcal{P}) = 2$ (see e.g. [18]). Notice that there may be other proper parametrizations of higher degree (the situation is different to the case of

curves). Under these conditions, we have the following theorem.

Theorem 8. It holds that

$$\mathbf{p}(\bar{t}) = (a_1\wp_1^1(\bar{t}), a_2\wp_2^1(\bar{t}), a_3\wp_3^1(\bar{t}), -a_4\wp_4^1(\bar{t})),$$

$$\mathbf{q}(\bar{t}) = (a_1\wp_1^2(\bar{t}), a_2\wp_2^2(\bar{t}), a_3\wp_3^2(\bar{t}), -a_4\wp_4^2(\bar{t})),$$

$$\mathbf{r}(\bar{t}) = (a_1(2\wp_1(\bar{t}) - t_1\wp_1^1(\bar{t}) - t_2\wp_1^2(\bar{t})), a_2(2\wp_2(\bar{t}) - t_1\wp_2^1(\bar{t}) - t_2\wp_2^2(\bar{t})), a_3(2\wp_3(\bar{t}) - t_1\wp_3^1(\bar{t}) - t_2\wp_3^2(\bar{t})), -a_4(2\wp_4(\bar{t}) - t_1\wp_4^1(\bar{t}) - t_2\wp_4^2(\bar{t})))$$

form a minimal μ -basis for any quadric parametrization

$$\mathcal{P}(\bar{t}) = (\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t})).$$

Proof. Since $F(\mathcal{P}) = 0$ and $\nabla(F(\mathcal{P}))\mathcal{P}^i(\bar{t}) = 0$, $i = 1, 2$, we get that $(a_1\wp_1)\wp_1 + (a_2\wp_2)\wp_2 + (a_3\wp_3)\wp_3 + (-a_4\wp_4)\wp_4 = (a_1\wp_1^1)\wp_1 + (a_2\wp_2^1)\wp_2 + (a_3\wp_3^1)\wp_3 + (-a_4\wp_4^1)\wp_4 = 0$ and $(a_1\wp_1^2)\wp_1 + (a_2\wp_2^2)\wp_2 + (a_3\wp_3^2)\wp_3 + (-a_4\wp_4^2)\wp_4 = 0$. From these three equalities we get that $a_1(2\wp_1(\bar{t}) - t_1\wp_1^1(\bar{t}) - t_2\wp_1^2(\bar{t}))\wp_1 + a_2(2\wp_2(\bar{t}) - t_1\wp_2^1(\bar{t}) - t_2\wp_2^2(\bar{t}))\wp_2 + a_3(2\wp_3(\bar{t}) - t_1\wp_3^1(\bar{t}) - t_2\wp_3^2(\bar{t}))\wp_3 - a_4(2\wp_4(\bar{t}) - t_1\wp_4^1(\bar{t}) - t_2\wp_4^2(\bar{t}))\wp_4 = 0$. Therefore, $\mathcal{P} \cdot \mathbf{p} = \mathcal{P} \cdot \mathbf{q} = \mathcal{P} \cdot \mathbf{r} = 0$.

In addition, since $\deg(\wp_i) \leq 2$, $i = 1, 2, 3$ and $\deg(\wp_j) = 2$ for some $j = 1, 2, 3$, we have that $\deg(2\wp_i - t_1\wp_i^1 - t_2\wp_i^2) \leq 1$ and $\deg(2\wp_j - t_1\wp_j^1 - t_2\wp_j^2) = 1$ for some $j = 1, 2, 3$. Thus, $\deg(\mathbf{p}) = \deg(\mathbf{q}) = \deg(\mathbf{r}) = 1$ and thus $\deg(\mathbf{p}) + \deg(\mathbf{q}) + \deg(\mathbf{r})$ is smallest. Therefore, by Definition 4, we get that $\mathbf{p}, \mathbf{q}, \mathbf{r}$ form a minimal μ -basis. \square

It is easy to prove that

$\mathcal{P}(\bar{t}) = (\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t})) = (2t_1\sqrt{a_3a_4} : 2t_2\sqrt{a_3a_4} : (t_1^2a_1 + t_2^2a_2 - a_3)\sqrt{a_4/a_3} : t_1^2a_1 + t_2^2a_2 + a_3)$ is a rational proper parametrization of \mathcal{V} . Remind that any proper parametrization of \mathcal{V} is expressed as $\mathcal{P}(R(\bar{t}))$, where $R(\bar{t}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ is birational. In this case, we get the following corollary.

Corollary 3. It holds that

$$\mathbf{p}(\bar{t}) = (a_1\sqrt{a_3a_4}, 0, t_1a_3\sqrt{a_4/a_3} : -t_1a_4),$$

$$\mathbf{q}(\bar{t}) = (0, a_2\sqrt{a_3a_4}, t_2a_3\sqrt{a_4/a_3}, -t_2a_4),$$

$$\mathbf{r}(\bar{t}) = (a_1t_1\sqrt{a_3a_4}, a_2t_2\sqrt{a_3a_4}, -a_3\sqrt{a_3a_4}, -a_3a_4)$$

form a minimal μ -basis for $\mathcal{P}(\bar{t}) = (2t_1\sqrt{a_3a_4} : 2t_2\sqrt{a_3a_4} : (t_1^2a_1 + t_2^2a_2 - a_3)\sqrt{a_4/a_3} : t_1^2a_1 + t_2^2a_2 + a_3)$.

Since $\bar{x} \cdot \mathbf{p} = \bar{x} \cdot \mathbf{q} = \bar{x} \cdot \mathbf{r} = 0$, where $\bar{x} \in \mathcal{V}$ and $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is the μ -basis of Corollary 3, we can also compute an inverse of the input proper parametrization. More precisely, we consider the polynomials $G_1(\bar{t}, \bar{x}) = \bar{x} \cdot \mathbf{p}$, $G_2(\bar{t}, \bar{x}) = \bar{x} \cdot \mathbf{q}$, $G_3(\bar{t}, \bar{x}) = \bar{x} \cdot \mathbf{r}$, and by applying elimination techniques, for instance [6, 16], we obtain an inverse of \mathcal{P} by solving the equations $G_i = 0$, $i = 1, 2, 3$ in the variables (t_1, t_2) (we remind that the inverse is unique modulo the implicit equation defining the algebraic surface). In this particular case, we have the following corollary.

Corollary 4. It holds that $\left(\frac{-x_1 \sqrt{a_3 a_4}}{a_1(x_3 \sqrt{a_4/a_3+x_4})}, \frac{-x_2 \sqrt{a_3 a_4}}{a_2(x_3 \sqrt{a_4/a_3+x_4})}\right)$ is an inverse of $\mathcal{P}(\bar{t})$.

Remark 7. Using Remarks 5 and 6 (see Lemmas 1 and 2) and Corollary 3, one gets a μ -basis for any proper parametrization $\mathcal{P}(\bar{t})$ of any quadric \mathcal{V} implicitly defined by some irreducible polynomial $F(\bar{x})$ of degree 2.

Example 3. We consider the quadric defined by the implicit homogeneous polynomial $3/2x_1^2 - x_1x_2 + 3/2x_2^2 + x_3^2 - 3x_4^2$. By well-known techniques in linear algebra, we can find a linear change of coordinates transformation $T(\bar{x}) = (1/2x_1 \sqrt{2} - 1/2x_3 \sqrt{2}, 1/2x_1 \sqrt{2} + 1/2x_3 \sqrt{2}, x_2, x_4)$, that transforms the input quadric to the quadric \mathcal{V} defined by the polynomial $F(\bar{x}) = x_1^2 + x_2^2 + 2x_3^2 - 3x_4^2$. From Corollary 3, we get that

$$\begin{aligned} \mathbf{p}(\bar{t}) &= (\sqrt{6}, 0, t_1 \sqrt{6}, -3t_1), \\ \mathbf{q}(\bar{t}) &= (0, \sqrt{6}, t_2 \sqrt{6}, -3t_2), \\ \mathbf{r}(\bar{t}) &= (t_1 \sqrt{6}, t_2 \sqrt{6}, -2\sqrt{6}, -6) \end{aligned}$$

form a μ -basis for the parametrization $\mathcal{P}(\bar{t}) = (2t_1 \sqrt{6} : 2t_2 \sqrt{6} : \sqrt{6}/2(t_1^2 + t_2^2 - 2) : t_1^2 + t_2^2 + 2)$ of \mathcal{V} . Using Remark 6 with the linear transformation $S(\bar{x}) = (1/2x_1 \sqrt{2} + 1/2x_2 \sqrt{2}, x_3, -1/2x_1 \sqrt{2} + 1/2x_2 \sqrt{2}, x_4)$ that can be easily derived from $S(\bar{x})$, one gets the μ -basis

$$\begin{aligned} &(-\sqrt{3}(-1+t_1), \sqrt{3}(1+t_1), 0, -3t_1), \\ &(-t_2 \sqrt{3}, t_2 \sqrt{3}, \sqrt{6}, -3t_2), \\ &(\sqrt{3}(2+t_1), \sqrt{3}(-2+t_1), t_2 \sqrt{6}, -6) \end{aligned}$$

for the input quadric. In addition (from Definition 4), for the input quadric we get the parametrization

$$(-\sqrt{3}/2(-4t_1+t_1^2+t_2^2-2) : \sqrt{3}/2(4t_1+t_1^2+t_2^2-2) : 2t_2 \sqrt{6}, t_1^2+t_2^2+2).$$

On the other hand, we can also find a linear change of coordinates over \mathbb{K} transforming the irreducible quadric \mathcal{V} to a quadric of the form $F(\bar{x}) = f_2(x_1, x_2, x_3) + x_4 f_1(x_1, x_2, x_3)$, where $f_i(x_1, x_2, x_3)$ are homogeneous polynomials of degree i . Note that we have assumed that \mathcal{V} passes through the origin. Under these conditions, we have that

$$\begin{aligned} \mathcal{P}(\bar{t}) &= (\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t})) \\ &= (-t_1 f_1(\bar{t}, 1) : -t_2 f_1(\bar{t}, 1) : -f_1(\bar{t}, 1) : f_2(\bar{t}, 1)) \end{aligned}$$

is a proper parametrization of \mathcal{V} (see e.g [18]).

Theorem 9. For a quadric defined by $f_2(x_1, x_2, x_3) + x_4 f_1(x_1, x_2, x_3) = 0$, it holds that

$$\mathbf{p}(\bar{t}) = (1, 0, -t_1, 0), \quad \mathbf{q}(\bar{t}) = (0, 1, -t_2, 0),$$

$$\mathbf{r}(\bar{t}) = (f_2^{x_1}(\bar{t}, 1), f_2^{x_2}(\bar{t}, 1), f_2^{x_3}(\bar{t}, 1), t_1 f_1^{x_1}(\bar{t}, 1) + t_2 f_1^{x_2}(\bar{t}, 1) + f_1(\bar{t}, 1) + f_1^{x_3}(\bar{t}, 1))$$

form a minimal μ -basis for the parametrization

$$\mathcal{P}(\bar{t}) = (-t_1 f_1(\bar{t}, 1), -t_2 f_1(\bar{t}, 1), -f_1(\bar{t}, 1), f_2(\bar{t}, 1)).$$

Proof. We first note that $\deg(\mathbf{p}) = \deg(\mathbf{q}) = \deg(\mathbf{r}) = 1$ and $\mathcal{P} \cdot \mathbf{p} = \mathcal{P} \cdot \mathbf{r} = 0$. We only need to prove that $\mathcal{P} \cdot \mathbf{r} = 0$ (see Definition 4).

Since $F(\mathcal{P}) = 0$, we derive w.r.t t_1 (similarly if one derives w.r.t t_2), and get that $0 = \nabla(F(\mathcal{P}))\mathcal{P}^{t_1}(\bar{t}) = (-f_1(\bar{t}, 1)f_2^{x_1}(\bar{t}, 1) + f_2(\bar{t}, 1)f_1^{x_1}(\bar{t}, 1))(-f_1(\bar{t}, 1) - t_1 f_1'(\bar{t}, 1)) + (-f_1(\bar{t}, 1)f_2^{x_2}(\bar{t}, 1) + f_2(\bar{t}, 1)f_1^{x_2}(\bar{t}, 1))(-t_2 f_1'(\bar{t}, 1)) + (-f_1(\bar{t}, 1)f_2^{x_3}(\bar{t}, 1) + f_2(\bar{t}, 1)f_1^{x_3}(\bar{t}, 1))(-f_1'(\bar{t}, 1)) - f_1(\bar{t}, 1)^2 f_2'(\bar{t}, 1)$ where $f_j'(\bar{t}, 1)$ is the derivative of the polynomial f_j w.r.t.

t_1 (note that $f_1^{x_1}, f_1^{x_2}, f_1^{x_3} \in \mathbb{K}$). Then, using that $f_j^{x_i}(\bar{t}, 1) = f_j'(\bar{t}, 1)$, $j = 1, 2$, and dividing the equality by $f_1'(\bar{t}, 1)$, we get that

$$0 = f_2^{x_1}(\bar{t}, 1)t_1 f_1(\bar{t}, 1) + f_2^{x_2}(\bar{t}, 1)t_2 f_1(\bar{t}, 1) + f_2^{x_3}(\bar{t}, 1)f_1(\bar{t}, 1) - f_2(\bar{t}, 1)(f_1(\bar{t}, 1) + t_1 f_1'(\bar{t}, 1) + t_2 f_1^{x_2}(\bar{t}, 1) + f_1^{x_3}(\bar{t}, 1)),$$

which implies that $\mathcal{P} \cdot \mathbf{r} = 0$. \square

Remark 8.

- Using Remarks 5 and 6 (see Lemmas 1 and 2) and Theorem 9, one gets a μ -basis for any proper parametrization $\mathcal{P}(\bar{t})$ of any quadric \mathcal{C} defined by some irreducible polynomial $F(\bar{x})$.
- Reasoning as in Corollary 4, we can easily compute the inverse of the input parametrization from the μ -based obtained in Theorem 9.
- We can also design μ -basis for implicit equations since the expressions obtained in Theorem 9 can be directly obtained from the implicit equation defining the quadric.

Theorem 9 can be generalized to surfaces having a point of maximum multiplicity. More precisely, in the following we assume we have a surface \mathcal{V} of degree d where $(0 : 0 : 0 : 1)$ is a point of multiplicity $d - 1$ (if $(0 : 0 : 0 : 1)$ is not the point of maximum multiplicity, we consider a linear change of coordinates). Thus, we get that \mathcal{V} is implicitly defined as $F(\bar{x}) = f_d(x_1, x_2, x_3) + x_4 f_{d-1}(x_1, x_2, x_3)$. Under these conditions, we have that

$$\begin{aligned} \mathcal{P}(\bar{t}) &= (\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t})) \\ &= (-t_1 f_{d-1}(\bar{t}, 1) : -t_2 f_{d-1}(\bar{t}, 1) : -f_{d-1}(\bar{t}, 1) : f_d(\bar{t}, 1)) \end{aligned}$$

is a proper parametrization of \mathcal{V} (see [18]).

Theorem 10. For a surface defined by $f_d(x_1, x_2, x_3) + x_4 f_{d-1}(x_1, x_2, x_3) = 0$, it holds that

$$\mathbf{p}(\bar{t}) = (1, 0, -t_1, 0), \quad \mathbf{q}(\bar{t}) = (0, 1, -t_2, 0),$$

$$\mathbf{r}(\bar{t}) = (f_d^{x_1}(\bar{t}, 1), f_d^{x_2}(\bar{t}, 1), f_d^{x_3}(\bar{t}, 1), t_1 f_{d-1}^{x_1}(\bar{t}, 1) + t_2 f_{d-1}^{x_2}(\bar{t}, 1) + f_{d-1}(\bar{t}, 1) + f_{d-1}^{x_3}(\bar{t}, 1))$$

is a minimal μ -basis for the parametrization

$$\mathcal{P}(\bar{t}) = (-t_1 f_{d-1}(\bar{t}, 1) : -t_2 f_{d-1}(\bar{t}, 1) : -f_{d-1}(\bar{t}, 1) : f_d(\bar{t}, 1)).$$

Proof. We first note that $\deg(\mathbf{p}) = \deg(\mathbf{q}) = 1$ and $\deg(\mathbf{r}) = d - 1$. In addition, $\mathcal{P} \cdot \mathbf{p} = \mathcal{P} \cdot \mathbf{q} = 0$.

To prove that $\mathcal{P} \cdot \mathbf{r} = 0$, since $F(\mathcal{P}) = 0$, we derive w.r.t t_1 (similarly if one derives w.r.t t_2), and get

that $0 = \nabla(F(\mathcal{P}))\mathcal{P}^1(\bar{t}) = (-f_{d-1}(\bar{t}, 1)f_d^{x_1}(\bar{t}, 1) + f_d(\bar{t}, 1)f_{d-1}^{x_1}(\bar{t}, 1))(-f_{d-1}(\bar{t}, 1) - t_1f_{d-1}'(\bar{t}, 1)) + (-f_{d-1}(\bar{t}, 1)f_d^{x_2}(\bar{t}, 1) + f_d(\bar{t}, 1)f_{d-1}^{x_2}(\bar{t}, 1))(-t_2f_{d-1}'(\bar{t}, 1)) + (-f_{d-1}(\bar{t}, 1)f_d^{x_3}(\bar{t}, 1) + f_d(\bar{t}, 1)f_{d-1}^{x_3}(\bar{t}, 1))(-f_{d-1}'(\bar{t}, 1)) - f_{d-1}(\bar{t}, 1)^2f_d'(\bar{t}, 1)$ where $f_j^{x_i}(\bar{t}, 1)$ is the derivative of the polynomial f_j w.r.t. t_i . Then, by $f_j^{x_i}(\bar{t}, 1) = f_j'(\bar{t}, 1)$, $j = d - 1, d$, and dividing the equality by $f_{d-1}'(\bar{t}, 1)$, we get that $0 = f_d^{x_1}(\bar{t}, 1)t_1f_{d-1}(\bar{t}, 1) + f_d^{x_2}(\bar{t}, 1)t_2f_{d-1}(\bar{t}, 1) + f_d^{x_3}(\bar{t}, 1)f_{d-1}(\bar{t}, 1) - f_d(\bar{t}, 1)(f_{d-1}(\bar{t}, 1) + t_1f_{d-1}'(\bar{t}, 1) + t_2f_{d-1}^{x_2}(\bar{t}, 1) + f_{d-1}^{x_3}(\bar{t}, 1))$, which implies that $\mathcal{P} \cdot \mathbf{r} = 0$. \square

Using Lemmas 1 and 2, Remarks 5 and 6 and Theorem 10, one gets a μ -basis for any general surface \mathcal{V} having a point of maximum multiplicity and computes the μ -basis from the implicit or from the parametric equations (see Section 5)

4.1. Numerical μ -bases of implicit monoid surfaces

For the numerical μ -basis, we generalize the results presented in Section 3 (Definitions 2 and 4, and Theorems 6 and 7) for the case of surfaces using the results in [18]. Namely, we could say that given a surface \mathcal{V} implicitly defined by $F(\bar{x})$ with an approximate point of maximum multiplicity (an ϵ -point of maximum multiplicity) and thus not necessarily rational, we could find μ -basis, $\bar{\mathbf{p}}(\bar{t})$, $\bar{\mathbf{q}}(\bar{t})$ and $\bar{\mathbf{r}}(\bar{t})$, such that $[\bar{\mathbf{p}}(\bar{t}), \bar{\mathbf{q}}(\bar{t}), \bar{\mathbf{r}}(\bar{t})]$ defines a rational parametrization $\bar{\mathcal{P}}(\bar{t})$ of a new rational surface $\bar{\mathcal{V}}$ (defined by a new polynomial $\bar{F}(\bar{x})$) such that $\bar{\mathcal{V}}$ and \mathcal{V} are "closed" enough in the sense given in [18]. Summarizing, we could compute μ -basis, $\bar{\mathbf{p}}(\bar{t})$, $\bar{\mathbf{q}}(\bar{t})$ and $\bar{\mathbf{r}}(\bar{t})$, for an input surface, \mathcal{V} , not necessarily rational.

To start with, we first summarize some notions and results presented in [18]. To be more precise, we first generalize the notion of ϵ -singularity introduced in 2 for the case of surfaces. We assume that $\bar{P} = (\bar{a}, \bar{b}, \bar{c}, 1)$ although we reason similarly if we dehomogenize in a different chart. Similarly as in the case of plane curves, we use $\|\cdot\|$ that denotes the ∞ -norm. More precisely, as the implicit equation defines univocally a surface up to a non-zero constant, we need to normalize it by considering $\|F(x_1, x_2, x_3, 1)\|$, where $F(\bar{x})$ is the defined polynomial of the surface.

Definition 5. We say that $\bar{P} = (\bar{a}, \bar{b}, \bar{c}, 1)$ is an ϵ -singularity of multiplicity r of an algebraic surface defined by a polynomial

$F(\bar{x})$ if it holds that $\frac{\partial^{i+j+k} F(\bar{P})}{\partial x_1^i \partial x_2^j \partial x_3^k} \leq \epsilon$ for $0 \leq i+j+k \leq r-1$, and $\frac{\partial^{i_0+j_0+k_0} F(\bar{P})}{\partial x_1^{i_0} \partial x_2^{j_0} \partial x_3^{k_0}} > \epsilon$ for some $i_0, j_0, k_0 \in \mathbb{N}$ with $i_0 + j_0 + k_0 = r$.

The following theorem shows that the implicit equation of the rational surface approximating an input not necessarily rational surface with an ϵ -singularity of maximum multiplicity can be obtained by Taylor expansions at the ϵ -singularity (see results in [18]). In fact, the theorem includes the result for quadrics as a particular case and it is a generalization of Theorem 6.

Theorem 11. Let $F(\bar{x})$ be the implicit equation of a surface \mathcal{V} of degree d with an ϵ -singularity of maximum multiplicity at \bar{P} . Let $\bar{\mathcal{V}}$ be the surface defined by the polynomial $\bar{F}(\bar{x}) = F(\bar{x}) - T_{\bar{P}}(\bar{x})$, where $T_{\bar{P}}(\bar{x})$ is the Taylor expansion up to order $d-1$ of

$F(\bar{x})$ at \bar{P} . Then, it holds that $\bar{\mathcal{V}}$ is a rational surface having at \bar{P} a singularity of maximum multiplicity. Furthermore, it holds that

1. For almost all point $\bar{Q} \in \bar{\mathcal{V}}$ there exists a point $Q \in \mathcal{V}$ (and reciprocally) such that $\|\bar{Q} - Q\|_2 \leq \sqrt{3}\epsilon^{\frac{1}{2d}} \exp(3)$.
2. \mathcal{C} is contained in the offset region of $\bar{\mathcal{V}}$ (and reciprocally) at distance $3\sqrt{3}\epsilon^{\frac{1}{2d}} \exp(3)$.

Definition 6. Let $F(\bar{x})$ be the implicit equation of a surface \mathcal{V} of degree d . We say that three polynomial vectors $\bar{\mathbf{p}}(\bar{t})$, $\bar{\mathbf{q}}(\bar{t})$ and $\bar{\mathbf{r}}(\bar{t})$ form an ϵ - μ -basis of the surface $\bar{\mathcal{V}}$, if $\bar{\mathbf{p}}(\bar{t})$, $\bar{\mathbf{q}}(\bar{t})$ and $\bar{\mathbf{r}}(\bar{t})$ form a μ -basis of a rational surface $\bar{\mathcal{V}}$ that is contained in the offset region of $\bar{\mathcal{V}}$ (and reciprocally) at distance ϵ .

From Theorems 10 and 11 and using Definition 6, we get the following theorem which generalizes Theorem 7.

Theorem 12. Let $F(\bar{x})$ be the implicit equation of a surface \mathcal{V} of degree d . Let $\bar{\mathbf{p}}(\bar{t})$, $\bar{\mathbf{q}}(\bar{t})$ and $\bar{\mathbf{r}}(\bar{t})$ be the μ -basis computed in Theorem 10 for the surface $\bar{\mathcal{V}}$ constructed in Theorem 11. It holds that $\bar{\mathbf{p}}(\bar{t})$, $\bar{\mathbf{q}}(\bar{t})$ and $\bar{\mathbf{r}}(\bar{t})$ form an $3\sqrt{3}\epsilon^{\frac{1}{2d}} \exp(3)$ - μ -basis of the surface \mathcal{V} .

Example 4. Let

$F(\bar{x}) = 4587775x_1^3 + 24841x_2^4 - 243324x_1x_2^2 + 345896x_3^4 + 0.0005x_3 + 0.0005x_1 - 0.0001x_1x_3 - 0.0002x_3^2 - 0.0001$ be the implicit equation of a surface \mathcal{V} of degree $d = 4$ (see Figure 2). One may check that \mathcal{V} is not a rational surface but it has an ϵ -singularity of maximum multiplicity at the origin $\bar{P} = (0 : 0 : 0 : 1)$, with $\epsilon = 2 \cdot 10^{-11}$. Let $\bar{\mathcal{V}}$ be the surface defined by the polynomial $\bar{F}(\bar{x}) = F(\bar{x}) - T_{\bar{P}}(\bar{x}) = 4587775x_1^3 + 24841x_2^4 - 243324x_1x_2^2 + 345896x_3^4$, where $T_{\bar{P}}(\bar{x})$ is the Taylor expansion up to order 3 of $F(\bar{x})$ at \bar{P} . Then, it holds that $\bar{\mathcal{V}}$ is a rational surface having a singularity at \bar{P} of maximum multiplicity.

Let $\bar{\mathbf{p}}(t) = (1, 0, -t_1, 0)$, $\bar{\mathbf{q}}(t) = (0, 1, -t_2, 0)$ and $\bar{\mathbf{r}}(t) = (0, 99364t_2^3, 1383584, 18351100t_1^3 - 973296t_1t_2^2)$ be the μ -basis computed in Theorem 10 for the surface $\bar{\mathcal{V}}$. It holds that $\bar{\mathbf{p}}(t)$, $\bar{\mathbf{q}}(t)$ and $\bar{\mathbf{r}}(t)$ form an 4.851- μ -basis of the surface \mathcal{V} (see Theorem 12).

By Definition 4, we conclude that $[\bar{\mathbf{p}}, \bar{\mathbf{q}}, \bar{\mathbf{r}}] = (-4587775t_1^4 + 243324t_1^2t_2^2 : -4587775t_2t_1^3 + 243324t_1t_2^3 : -4587775t_1^3 + 243324t_1t_2^2 : 24841t_2^4 + 345896)$ is a parametrization of $\bar{F}(\bar{x})$ and so an approximate parametrization of $F(\bar{x})$.

5. Algorithms and examples

In this section, we present two general algorithms and we illustrate them with examples. More precisely, the first algorithm compute a μ -basis for a plane curve C (defined implicitly or parametrically) having a point of maximum multiplicity. Similarly, the first algorithm compute a μ -basis for a surface \mathcal{V} (defined implicitly or parametrically) having a point of maximum multiplicity. These algorithms are based on Lemmas 1 and 2, Remarks 5 and 6 and Theorems 5 and 10.



Fig. 2. Input surface (red color with 25% transparency), output surface (blue color with 25% transparency), both surfaces in same figure and enlarged surfaces near the origin

Algorithm 1 (Compute a μ -Basis for a Monoid Curve).

Input a plane algebraic monoid curve C .

Output a μ -basis of a parametrization $Q(t)$.

Steps

1. If C is defined implicitly:

1.1. Apply a linear change of coordinates $T(\bar{x})$ such that the new curve \mathcal{D} has in $(0 : 0 : 1)$ a point of maximum multiplicity. Thus \mathcal{D} is defined as $F(\bar{x}) = f_d(x_1, x_2) + x_3 f_{d-1}(x_1, x_2)$.

1.2. Compute

$$\mathbf{p}(t) = (1, -t, 0),$$

$$\mathbf{q}(t) = (f_d^{x_1}(t, 1), f_d^{x_2}(t, 1), t f_{d-1}^{x_1}(t, 1) + f_{d-1}(t, 1) + f_{d-1}^{x_2}(t, 1))$$

that form a μ -basis for the proper parametrization of \mathcal{D} defined as $\mathcal{P}(t) = (-t f_{d-1}(t, 1), -f_{d-1}(t, 1), f_d(t, 1))$.

1.3. Let $S(\bar{x})$ computed from $T(\bar{x})$ as in Lemma 2. Return $S(\mathbf{p}(t)), S(\mathbf{q}(t))$ form a μ -basis for the proper parametrization of C defined as $Q(t) := S(\mathcal{P}(t))$.

2. If C is defined parametrically by $Q(t)$:

2.1. Apply a linear change of coordinates $T(\bar{x})$ such that the new curve \mathcal{D} has in $(0 : 0 : 1)$ a point of maximum multiplicity. Thus \mathcal{D} is defined as

$$\mathcal{P}(t) = (-r_1 f_{d-1}(r_1, r_2), -f_{d-1}(r_1, r_2), f_d(r_1, r_2)),$$

where $R(t) := r_1/r_2 \in \mathbb{K}(t) \setminus \mathbb{K}$.

2.2. Compute

$$\mathbf{p}(t) = (1, -r_1, 0),$$

$$\mathbf{q}(t) = (f_d^{x_1}(r_1, r_2), f_d^{x_2}(r_1, r_2), r_1 f_{d-1}^{x_1}(r_1, r_2) + f_{d-1}(r_1, r_2) + f_{d-1}^{x_2}(r_1, r_2))$$

that form a μ -basis for $\mathcal{P}(t)$.

2.3. Let $S(\bar{x})$ computed from $T(\bar{x})$ as in Lemma 2. Return $S(\mathbf{p}(t)), S(\mathbf{q}(t))$ form a μ -basis for the input parametrization $Q(t)$.

Remark 9. • In order to determine if C has a point P with the multiplicity $d-1$. If C is defined implicitly by a polynomial of degree d , one check whether all partial derivatives of the defining polynomial of the curve, till order $d-2$, vanish at P . If C is defined parametrically, one may apply the results in [19].

• Observe that the parametrization obtained in Step 2.1 is not necessarily the used in Theorem 5 but with a change of variable $r(t) \in \mathbb{K}(t) \setminus \mathbb{K}$.

• The output parametrization in Step 1.3 can also be computed using Theorem 2.

Example 5. We consider the curve of degree $d = 3$ defined by the implicit homogeneous polynomial

$$x_1^3 - 2x_3x_1^2 + x_3^2x_1 + x_2^3 + x_1^2x_2 - 2x_2x_1x_3 + x_3^2x_2 - 3x_3x_2^2.$$

We have that $(1 : 0 : 1)$ is a point of multiplicity 2 of this curve. Thus, we consider the linear change of coordinates transformation $T(\bar{x}) = (x_1 - x_3, x_2, x_3)$ which transforms the given input curve onto the curve C defined by the polynomial $F(\bar{x}) = f_3(x_1, x_2) + x_3 f_2(x_1, x_3) = x_1^3 + x_2^3 + x_1^2x_2 + x_3(x_1^2 - 3x_2^2)$ (Step 1.1 in Algorithm 1). Observe that $(0 : 0 : 1)$ is a point of multiplicity 2 of C .

From Theorem 5 (Step 1.2 in Algorithm 1), we get that $\mathbf{p}(t) = (1, -t, 0)$ and $\mathbf{q}(t) = (3t^2 + 2t, t^2 + 3, 3t^2 - 9)$ is a μ -basis for the parametrization $\mathcal{P}(t) = (-t(t^2 - 3) : -t^2 + 3 : t^3 + t^2 + 1)$ of C . One can get $S(\bar{x}) = (x_1 - x_3, x_2, x_3)$ from $T(\bar{x})$ (see Lemma 2), and we obtain the μ -basis $(1, -t, -1), (3t^2 + 2t, t^2 + 3, -2t - 9)$ for the input curve. In addition (from Theorem 2), for the input curve we get the parametrization $(3t + t^2 + 1 : -t^2 + 3 : t^3 + t^2 + 1)$ (Step 1.3 in Algorithm 1).

Algorithm 2 (Compute a μ -Basis for a Monoid Surface).

Input an algebraic monoid surface \mathcal{V} .

Output a μ -basis of a parametrization $Q(\bar{t})$.

Steps

1 1. If \mathcal{V} is defined implicitly:

2 1.1. Apply a linear change of coordinates $T(\bar{x})$ such that
3 the new surface \mathcal{W} has in $(0 : 0 : 0 : 1)$ a point of
4 maximum multiplicity. Thus \mathcal{W} is defined as $F(\bar{x}) =$
5 $f_d(x_1, x_2, x_3) + x_4 f_{d-1}(x_1, x_2, x_3)$.

6 1.2. Compute

$$\mathbf{p}(\bar{t}) = (1, 0, -t_1, 0), \quad \mathbf{q}(\bar{t}) = (0, 1, -t_2, 0),$$

$$\mathbf{r}(\bar{t}) = (f_d^{x_1}(\bar{t}, 1), f_d^{x_2}(\bar{t}, 1), f_d^{x_3}(\bar{t}, 1), t_1 f_{d-1}^{x_1}(\bar{t}, 1) +$$

$$t_2 f_{d-1}^{x_2}(\bar{t}, 1) + f_{d-1}(\bar{t}, 1) + f_{d-1}^{x_3}(\bar{t}, 1))$$

7 that form a μ -basis for the proper parametrization of
8 \mathcal{W} defined as $\mathcal{P}(\bar{t}) = (-t_1 f_{d-1}(\bar{t}, 1) : -t_2 f_{d-1}(\bar{t}, 1) :$
9 $-f_{d-1}(\bar{t}, 1) : f_d(\bar{t}, 1))$.

10 1.3. Let $S(\bar{x})$ computed from $T(\bar{x})$ as in Remark 6. Re-
11 turn $S(\mathbf{p})(\bar{t}), S(\mathbf{q})(\bar{t}), S(\mathbf{r})(\bar{t})$ form a μ -basis for the
12 proper parametrization of \mathcal{V} defined as $\mathcal{Q}(\bar{t}) :=$
13 $S(\mathcal{P})(\bar{t})$.

14 2. If \mathcal{V} is defined parametrically by $\mathcal{Q}(\bar{t})$:

15 2.1. Apply a linear change of coordinates $T(\bar{x})$ such
16 that the new curve \mathcal{W} has in $(0 : 0 : 0 : 1)$ a
17 point of maximum multiplicity. Thus \mathcal{W} is defined
18 as $\mathcal{P}(\bar{t}) = (-r_1 f_{d-1}(r_1, r_2, r_3) : -r_2 f_{d-1}(r_1, r_2, r_3) :$
19 $-f_{d-1}(r_1, r_2, r_3) : f_d(r_1, r_2, r_3))$,
20 where $R(\bar{t}) := (r_1/r_3, r_2/r_3) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$.

21 2.2. Compute

$$\mathbf{p}(\bar{t}) = (1, 0, -r_1, 0), \quad \mathbf{q}(\bar{t}) = (0, 1, -r_2, 0),$$

$$\mathbf{r}(\bar{t}) = (f_d^{x_1}(r_1, r_2, r_3), f_d^{x_2}(r_1, r_2, r_3), f_d^{x_3}(r_1, r_2, r_3),$$

$$r_1 f_{d-1}^{x_1}(r_1, r_2, r_3) + r_2 f_{d-1}^{x_2}(r_1, r_2, r_3) + f_{d-1}(r_1, r_2, r_3) +$$

$$f_{d-1}^{x_3}(r_1, r_2, r_3))$$

22 that form a μ -basis for $\mathcal{P}(\bar{t})$.

23 2.3. Let $S(\bar{x})$ computed from $T(\bar{x})$ as in Remark 6. Re-
24 turn $S(\mathbf{p})(\bar{t}), S(\mathbf{q})(\bar{t}), S(\mathbf{r})(\bar{t})$ form a μ -basis for the
25 parametrization $\mathcal{Q}(\bar{t})$.

26 **Remark 10.** • Similarly to the curves, one can determine if
27 \mathcal{V} has a point P with the maximum multiplicity. If \mathcal{V} is
28 defined implicitly by a polynomial of degree d , one check
29 whether all partial derivatives of the defining polynomial
30 of the surface, till order $d - 2$, vanish at P . If \mathcal{V} is defined
31 parametrically, one may apply the results in [22].

32 • The parametrization obtained in Step 2.1 is not necessar-
33 ily the used in Theorem 10 but with a change of variable
34 $R(\bar{t}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$.

35 • The output parametrization in Step 1.3 can also be com-
36 puted using Definition 4.

37 **Example 6.** We consider the surface \mathcal{V} defined by the rational
38 parametrization

$$\mathcal{Q}(\bar{t}) = (3t_1 t_2^3 + t_1 + t_2^3 + 4 : -t_2 t_1^3 + 3t_2^4 + t_2 + t_1^4 + t_2^3 + 4 :$$

$$-t_1^3 + 3t_2^3 + 1 : t_1^4 + t_2^3 + 4).$$

39 We have that $(1 : 1 : 0 : 1)$ is a point of multiplicity 3 of this
40 surface which has degree $d = 4$ (see [22]). Thus, we consider
41 the linear change of coordinates transformation $T(\bar{x}) = (x_1 -$
42 $x_4, x_2 - x_4, x_3, x_4)$ which transforms the given input surface onto

43 the surface \mathcal{W} defined by the parametrization $\mathcal{P}(\bar{t}) = (-t_1(t_1^3 -$
44 $3t_2^3 - 1) : -t_2(t_1^3 - 3t_2^3 - 1) : -t_1^3 + 3t_2^3 + 1 : t_1^4 + t_2^3 + 4)$ (Step 2.1 in
45 Algorithm 2). Observe that $\mathcal{P}(\bar{t})$ has the standard proper form
46 and that $(0 : 0 : 0 : 1)$ is a point of multiplicity 3 of \mathcal{W} .

47 From Theorem 10 (Step 2.2 in Algorithm 2), we get that
48 $\mathbf{p}(t) = (1, 0, -t_1, 0)$, $\mathbf{q}(t) = (0, 1, -t_2, 0)$ and $\mathbf{r}(t) = (4t_1^3, 3t_2^3, t_2^3 +$
49 $16, 4t_1^3 - 12t_2^3 - 4)$ is a μ -basis for the parametrization $\mathcal{P}(\bar{t})$
50 of \mathcal{W} . One can get $S(\bar{x}) = (x_1, x_2, x_3, x_4 - x_1 - x_2)$ from
51 $T(\bar{x})$ (see Remark 6), and we obtain the μ -basis $(1, 0, -t_1, -1),$
52 $(0, 1, -t_2, -1)$ and $(4t_1^3, 3t_2^3, t_2^3 + 16, -3t_2^3 - 12t_2^3 - 4)$ for the input
53 parametrization $\mathcal{Q}(\bar{t})$ (Step 2.3 in Algorithm 2).

54 6. Conclusion

55 The μ -basis has shown some interesting and useful propo-
56 sitions. However, there are two aspects, finding μ -bases from
57 implicit equations and computing in approximate way, need
58 much more attentions. We derive the explicit μ -bases for im-
59 plicit monoid curves and surfaces including conics and quadrics
60 defined implicitly or by any rational parametrization (not neces-
61 sarily in standard proper form). The way is based on the simple
62 investigations and basic algebraic results, hence the computa-
63 tions are efficient. The technique introduces only linear trans-
64 formation and so the proposed methods can be naturally gen-
65 eralized to deal with the numerical situations. By combining
66 the current technique on ϵ -singularity analysis, we can find the
67 approximate μ -bases. More important, we propose the error es-
68 timation formula while only one paper gave tentative definitions
69 of approximate μ -bases before.

70 References

- 71 [1] Corless R.M., Gianni P.M., Trager B.M., Watt S.M. (1995). *The Singu-
72 lar Value Decomposition for Polynomial Systems*. Proc. ISSAC 1995, pp.
73 195-207 ACM Press.
- 74 [2] Cortadellas Benítez T., D'Andrea C. (2010). *Minimal Generators of the
75 Defining Ideal of the Rees Algebra Associated to Monoid Parameteriza-
76 tions*. Computer Aided Geometric Design, 27 (6), pp. 461-473.
- 77 [3] Chen F., Wang W. (2003). *The μ -Basis of a Planar Rational Curve—
78 Properties and Computation*. Graphical Models, 64 (14), pp. 368-381.
- 79 [4] Chen F., Shen L., Deng J. (2007). *Implicitization and parametrization of
80 quadratic and cubic surfaces by μ -Basis*. Computing 79, pp. 131-142.
- 81 [5] Chen F., Cox D., Liu Y. (2005). *The μ -Basis and Implicitization of Ra-
82 tional Parametric Surfaces*. Journal of Symbolic Computation, 39, pp.
83 689-706.
- 84 [6] Chou S.C., Gao X.-S. (1992). *Solving Parametric Algebraic Systems*, in
85 the Proceedings of ISSAC'92, Berkeley, 335-341, ACM Press.
- 86 [7] Cox D.A., Sederberg T.W., Chen F. (1998). *The Moving Line Ideal Basis
87 of Planar Rational Curves*. Computer Aided Geometric Design, 15, pp.
88 803-827.
- 89 [8] Deng J., Chen F., Shen L. (2005) *Computing μ -Basis of Rational Curves
90 and Surfaces Using Polynomial Matrix Factorization*, in: Proceedings of
91 the 2005 International Symposium on Symbolic and Algebraic Computa-
92 tion, ACM, pp. 132-139.
- 93 [9] Feng R., Shen L. (2016). *Computing the intersections of three conics ac-
94 cording to their Jacobian curve*. Journal of Symbolic Computation, 73,
95 pp.175-191.
- 96 [10] Fortune S. (2001). *Polynomial root-finding using iterated eigenvalue
97 Computation*. Proc. ISSAC 2001 pp. 121-128, ACM Press.
- 98 [11] Golub G.H., Van Loan C.F. (1989). *Matrix Computations*. The Johns
99 Hopkins University Press, Baltimore and London.
- 100 [12] Hosaka M. (1992). *Conics and Quadrics*. In: Modeling of Curves and
101 Surfaces in CAD/CAM. Computer Graphics—Systems and Applications.
102 Springer, Berlin, Heidelberg, pp. 45-54.

- 1 [13] Jia X., Tu C, Mourrain B., Wang W.(2020). *Complete Classification and*
2 *Efficient Determination of Arrangements Formed by Two Ellipsoids*. ACM
3 Transactions on Graphics, 39(3): 27:1-27:12.
- 4 [14] Jia X., Shi X., Chen F. (2018). *Survey on the Theory and Applications of*
5 *μ -Basis for Rational Curves and Surfaces*. Journal of Computational and
6 Applied Mathematics, 329, pp. 2–23.
- 7 [15] Johansen P.H., Løberg M., Piene R. (2008). *Monoid Hypersurfaces*. In:
8 Jüttler B., Piene R. (eds) Geometric Modeling and Algebraic Geometry.
9 Springer, Berlin, Heidelberg. pp. 55-77.
- 10 [16] Pérez-Díaz S., Schicho J., Sendra J.R. (2002). *Properness and Inversion*
11 *of Rational Parametrizations of Surfaces*. Applicable Algebra in Engi-
12 neering, Communication and Computing, 13, pp. 29–51.
- 13 [17] Pérez-Díaz S., Sendra J., Sendra J.R. (2004). *Parametrization of approx-*
14 *imate algebraic curves by lines*. Theoretical Computer Science on Alge-
15 braic - Numeric Algorithms, 315/2-3, pp. 627–650.
- 16 [18] Pérez-Díaz S., Sendra J., Sendra J.R. (2005). *Parametrization of approx-*
17 *imate algebraic surfaces by lines*. Computer Aided Geometric Design,
18 22/2, pp. 147–181.
- 19 [19] Pérez-Díaz S. (2007). *Computation of the Singularities of Parametric*
20 *Plane Curves*. Journal of Symbolic Computation, 42/8, pp. 835–857.
- 21 [20] Pérez-Díaz S., Shen L., (2021). *Inversion, Degree, Reparametrization*
22 *and Implicitization of Rational Planar Curves Using μ -Basis*. [Computer](#)
23 [Aided Geometric Design](#), Article 101957, 84(3/4).
- 24 [21] Pérez-Díaz S., Shen L., (2021). *The μ -Basis of Improper Rational Para-*
25 *metric Surface and Its Application*. [Mathematics](#). Article 640, 9(6).
- 26 [22] Pérez-Díaz S., Sendra J.R., Villarino C. (2015). *Computing the Singu-*
27 *larities of Rational Surfaces*. Mathematics of Computation, 84 (294), pp.
28 1991–2021.
- 29 [23] Sendra J.R., Winkler F., Pérez-Díaz S. (2007). *Rational Algebraic Curves:*
30 *A Computer Algebra Approach*. Series: Algorithms and Computation in
31 Mathematics, 22, Springer Verlag.
- 32 [24] Shen L., Chen F., Jüttler B., Deng L. (2006). *Approximate μ -Bases of*
33 *Rational Curves and Surfaces*. Proceedings of GMP'2006, M.S. Kim and
34 K. Shimada eds., LNCS, 4077, pp. 175–188.
- 35 [25] Shen L. (2016). *Computing μ -bases from algebraic ruled surfaces*. Com-
36 puter Aided Geometric Design, 46, pp. 125–130.
- 37 [26] Sederberg T.W., Zheng J., Klimaszewski K., Dokken T. (1999). *Approx-*
38 *imate implicitization using monoid curves and surfaces*. Graphical Models
39 and Image Processing, 61(4), pp. 177–198.
- 40 [27] Song, N., Chen, F., Goldman, R. (2007). *Axial moving lines and singular-*
41 *ities of rational planar curves*. Computer Aided Geometric Design, 24(4),
42 pp. 200-209.
- 43 [28] Wang, H., Jia, X., Goldman, R. (2009). *Axial moving planes and singular-*
44 *ities of rational space curves*. Computer Aided Geometric Design, 26(3),
45 pp. 300-316.
- 46 [29] Wang X., Chen F., Deng J. (2008). *Implicitization and parametrization of*
47 *quadratic surfaces with one simple base point*. Symbolic and Algebraic
48 Computation, International Symposium, ISSAC 2008, Linz/Hagenberg,
49 Austria, July 20–23, 2008, Proceedings. pp. 31–38.

The two authors share the equivalent contributions to this paper.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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