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ROBUST STABILIZABILITY AND STABILIZATION OF THREE-TIME-SCALE LINEAR TIME-INVARIANT SINGULARLY PERTURBED SYSTEMS WITH DELAY

Abstract. The objective of this study is to obtain the stabilizability conditions and a stabilizing composite state feedback control for the exponential stabilization of three-time-scale singularly perturbed linear time-invariant systems with multiple commensurate delays in the slow state variables and with two small parameters of perturbation (TSPLTISD). The stabilizability conditions and the stabilizing feedback do not depend on the small parameters and are valid for all of their sufficiently small values. The approach used in this work is the nondegenerate decoupling transformation that splits the TSPLTISD into three regularly dependent on the small parameters subsystems, which are lower in dimensions than the TSPLTISD. Further, the decoupled subsystems are approximated by three subsystems that do not depend on the small parameters. It is proven that the stabilizability of the approximating subsystems guarantees the robust (with respect to small parameters) stabilizability of the original TSPLTISD. Finally, we obtain a representation of a parameter free composite feedback control for the TSPLTISD, stabilizing it for all sufficiently small values of the parameters. A numerical example is given.

Keywords: singularly perturbed systems, delay, decomposition, stabilizability, composite state-feedback

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РОБАСТНАЯ СТАБИЛИЗИРУЕМОСТЬ И СТАБИЛИЗАЦИЯ ТРЕХТЕМПОВЫХ ЛИНЕЙНЫХ СТАЦИОНАРНЫХ СИНГУЛЯРНО ВОЗМУЩЕННЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

Аннотация. Целью работы является получение условий стабилизируемости и построение композитной стабилизирующей обратной связи по состоянию для трехтемповых линейных стационарных сингулярно возмущенных систем с кратными соизмеримыми запаздываниями в медленных переменных состояния и с двумя малыми параметрами при части старших производных (ТСВЛССЗ). Условия стабилизируемости и стабилизирующая обратная связь не зависят от малых параметров и действительны для всех их достаточно малых значений. Применяемый в работе подход использует невырожденное преобразование, которое полностью расщепляет зависящую от двух малых параметров сингулярно возмущенную систему на три регулярно зависящие от параметров подсистемы меньших размерностей, чем исходная система, которые аппроксимируются подсистемами, не зависящими от малых параметров. Доказано, что стабилизируемость аппроксимирующих подсистем гарантирует робастную (по малым параметрам) стабилизируемость исходной ТСВЛССЗ. Получено представление не зависящего от параметра композитного управления с обратной связью для ТСВЛССЗ, стабилизирующего ее при всех достаточно малых значениях параметров. Приведен численный пример.

Ключевые слова: сингулярно возмущенные системы, запаздывание, декомпозиция, стабилизируемость, стабилизирующее композитное управление с обратной связью

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Introduction. Singularly perturbed systems (SPS) are found in a broad class of applications ranging from engineering applications to areas such as quantum mechanics [1–3]. Whenever parameters of perturbation that represent small parasitics such as electrical resistance, viscosity, friction, etc. are present in classes of dynamic systems, this can lead to SPSs with high dimensionality. SPSs are composed of variables with different tempos, which leads to solutions rapidly varying at different tempos in different domains. Thus, in practical applications, due to the stiffness of such systems, accurate analysis and control of such systems are challenging and complex [4, 5]. But, due to the presence of variables with different tempos, the system can be decomposed into its subsystems according to the tempo of the variables [4]. Decomposed subsystems are lower in dimensionality compared to the original system and do not depend on parameters. As a result, analysis of the system properties such as stabilizability, controllability etc, and the design of the controller for such systems are less complex compared to that of the original SPS [4, 5].

For systems with delays, the conditions for the stabilizability are established in the literature [6–9]. Even though there are previous studies [10–14] (see also references in [10]) on the stabilizability of linear two- and three-time-scale SPSs with single state delays, no work has been published on more complex and generalized higher order SPSs with multiple delays. In [14] Chang-type [15] non-degenerate transformation is used in a stabilization problem for a two-time-scale singularly perturbed system with a single state delay in the slow state variable. To the authors' knowledge, the stabilization of TSPLTISDs has not yet been well explained. In [16] a three-time-scale SPS without delay is considered and a non-singular linear transformation is introduced in such a way that the original system is diagonalized and simplified, the analysis of the stability properties of the overall system is reduced to analyzing the reduced subsystems. A similar study has been conducted in [17] where the time-scale separation and stability of linear time-varying and time-invariant multiparameter SPSs are discussed.

This study has also been greatly influenced by the results of [12, 14, 16, 17], and the major objective of this study is to obtain solutions for the stabilization problem for complex generalized higher order SPSs with multiple delays. Hence, this study considers a more generalized class of the problem, a linear time invariant three-time-scale singularly perturbed system with multiple commensurate delays in the slow state variable (TSPLTISD). In the context of decomposition of the considered SPS, in [18] the generalization of the Chang-type transformation [15] on the three-time scale singularly perturbed time-invariant system with multiple commensurate delays in slow state variables is carried out which decouples the SPS into its lower dimensional subsystems according to the tempo of the variables. The simulation results of study [19] indicate that the asymptotic method is effective in the approximation of sub-systems of SPS without compromising the qualitative behaviour of the solutions. Thus, a similar schema to that of study [18] is used in the asymptotic decomposition of the considered three-time-scale SPS. In [20], based on the first Lyapunov method and decomposition, it is proven that the exponential stability of the asymptotically approximated subsystems guarantees the exponential stability of the original three-time-scale system with delay and the established conditions are robust with respect to the small parameters.

Considering the results of studies [10–14], on similar DS and BLSs, theorems are proven for the DS and BLSs of the considered three-time-scale SPS with respect to the stabilizability of the subsystems. It is further proven that the decomposed exact slow and fast subsystems are $\mathcal{O}(\mu)$ closer to the DS and BLSs, respectively. Based on the preliminary results, the robust sufficient conditions for the stabilizability of the obtained lower dimensional subsystems with respect to small parameters $\varepsilon_1, \varepsilon_2 > 0$ and for the overall stabilizability of the considered TSPLTISD are extensive discussed with relevant illustrative examples. A representation of a parameter free composite stabilizing feedback control for the system is obtained.

Statement of the problem. Let us consider a three-time-scale singularly perturbed linear time-invariant system with multiple commensurate delays in the slow state variables (TSPLTISD):

$$\dot{x}(t) = \sum_{j=0}^{k} A_{11j}x(t-jh) + A_{12}y(t) + A_{13}z(t) + B_1u(t), \quad x \in \mathbb{R}^{n1}, \tag{1}$$

$$\varepsilon_1\dot{y}(t) = \sum_{j=0}^{k} A_{21j}x(t-jh) + A_{22}y(t) + A_{23}z(t) + B_2u(t), \quad y \in \mathbb{R}^{n2}, \tag{2}$$

$$\varepsilon_2\dot{z}(t) = \sum_{j=0}^{k} A_{31j}x(t-jh) + A_{32}y(t) + A_{33}z(t) + B_3u(t), \quad z \in \mathbb{R}^{n3}, \quad u \in \mathbb{R}^{r}, \quad t > 0, \tag{3}$$

here $A_{i1j}, A_{i2}, A_{i3}, B_i$, $i=\overline{1,3}$, $j=\overline{0,k}$ are constant matrices with appropriate dimensions, $h=\text{const}>0$ is a delay, $0<\varepsilon_2\ll\varepsilon_1\ll 1$, are small parameters, $\varepsilon_1\in(0,\varepsilon_1^0]$, $\varepsilon_2\in(0,\varepsilon_2^0]$, that describe the time-scale separation, x is the slow variable, y is the fast variable and z is the fastest variable, $u\in U$, U is a set of piecewise continuous vector functions for $t\geq 0$. Let $\varphi(0)\triangleq x_0$. Note that since $\varepsilon_2\ll\varepsilon_1\ll 1$, then $\varepsilon_1\varepsilon_2\ll\varepsilon_2$, $\frac{\varepsilon_2}{\varepsilon_1}\ll 1$, $\mu\triangleq\frac{\varepsilon_2}{\varepsilon_1}\underset{\varepsilon_1\to 0}{\to}0$. Let $\mu<\varepsilon_1$.

Let $\|r\|_{[a,b]}=\sup_{\theta\in[a,b]}\|r(\theta)\|$ be the uniform norm defined in the space of piecewise continuous functions, where $\|r(\theta)\|$ is the Euclidean norm. Let $\mathbb{R}^{r\times n}[z]$ be a set of $(r\times n)$-matrices with entries from a ring of polynomials of $\mathbb{R}$. Let $p\triangleq\frac{d}{dt}$ be a differentiation operator, e^{-ph} be a delay operator:

$$e^{-ph}v(t)=v(t-h),\quad e^{-jph}v(t)=v(t-jh);\quad \mathrm{A}_{i1}(e^{-ph})\triangleq\sum_{j=0}^{l}A_{i1j}e^{-jph}\in\mathbb{R}^{n_i\times n_1}[e^{-ph}],\quad i=\overline{1,3}, B_j,\ j=\overline{1,3},$$

be the matrix operators.

D e f i n i t i o n 1. For a given $\varepsilon_1>0$, $\varepsilon_2>0$, unforced $(u(t)\equiv 0)$ TSPLTISD (1)–(3) is considered to be *exponentially stable* if constants exist $\alpha>0$ and $K\neq 1$ such that for any $x_0\in\mathbb{R}^{n_1}$, $y_0\in\mathbb{R}^{n_2}$, $z_0\in\mathbb{R}^{n_3}$ and a piecewise continuous n_1-vector function $\varphi(\theta)$, $\theta\in[-kh,0)$, the solution $(x(t),y(t),z(t))$, $t\geq 0$, of system (1)–(3) with initial conditions:

$$x(0)=x_0,\quad y(0)=y_0,\quad z(0)=z_0,\quad x(\theta)=\varphi(\theta),\quad \theta\in[-kh,0),$$

satisfies the inequality $\|\{x(t),y(t),z(t)\}\|\leq Ke^{-\alpha t}\left(\|\varphi\|_{[-kh,0]}+\|y(0)\|+\|z(0)\|\right)$.

D e f i n i t i o n 2. For a given $\varepsilon_1>0$, $\varepsilon_2>0$, TSPLTISD (1)–(3) is considered to be *stabilizable by feedback* if the linear state feedback control exists:

$$u(t)=\mathrm{F}_1(e^{-ph})x(t)+F_2y(t)+F_3z(t),\tag{4}$$

with the feedback gain $\left(\mathrm{F}_1(e^{-ph}),F_2,F_3\right)\in\mathbb{R}^{r\times n_1}[z]\oplus\mathbb{R}^{r\times n_2}\oplus\mathbb{R}^{r\times n_3}$, such that closed loop system (1)–(3), (4) is exponentially stable for the given $\varepsilon_1>0$, $\varepsilon_2>0$.

In this case control law (4) is called the *stabilizing composite state-feedback*.

D e f i n i t i o n 3. If numbers $\varepsilon_1^*>0$, $\varepsilon_2^*>0$ exist such that TSPLTISD (1)–(3) is stabilizable for any $\varepsilon_1\in(0,\varepsilon_1^*]$ and $\varepsilon_2\in(0,\varepsilon_2^*]$, we say that *stabilizability is robust with respect to parameters* ε_1, ε_2.

O b j e c t i v e. The main objective of this paper is to obtain sufficient conditions for the robust stabilizability of TSPLTISD (1)–(3) and to consruct representation of a parameter free stabilizing feedback gain for the TSPLTISD.

Preliminaries. For simplicity in the sequel, where this does not lead to an ambiguous understanding, the argument e^{-ph} in the matrix functions $\mathrm{A}_{i1}(e^{-ph})$, $i=\overline{1,3}$, etc. will be omitted.

By $\mathbb{C}$, the set of complex numbers is denoted. Let us consider $S(\mathbb{C})$ to be a set of all complex numbers $\mathbb{C}$ with a negative real part: $S(\mathbb{C})=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$ is the stability region.

For $\varepsilon_1>0$, $\varepsilon_2>0$ let us introduce the matrix operators:

$$\mathrm{A}\left(\varepsilon_1,\varepsilon_2,e^{-ph}\right)=\begin{bmatrix}\mathrm{A}_{11}(e^{-ph}) & A_{12} & A_{13}\\ \varepsilon_1^{-1}\mathrm{A}_{21}(e^{-ph}) & \varepsilon_1^{-1}A_{22} & \varepsilon_1^{-1}A_{23}\\ \varepsilon_2^{-1}\mathrm{A}_{31}(e^{-ph}) & \varepsilon_2^{-1}A_{32} & \varepsilon_2^{-1}A_{33}\end{bmatrix},\quad B(\varepsilon_1,\varepsilon_2)=\begin{bmatrix}B_1\\ \varepsilon_1^{-1}B_2\\ \varepsilon_2^{-1}B_3\end{bmatrix},$$

and define a matrix-valued function as follows:

$$\mathrm{N}\left(\varepsilon_1,\varepsilon_2,\lambda,e^{-\lambda h}\right)\triangleq\left[\lambda I_{n_1+n_2+n_3}-\mathrm{A}\left(\varepsilon_1,\varepsilon_2,e^{-\lambda h}\right),B(\varepsilon_1,\varepsilon_2)\right],\quad \lambda\in\mathbb{C}.$$

Applying the results of [8, 14] to the system (1)–(3) for a given $\varepsilon_1>0$, $\varepsilon_2>0$, we obtain:

S t a t e m e n t 1. *For a fixed* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$ *system* (1)–(3) *is stabilizable if and only if:*

$$\operatorname{rank} \mathrm{N}\left(\varepsilon_1, \varepsilon_2, \lambda, e^{-\lambda h}\right) = n_1 + n_2 + n_3, \quad \forall \lambda \in \mathbb{C} \setminus S(\mathbb{C}).$$

Note that it suffices to verify the condition of Statement 1 only for a λ from a TSPLTISD (1)–(3) spectrum (the set of eigenvalues)

$$\sigma(\varepsilon_1, \varepsilon_2) = \left\{\lambda \in \mathbb{C} : \det\left[\lambda I_{n_1+n_2+n_3} - \mathrm{A}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right)\right] = 0\right\}.$$

The application of Statement 1 depends on the values of the small parameters, while in various real-life problems these values are unknown, i. e. these problems are uncertain with respect to the parameters. To obtain other conditions for the TSPLTISD stabilization, robust in small parameters, in this work we apply the decomposition procedure.

A degenerate system and two boundary layer systems are associated with $(n_1 + n_2 + n_3)$-dimensional TSPLTISD (1)–(3), which are parameter free and smaller dimensional than the original system [21, 18]. Let

$$\det A_{33} \neq 0, \qquad \det\left[A_{22} - A_{23} A_{33}^{-1} A_{32}\right] \neq 0. \tag{5}$$

The degenerate system (slow subsystem, DS) can be obtained from TSPLTISD (1)–(3) by setting $\varepsilon_1 = \varepsilon_2 = 0$, which leads to:

$$\begin{aligned} \dot{x}_s(t) &= \mathrm{A}_{11}(e^{-ph}) x_s(t) + A_{12} y_s(t) + A_{13} z_s(t) + B_1 u_s(t), \\ 0 &= \mathrm{A}_{21}(e^{-ph}) x_s(t) + A_{22} y_s(t) + A_{23} z_s(t) + B_2 u_s(t), \\ 0 &= \mathrm{A}_{31}(e^{-ph}) x_s(t) + A_{32} y_s(t) + A_{33} z_s(t) + B_2 u_s(t). \end{aligned} \tag{6}$$

Here $x_s(t)$, $y_s(t)$, $z_s(t)$ are variables of TSPLTISD (1)–(3) with $\varepsilon_1 = \varepsilon_2 = 0$, $u_s(t)$ is a control for DS (6). Considering the last two equations of obtained system (6) under conditions (5), expressing y_s, z_s in terms of x_s and substituting the resulting expressions into the first equation of system (6), the DS is obtained in the matrix operator form:

$$\dot{x}_s(t) = \mathrm{A}_s(e^{-ph}) x_s(t) + B_s u_s(t), \tag{7}$$

where

$$\mathrm{A}_s(e^{-ph}) = \mathrm{A}_{11}(e^{-ph}) - A_{13} A_{33}^{-1} \mathrm{A}_{31}(e^{-ph}) - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{A}_{21}, \qquad B_s = B_1 - A_{13} A_{33}^{-1} B_3 - \hat{A}_{12} \hat{A}_{22}^{-1} \hat{B}_2,$$

$$\hat{A}_{12} = A_{12} - A_{13} A_{33}^{-1} A_{32}, \quad \hat{B}_2 = B_2 - A_{23} A_{33}^{-1} B_3, \quad \hat{A}_{22} = A_{22} - A_{23} A_{33}^{-1} A_{32}, \quad \hat{\mathrm{A}}_{21} = \mathrm{A}_{21} - A_{23} A_{33}^{-1} \mathrm{A}_{31}. \tag{8}$$

DS (7) is a n_1-dimensional system with multiple commensurate delays.

The ε_1-boundary layer system (the fast subsystem, ε_1-BLS), corresponding to TSPLTISD (1)–(3), is obtained by setting $\varepsilon_2 = 0$ in TSPLTISD (1)–(3) with $x_{f1}(t)$, $y_{f1}(t)$, $z_{f1}(t)$ as a variables of a TSPLTISD system with $\varepsilon_2 = 0$, expressing $z_{f1}(t)$ from the last equation of the obtained system under conditions (5) in terms of y_{f1}, x_{f1} and substituting into the second equation of the system, the following equation is obtained:

$$\varepsilon_1 \dot{y}_{f1}(t) = \hat{\mathrm{A}}_{21}(e^{-ph}) x_{f1}(t) + \hat{A}_{22} y_{f1}(t) + \hat{B}_2 u_{f1}(t).$$

Then by "stretching" the time scale with the transformation $\tau_1 = \dfrac{t}{\varepsilon_1}$, and "freezing" in this equation the slow variables $x_{f1} : \dfrac{dx_{f1}}{d\tau_1} = 0$, $x_{f1} \equiv x_0$, and defining $\hat{y}(\tau_1) \triangleq \hat{\mathrm{A}}_{22}^{-1} \hat{\mathrm{A}}_{21}(e^{-ph}) \varphi(0) + y_f(\varepsilon_1 \tau_1)$, following n_2-dimensional ε_1-BLS without delay can be obtained:

$$\frac{d\hat{y}(\tau_1)}{d\tau_1} = A_{f1} \hat{y}(\tau_1) + B_{f1} u_{f1}(\tau_1), \tag{9}$$

where $A_{f1} = \hat{A}_{22}$, $B_{f1} = \hat{B}_2$, $u_{f1}(\tau_1) = u(\varepsilon_1\tau_1) - u_s(t)$ is a control for ε_1-BLS (9), which we will consider with the initial conditions $\hat{y}(0) = y_0 + \hat{A}_{22}{}^{-1}\hat{A}_{21}(e^{-ph})\varphi(0) + y_0$.

The ε_2-boundary layer system (the fastest subsystem, ε_2-BLS), corresponding to TSPLTISD (1)–(3) is obtained by "stretching" the time scale with the transformation $\tau_2 = \dfrac{t}{\varepsilon_2}$ and "freezing" in (1)–(3) slow x_{f2} and fast y_{f2} variables $\dfrac{dx_{f2}}{d\tau_2} = 0$, $\dfrac{dy_{f2}}{d\tau_2} = 0$, $x_{f2} \equiv x_0$, $y_{f2} \equiv y_0$, and defining $\hat{z}(\tau_2) \triangleq A_{33}^{-1}\hat{A}_{31}\varphi(0) + z_{f2}(\varepsilon_2\tau_2)$. The following n_3-dimensional ε_2-BLS without delay can be obtained:

$$\frac{d\hat{z}(\tau_2)}{d\tau_2} = A_{f2}\hat{z}(\tau_2) + B_{f2}u_{f2}(\tau_2), \tag{10}$$

where $A_{f2} = A_{33}$, $B_{f2} = B_3$, $u_{f2}(\tau_2) = u(t) - u_{f1}(\tau_1) - u_s(t)$ is a control for ε_2-BLS (10), which we will consider with the initial conditions $\hat{z}(0) = z_0 + A_{33}^{-1}[A_{310}x_0 + A_{311}\varphi(-h) + A_{32}y_0]$.

With the introducing of Chang's type [15] non-degenerate transformation [18, 19], TSPLTISD (1)–(3) can be completely split into its subsystems with separated motions without compromising the qualitative behaviours of original TSPLTISD (1)–(3). Further in [19], it is proven that, under conditions (5) the separated subsystems can be asymptotically approximated for sufficiently small ε_1, $\varepsilon_2 > 0$ to any desired degree of accuracy with respect to small parameter of perturbation.

From [18] and [19] follows the next statement:

Theorem 1. *Let* (5) *be satisfied, then parameters* $\varepsilon_1^*, \varepsilon_2^* \geq 0$ *exist that for all* $\varepsilon_1 \in [0, \varepsilon_1^*)$, $\varepsilon_2 \in [0, \varepsilon_2^*)$, $\varepsilon_2 \ll \varepsilon_1$, *as a result of applying the Chang's type decoupling transformation* $\mathrm{T}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right)$

$$\begin{bmatrix} \xi(t) \\ \eta(t) \\ \beta(t) \end{bmatrix} = \mathrm{T}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right)\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \quad \xi(t) \in R^{n1}, \ \eta(t) \in R^{n2}, \ \beta(t) \in R^{n3}, \ t \geq 0,$$

TSPLTISD (1)–(3) *is transformed into an equivalent system with separated motions:*

$$\dot{\xi}(t) = \mathrm{A}_\xi\left(\varepsilon_1, \mu, e^{-ph}\right)\xi(t) + \mathrm{B}_\xi\left(\varepsilon_1, \mu, e^{-ph}\right)u(t),$$

$$\varepsilon_1\dot{\eta}(t) = \mathrm{A}_\eta\left(\varepsilon_1, \mu, e^{-ph}\right)\eta(t) + \mathrm{B}_\eta\left(\varepsilon_1, \mu, e^{-ph}\right)u(t),$$

$$\varepsilon_2\dot{\beta}(t) = \mathrm{A}_\beta\left(\varepsilon_1, \mu, e^{-ph}\right)\eta(t) + \mathrm{B}_\beta\left(\varepsilon_1, \mu, e^{-ph}\right)u(t),$$

where for all sufficiently small parameters ε_1 *and* ε_2, $\mu < \varepsilon_1$:

$$\mathrm{A}_\xi\left(\varepsilon_1, \mu, e^{-ph}\right) = \mathrm{A}_s(e^{-ph}) + \mathrm{O}(\varepsilon_1), \quad \mathrm{A}_\eta\left(\varepsilon_1, \mu, e^{-ph}\right) = \mathrm{A}_{f1} + \mathrm{O}(\varepsilon_1), \quad \mathrm{A}_\beta\left(\varepsilon_1, \mu, e^{-ph}\right) = \mathrm{A}_{f2} + \mathrm{O}(\varepsilon_1),$$

$$\mathrm{B}_\xi\left(\varepsilon_1, \mu, e^{-ph}\right) = \mathrm{B}_s + \mathrm{O}(\varepsilon_1), \quad \mathrm{B}_\eta\left(\varepsilon_1, \mu, e^{-ph}\right) = \mathrm{B}_{f1} + \mathrm{O}(\varepsilon_1), \quad \mathrm{B}_\beta\left(\varepsilon_1, \mu, e^{-ph}\right) = \mathrm{B}_{f2} + \mathrm{O}(\varepsilon_1).$$

Note that since $\det T\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right) \equiv 1$ [18], then the time-independent transformation $\mathrm{T}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right)$ is the Lyapunov transformation.

Stabilizability of the subsystems of the TSPLTISD.

Definition 4. The unforced DS (7) $\left(u_s(t) \equiv 0\right)$ is said to be *exponentially stable* if constants $\alpha_s > 0$ and $K_s \neq 1$ exist that for any $x_0 \in \mathbb{R}^{n1}$, $\varphi(\theta) \in \mathrm{PC}([-kh, 0); \mathbb{R}^{n1})$ the solution $x_s(t)$, $t \geq 0$, of DS (7) with initial conditions $x_s(0) = x_0, x_s(\theta) = \varphi(\theta)$, $\theta \in [-kh, 0)$, satisfies the inequality $\|x_s(t)\| \leq K_s e^{-\alpha_s t}\|\varphi\|_{[-kh,0]}$. The unforced ε_1-BLS (9) $\left(u_{f1}(\tau_1) \equiv 0\right)$ is said to be *exponentially stable* if constants $\alpha_{f1} > 0$ and $K_{f1} \neq 1$ exist that for any $\hat{y}_0 \in \mathbb{R}^{n2}$, the solution $\hat{y}(\tau_1)$, $\tau_1 \geq 0$, of ε_1-BLS (9) with initial conditions $\hat{y}(0)=\hat{y}_0$ satisfies the inequality $\|\hat{y}(\tau_1)\| \le K_{f1}e^{-\alpha_{f1}\tau_1}\|\hat{y}(0)\|$. Unforced ε_2-BLS (10) $\left(u_{f2}(\tau_2)\equiv 0\right)$ is said to be *exponentially stable* if constants $\alpha_{f2}>0$ and $K_{f2}\neq 1$ exist that for any $\hat{z}_0\in\mathbb{R}^{n_3}$, the solution $\hat{z}(\tau_2)$, $\tau_2\ge 0$, of ε_2-BLS (10) satisfies the inequality $\|\hat{z}(\tau_2)\| \le K_{f2}e^{-\alpha_{f2}\tau_2}\|\hat{z}(0)\|$.

D e f i n i t i o n 5. DS (7) is said to be *stabilizable* if a linear state feedback $u_s(t)=\mathrm{F}_s(e^{-ph})x_s(t)$, exist that the closed loop DS $\dot{x}_s(t)=\left[\mathrm{A}_s(e^{-ph})+B_s\mathrm{F}_s(e^{-ph})\right]x_s(t)$ is exponentially stable. ε_1-BLS (9) is said to be *stabilizable* if the linear state feedback $u(\tau_1)=F_{f1}\hat{y}(\tau_1)$ exists that the closed loop system $\dfrac{d\hat{y}(\tau_1)}{d\tau_1}=[A_{f1}+B_{f1}F_{f1}]\hat{y}(\tau_1)$ is exponentially stable. ε_2-BLS (10) is said to be *stabilizable* if the linear state feedback $u(\tau_2)=F_{f2}\hat{z}(\tau_2)$ exists that the closed loop system $\dfrac{d\hat{z}(\tau_2)}{d\tau_2}z(\tau_2)=[A_{f2}+B_{f2}F_{f2}]\hat{z}(\tau_2)$ is exponentially stable.

Applying the results in [8, 14] on DS (7) and BLSs (9), (10) the following theorem can be proven [20].

S t a t e m e n t 2.

a) The DS (7) *is stabilizable if and only if:*

$$\operatorname{rank}\mathrm{N}_s(\lambda,e^{-\lambda h})\triangleq\operatorname{rank}\left[\lambda I_{n_1}-\mathrm{A}_s(e^{-\lambda h}),B_s\right]=n_1,\qquad \forall\lambda\in\mathbb{C}\setminus S(\mathbb{C}).$$

b) The ε_1*-BLS* (9) *is stabilizable if and only if:*

$$\operatorname{rank}N_{f1}(\lambda)\triangleq\operatorname{rank}[\lambda I_{n_2}-A_{f1},B_{f1}]=n_2,\qquad \forall\lambda\in\mathbb{C}\setminus S(\mathbb{C}).$$

c) The ε_2*-BLS* (10) *is stabilizable if and only if:*

$$\operatorname{rank}N_{f2}(\lambda)\triangleq\operatorname{rank}[\lambda I_{n_3}-A_{f2},B_{f2}]=n_3,\qquad \forall\lambda\in\mathbb{C}\setminus S(\mathbb{C}).$$

Note that it suffices to verify conditions of Statement 2 only for a λ from the spectra $\sigma_s=\{\lambda\in\mathbb{C}:\det[\lambda I_{n_1}-\mathrm{A}_s(e^{-\lambda h})]=0\}$ of DS (7), $\sigma_{f1}=\{\lambda\in\mathbb{C}:\det[\lambda I_{n_2}-A_{f1}]=0\}$ of ε_1-BLS (9) and $\sigma_{f2}=\{\lambda\in\mathbb{C}:\det[\lambda I_{n_3}-A_{f2}]=0\}$ of ε_2-BLS (10).

Let (5) be satisfied and let $\mathrm{F}_s(e^{-ph})$, F_{f1}, F_{f2} be stabilizing linear state feedback gains for DS (7), ε_1-BLS (9) and ε_2-BLS (10) , respectively:

$$u_s(t)=\mathrm{F}_s(e^{-ph})x_s(t),\qquad u_{f1}(\tau_1)=F_{f1}\hat{y}_{f1}(\tau_1),\qquad u_{f2}(\tau_2)=F_{f2}\hat{z}_{f2}(\tau_2). \tag{11}$$

Let us define matrices as follows:

$$\begin{gathered}\mathrm{F}_1(e^{-ph})=\mathrm{F}_s(e^{-ph})+F_{f1}\hat{A}_{22}^{-1}\left[\hat{\mathrm{A}}_{21}(e^{-ph})+\hat{B}_2\mathrm{F}_s(e^{-ph})\right]+\\ +F_{f2}A_{33}^{-1}\left[\mathrm{A}_{31}(e^{-ph})+B_3\mathrm{F}_s(e^{-ph})+B_3F_{f1}\hat{A}_{22}^{-1}\left[\hat{\mathrm{A}}_{21}(e^{-ph})+\hat{B}_2\mathrm{F}_s(e^{-ph})\right]\right],\\ F_2=F_{f1}+F_{f2}A_{33}^{-1}[A_{32}+B_3F_{f1}],\qquad F_3=F_{f2}.\end{gathered}\tag{12}$$

T h e o r e m 2. *Let* (5) *be satisfied, DS* (7), ε_1*-BLS* (7) *and* ε_2*-BLS* (10) *be stabilizable by a linear state feedback with gains* $\mathrm{F}_s(e^{-ph})$, F_{f1} *and* F_{f2}, *respectively, then the following conditions are satisfied:*

$$\det\left[A_{22}+B_2F_3-(A_{23}+B_2F_3)(A_{33}+B_3F_3)^{-1}(A_{32}+B_3F_2)\right]\neq 0,\qquad \det[A_{33}+B_3F_3]\neq 0. \tag{13}$$

Then, $\varepsilon_1^*\in(0,\varepsilon_1^0]$ *and* $\varepsilon_2^*\in(0,\varepsilon_2^0]$ *exist that TSPLTISD* (1)–(3) *is stabilizable for all* $\varepsilon_1\in(0,\varepsilon_1^*]$ *and* $\varepsilon_2\in(0,\varepsilon_2^*]$, $\varepsilon_2\ll\varepsilon_1$, *i. e. stabilizability is robust with respect to small parameters* ε_1, ε_2 *by a composite linear state feedback* (4)*, where* $\mathrm{F}_1(e^{-ph})$, F_2 *and* F_3 *are defined in* (12).

P r o o f. It follows from [20] that, under the conditions of Theorem 2, TSPLTISD (1)–(3) is stabilizable for sufficiently small ε_1, $\varepsilon_2>0$, preserving the order of the smallness.

Let us consider a feedback low of form (4), where $\mathrm{F}_1(e^{-ph})$, F_2 and F_3 are not yet defined. Then the following closed-loop system can be obtained:

$$\begin{bmatrix} \dot{x}(t) \\ \varepsilon_1 \dot{y}(t) \\ \varepsilon_2 \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{F}_{11} & F_{12} & F_{13} \\ \mathrm{F}_{21} & F_{22} & F_{23} \\ \mathrm{F}_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} &\mathrm{F}_{11}(e^{-ph}) = \mathrm{A}_{11}(e^{-ph}) + B_1\mathrm{F}_1(e^{-ph}), \quad F_{12} = A_{12} + B_1F_2, \quad F_{13} = A_{13} + B_1F_3, \\ &\mathrm{F}_{21}(e^{-ph}) = \mathrm{A}_{21}(e^{-ph}) + B_2\mathrm{F}_1(e^{-ph}), \quad F_{22} = A_{22} + B_2F_2, \quad F_{23} = A_{23} + B_2F_3, \\ &\mathrm{F}_{31}(e^{-ph}) = \mathrm{A}_{31}(e^{-ph}) + B_3\mathrm{F}_1(e^{-ph}), \quad F_{32} = A_{32} + B_3F_2, \quad F_{33} = A_{33} + B_3F_3. \end{aligned} \tag{15}$$

Since the closed loop system (14), (15) is of the form of TSPLTISD (1)–(3), then (see Theorem 1) under conditions (13), a Chang's type [15] decoupling transformation $\tilde{\mathrm{T}}(\varepsilon_1, \varepsilon_2, e^{-ph})$ with $\tilde{\mathrm{L}}_i(\varepsilon_1, \mu, e^{-ph})$ and $\tilde{\mathrm{H}}_i(\varepsilon_1, \mu, e^{-ph})$, $i = 1,2,3$, exists [18] satisfiying the equations:

$$\begin{aligned} &\mathrm{A}_{31} + B_3\mathrm{F}_1 - [A_{33} + B_3F_3]\tilde{\mathrm{L}}_2 + \varepsilon_2\tilde{\mathrm{L}}_2\left([\mathrm{A}_{11} + B_1\mathrm{F}_1] - [A_{13} + B_1F_3]\tilde{\mathrm{L}}_2\right) + \\ &+\mu\tilde{\mathrm{L}}_3\left([\mathrm{A}_{21} + B_2\mathrm{F}_1] - [A_{23} + B_2F_3]\tilde{\mathrm{L}}_2\right) = 0, \\ &A_{32} + B_3F_2 - [A_{33} + B_3F_3]\tilde{\mathrm{L}}_3 + \varepsilon_2\tilde{\mathrm{L}}_2\left([A_{12} + B_1F_2] - [A_{13} + B_1F_3]\tilde{\mathrm{L}}_3\right) + \\ &+\mu\tilde{\mathrm{L}}_3\left([A_{22} + B_2F_2] - [A_{23} + B_2F_3]\tilde{L}_3\right) = 0, \\ &\mathrm{A}_{21} + B_2\mathrm{F}_1 + \varepsilon_1\tilde{\mathrm{L}}_1[\mathrm{A}_{11} + B_1\mathrm{F}_1] - \varepsilon_1\tilde{\mathrm{L}}_1[A_{12} + B_1F_2]\tilde{\mathrm{L}}_1 - [A_{22} + B_2F_2]\tilde{\mathrm{L}}_1 - \\ &-[A_{23} + B_2F_3]\tilde{\mathrm{L}}_2 + [A_{23} + B_2F_3]\tilde{L}_3\tilde{\mathrm{L}}_1 + \varepsilon_1\tilde{\mathrm{L}}_1\left([A_{13} + B_1F_3]\tilde{\mathrm{L}}_3\tilde{\mathrm{L}}_1 - [A_{13} + B_1F_3]\tilde{\mathrm{L}}_2\right) = 0. \end{aligned} \tag{16}$$

The transformation $\tilde{\mathrm{T}}\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right)$ brings closed-loop system (14), (15) to the decoupled one:

$$\dot{\bar{\xi}}(t) = \bar{\mathrm{A}}_\xi\left(\varepsilon_1, \mu, e^{-ph}\right)\bar{\xi}(t), \quad \varepsilon_1\dot{\bar{\eta}}(t) = \bar{\mathrm{A}}_\eta\left(\varepsilon_1, \mu, e^{-ph}\right)\bar{\eta}(t), \quad \varepsilon_2\dot{\bar{\beta}}(t) = \bar{\mathrm{A}}_\beta\left(\varepsilon_1, \mu, e^{-ph}\right)\bar{\beta}(t), \tag{17}$$

where

$$\bar{\mathrm{A}}_\xi\left(\varepsilon_1, \mu, e^{-ph}\right), \ \bar{\mathrm{A}}_\eta\left(\varepsilon_1, \mu, e^{-ph}\right), \ \bar{\mathrm{A}}_\beta\left(\varepsilon_1, \mu, e^{-ph}\right)$$

are defined by:

$$\begin{aligned} &\bar{\mathrm{A}}_\xi(e^{-ph}) = \mathrm{A}_{11} + B_1\mathrm{F}_1 - (A_{12} + B_1F_2)\tilde{\mathrm{L}}_1 - (A_{13} + B_1F_3)\tilde{\mathrm{L}}_2 + (A_{13} + B_1F_3)\tilde{\mathrm{L}}_3\tilde{\mathrm{L}}_1, \\ &\bar{\mathrm{A}}_\eta(e^{-ph}) = A_{22} + B_2F_2 + \varepsilon_1\tilde{\mathrm{L}}_1(A_{12} + B_1F_2) - (A_{23} + B_2F_3)\tilde{\mathrm{L}}_3 - \varepsilon_1\tilde{\mathrm{L}}_1(A_{13} + B_1F_3)\tilde{\mathrm{L}}_3, \\ &\bar{\mathrm{A}}_\beta(e^{-ph}) = A_{33} + B_3F_3 + \varepsilon_1\mu\tilde{\mathrm{L}}_2(A_{13} + B_1F_3) + \mu\tilde{\mathrm{L}}_3(A_{23} + B_2F_3). \end{aligned} \tag{18}$$

We will look for $\mathrm{F}_1(e^{-ph})$, F_2 and F_3:

$$\mathrm{F}_1(e^{-ph}) = \mathrm{F}_s(e^{-ph}) + F_{f1}\tilde{\mathrm{L}}_1(e^{-ph}) + F_{f2}\tilde{\mathrm{L}}_2(e^{-ph}), \quad F_2 = F_{f1} + F_{f2}\tilde{\mathrm{L}}_3, \quad F_3 = F_{f2}. \tag{19}$$

Substituting (19) into (16) and taking (8) into account $\mu < \varepsilon_1$, under conditions (5) we obtain:

$$\begin{aligned} &\tilde{\mathrm{L}}_1(e^{-ph}) = \hat{A}_{22}^{-1}\left[\hat{\mathrm{A}}_{21}(e^{-ph}) + \hat{B}_2\mathrm{F}_s(e^{-ph})\right] + \mathrm{O}(\varepsilon_1), \\ &\tilde{\mathrm{L}}_3(e^{-ph}) = A_{33}^{-1}[A_{32} + B_3F_{f_{\varepsilon_1}}] + \mathrm{O}(\varepsilon_1), \\ &\tilde{\mathrm{L}}_2(e^{-ph}) = A_{33}^{-1}\left[\mathrm{A}_{31}(e^{-ph}) + B_3\mathrm{F}_s(e^{-ph}) + B_3F_{f1}\tilde{\mathrm{L}}_1(e^{-ph})\right] + \mathrm{O}(\varepsilon_1). \end{aligned} \tag{20}$$

Substituting (19) and (20) into (18), we obtain after transformation that decoupled closed loop control system (17) with feedback (4), (19) has the form:

$$\dot{\bar{\xi}}(t)=\left[\mathrm{A}_s(e^{-ph})+B_s\mathrm{F}_s(e^{-ph})+\mathrm{O}(\varepsilon_1)\right]\bar{\xi}(t),$$
$$\varepsilon_1\dot{\bar{\eta}}(t)=[A_{f1}+B_{f1}F_{f1}+\mathrm{O}(\varepsilon_1)]\bar{\eta}(t), \tag{21}$$
$$\varepsilon_2\dot{\bar{\beta}}(t)=[A_{f2}+B_{f2}F_{f2}+\mathrm{O}(\varepsilon_1)]\bar{\beta}(t),$$

i. e. decoupled closed loop subsystems (17) are $\mathrm{O}(\varepsilon_1)$-close to the closed loop DS, ε_1-BLS, ε_2-BLS, with feedback (11), respectively.

Since according to the assumptions of Theorem 2 DS, ε_2-BLS, ε_1-BLS are stabilizable by the feedback with gains $\mathrm{F}_s(e^{-ph})$, F_{f2}, F_{f1}, respectively, so the closed loop DS, ε_1-BLS, ε_2-BLS are stable, and taking into account the preservation of the stability under the small additive perturbations of the system parameters it follows that system (21) is stable for all sufficiently small values $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.

Original closed-loop system (14), (15) is obtained from (17) by the non-degenerate transformation $\tilde{\mathrm{T}}^{-1}(\varepsilon_1,\mu,e^{-ph})$ [18]. Since the stability is preserved under the non-degenerate transformation, then closed loop systems (14), (15) are also stable for all sufficiently small parameters ε_1 and ε_2, and therefore TSPLTISD (1)–(3) is stabilizable by (4), (19).

Substituting the constructed $\mathrm{F}_s(e^{-ph})$, F_{f2} and F_{f1} into (4) and considering approximations (20) for $\tilde{\mathrm{L}}_i(\varepsilon_1,\mu,e^{-ph})$, $i=1,2,3,$ we obtain:

$$u(t)=\Big[\mathrm{F}_s(e^{-ph})+F_{f1}\hat{A}_{22}^{-1}\Big(\hat{\mathrm{A}}_{21}(e^{-ph})+\hat{B}_2\mathrm{F}_s(e^{-ph})\Big)+F_{f2}A_{33}^{-1}\Big(A_{31}(e^{-ph})+B_3\mathrm{F}_s(e^{-ph})+B_3F_{f2}\hat{A}_{22}^{-1}\times$$
$$\times\Big[\hat{\mathrm{A}}_{21}+\hat{B}_2\mathrm{F}_s(e^{-ph})\Big]\Big)+\mathrm{O}(\varepsilon_1)\Big]x(t)+\Big[F_{f1}+F_{f2}A_{33}^{-1}(A_{32}+B_3F_{f1})+\mathrm{O}(\varepsilon_1)\Big]y(t)+\big(F_{f2}+\mathrm{O}(\varepsilon_1)\big)z(t).$$

Comparing the last representation with (12) we obtain:

$$u(t)=\Big[\mathrm{F}_1(e^{-ph})+\mathrm{O}(\varepsilon_1)\Big]x(t)+[F_2+\mathrm{O}(\varepsilon_1)]y(t)+[F_3+\mathrm{O}(\varepsilon_1)]z(t).$$

Taking into account the preservation of the stability under the small additive perturbations of the system parameters, we conclude that state feedback (4), (12) stabilizes TSPLTISD (1)–(3). End of Proof.

C o r o l l a r y. *Let* (5) *and the conditions of Statement* 2 *be satisfied. Then* $\varepsilon_1^{\,*}\in(0,\varepsilon_1^0]$ *and* $\varepsilon_2^{\,*}\in(0,\varepsilon_2^0]$ *exist that TSPLTISD* (1)–(3) *is stabilizable for all* $\varepsilon_1\in(0,\varepsilon_1^*]$ *and* $\varepsilon_2\in(0,\varepsilon_2^*]$, $\varepsilon_2\ll\varepsilon_1$, *i. e. stabilizability is robust with respect to the small parameters* ε_1, ε_2.

Results and verifications. An illustrative example of TSPLTISD is considered:

$$\dot{x}(t)=x(t)-y(t)+u(t),\qquad x,y,z\in\mathbb{R},$$
$$\varepsilon_1\dot{y}(t)=x(t)+y(t)+0.5u(t), \tag{22}$$
$$\varepsilon_2\dot{z}(t)=-x(t-1)+z(t)+0.1u(t),\qquad u(t)\in\mathbb{R},$$

with the initial conditions:

$$x(0)=1,\quad y(0)=0,\quad z(0)=1,\quad x(\theta)=1,\quad \theta\in[-1,0). \tag{23}$$

Considering system (22) parameters in the form of (1)–(3) can be denoted as:

$$h=1,\quad n_1=n_2=n_3=r=1,\quad A_{11,0}=1,\quad A_{11,1}=0,\quad A_{12}=-1,\quad A_{13}=0,\quad B_1=1,\quad A_{21,0}=1,$$
$$A_{21,1}=0,\quad A_{22}=1,\quad A_{23}=0,\quad B_2=0.5,\quad A_{31,0}=0,\quad A_{31,1}=-1,\quad A_{32}=0,\quad A_{33}=1,\quad B_3=0.1.$$

The characteristic equation of (22)

$$\omega(\varepsilon_1,\varepsilon_2,\lambda)=\varepsilon_1^{-1}\varepsilon_2^{-1}(\lambda\varepsilon_2-1)\Big(\lambda^2\varepsilon_1-\lambda\varepsilon_1-\lambda+2\Big)=0,\ \ \lambda\in\mathbb{C},$$

the spectrum of system (22)

$$\sigma(\varepsilon_1,\varepsilon_2)=\det\Big\{\lambda\in\mathbb{C}:\omega\Big(\varepsilon_1,\varepsilon_2,\lambda,e^{-\lambda h}\Big)=0\Big\}=$$
$$=\Big\{\varepsilon_2^{-1}\Big(1+\varepsilon_1-\sqrt{1-6\varepsilon_1+\varepsilon_1^2}\Big),\varepsilon_2^{-1}\Big(1+\varepsilon_1+\sqrt{1-6\varepsilon_1+\varepsilon_1^2}\Big),\varepsilon_2^{-1}\Big\}. \tag{24}$$

According to (24) the real parts of all the elements of the spectrum $\sigma(\varepsilon_1, \varepsilon_2)$ are positive, i.e. system (22) is unstable for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, as shown in Fig. 1, considering $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.001$ and $u(t) \equiv 0, t \geq 0$.

The DS for (22) has the form $\dot{x}_s(t) = 2x(t) + 1.5u(t)$, $x_s(\theta) = 1$, $\theta \in [-1, 0]$, and its spectrum is $\sigma_s = \{2\}$. The real part of the element of the spectrum σ_s is positive. i. e. DS for (22) is unstable. Since for $\mathrm{N}_s(\lambda, e^{-\lambda}) = [\lambda - 2 \quad 1.5]$ the rank $\mathrm{N}_s(\lambda, e^{-\lambda h}) = 1$ for $\forall \lambda \in \mathbb{C}$, as per the results of Statement 2, DS is stabilizable, namely by a stabilizing feedback with the gain $\mathrm{F}_s(e^{-ph}) \equiv F_s < -4/3$.

The ε_2-BLS for (22) has the form $\dfrac{d\hat{z}(\tau_2)}{d\tau_2} = \hat{z}(\tau_2) + 0.1u(\tau_2)$, $\hat{z}(0) = 0$, and its spectrum is $\sigma_{f2} = \{1\}$. The real part of the element of the spectrum σ_{f2} is positive. i.e. ε_2-BLS for (22) is unstable. Since for $N_{f2}(\lambda) = [\lambda - 1 \quad 0.1]$ the rank $N_{f2}(\lambda) = 1$ for $\forall \lambda \in \mathbb{C}$, as per the results of Statement 2, ε_2-BLS is stabilizable, namely by a stabilizing feedback with the gain F_{f2} if $F_{f2} < -10$.

The ε_1-BLS for (22) has the form $\dfrac{d\hat{y}(\tau_1)}{d\tau_1} = \hat{y}(\tau_1) + 0.5u(\tau_1)$, $\hat{y}(0) = 0$, and its spectrum is $\sigma_{f1} = \{1\}$. The real part of the element of the spectrum σ_{f1} is positive. i.e. ε_1-BLS for (22) is unstable. But since for, $N_{f1}(\lambda) = [\lambda - 1 \quad 0.5]$ the rank $N_{f1}(\lambda) = 1$ for $\forall \lambda \in \mathbb{C}$, as per the results of Statement 2, ε_1-BLS is stabilizable, namely by a stabilizing feedback with gain F_{f1} if $F_{f1} < -2$.

Considering the fact that the subsystems of TSPLTISD (22) are stabilizable, and also that (22) satisfies the conditions in Theorem 2, by Theorem 2 it can be stated that TSPLTISD (22) is stabilizable, which confirms the previous conclusion for all sufficiently small $\varepsilon_1 > 0$, $\varepsilon_2 > 0$.

Let $F_s = -2$, $F_{f1} = -10$ and $F_{f2} = -20$. By (12) for system (22) stabilizing feedback can be chosen in composite form (4):

$$u(t) = \mathrm{F}_1(e^{-ph})x(t) + F_2 y(t) + F_3 z(t), \quad \mathrm{F}_1 = 2 + 20e^{-ph}, \quad F_2 = 10, \quad F_3 = -20. \tag{25}$$

The explicit form of the closed-loop system for system (22), (25):

$$\begin{aligned} \dot{x}(t) &= (3 + 20e^{-ph})x(t) + 9y(t) - 20z(t), \\ \varepsilon_1 \dot{y}(t) &= (2 + 10e^{-ph})x(t) + 6y(t) - 10z(t), \\ \varepsilon_2 \dot{z}(t) &= (0.2 + e^{-ph})x(t) + y(t) - z(t). \end{aligned} \tag{26}$$

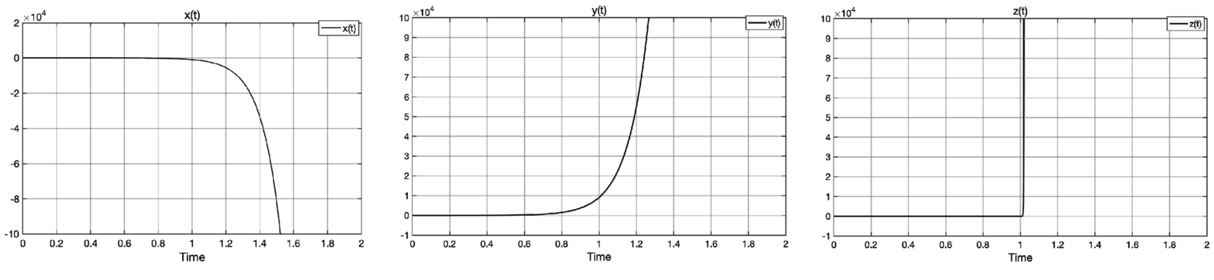

Fig. 1. Solutions of TSPLTISD (22) for $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.001$, $u(t) \equiv 0$, $t \geq 0$

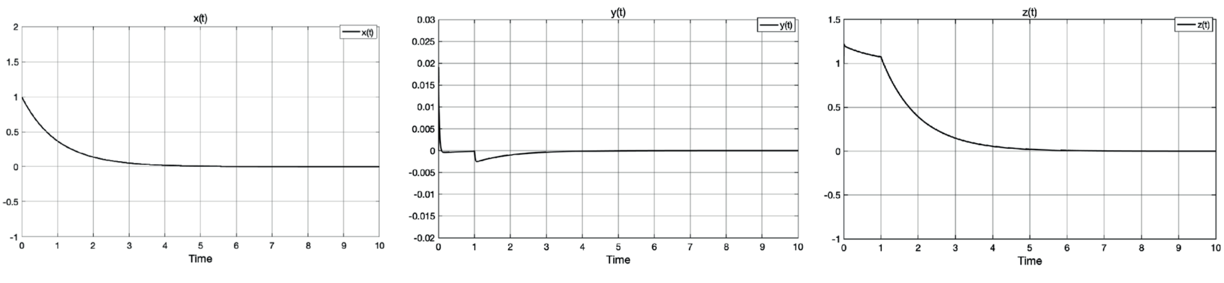

Fig. 2. Solutions of closed-loop system (26) for $\varepsilon_1 = 0.1$, $\varepsilon_1 = 0.001$

Resulting closed loop system (26) is stable. The characteristic equation of (26) is

$$\omega_c(\varepsilon_1,\varepsilon_2,\lambda) \triangleq \det\left[\lambda I - A_c\left(\varepsilon_1,\varepsilon_2,e^{-\lambda h}\right)\right] = 0, \lambda \in \mathbb{C},$$

where

$$\mathrm{A}_c\left(\varepsilon_1,\varepsilon_2,e^{-\lambda h}\right) = \begin{bmatrix} 3+20e^{-\lambda h} & 9 & -20 \\ (2+10e^{-\lambda h})\varepsilon_1^{-1} & 6\varepsilon_1^{-1} & -10\varepsilon_1^{-1} \\ (0.2+e^{-\lambda h})\varepsilon_2^{-1} & \varepsilon_2^{-1} & -\varepsilon_2^{-1} \end{bmatrix},$$

$\omega_c(\varepsilon_1,\varepsilon_2,\lambda) = \lambda^3 - 20\lambda^2\,\mathrm{e}^{-\lambda} - 8\lambda^2 + 30\lambda^2\,\mathrm{e}^{-\lambda} + 5\lambda + 4.$ For stabilizing control of form (25), let us solve closed system (26) with initial conditions (23) and for $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.001$. The solutions to system (26) are shown in Fig. 2, which demonstrates the stability of a closed-loop system.

Conclusion. For the considered three-time-scale singularly perturbed linear time-invariant systems with multiple commensurate delays in slow state variable (1)–(3), Statement 1 establishes the conditions for the stabilizabilty of TSPLTISD (1)–(3), but the conditions are dependent on parameters of perturbation (ε_1, ε_2), i.e. are complex due to the high dimensionality and the stiffness of the system. With the introduction of the Chang-type transformation [18] TSPLTISD is decomposed into its subsystems, which are lower dimensional, devided by tempos and are asymptotically close to three small parameter-free subsystems (7), (9), (10). Due to the results of Theorems 1, 2 robust sufficient conditions for the stabilizability of TSPLTISD (1)–(3) can be determined based on the stabilizability criteria of its Degenerate System and the two Boundary Layer Systems, i. e. ε-free sufficient conditions. Namely, the stabilizability of the slow and the fast subsystems of TSPLTISD (1)–(3) yields the stabilizability of original system (1)–(3) for all sufficiently small values of the parameters of singular perturbation, preserving the order of the smallness, i.e. robustly stabilizable with respect to the small perturbations. Moreover, by having a stabilizing state feedback for the Degenerate System and the two Boundary Layer Systems, the stabilization of original TSPLTISD (1)–(3) can be performed robustly and independently of the small parameters, with a composite state feedback of type (4), (12). The results obtained through the illustrative example considered also support the results.

Since subregulator problems are mutually independent, a parallel algorithm can be applied to the design of subsystems' subregulators separately. The knowledge of the small parameters is not required in a composite design. This expands the possibilities of practical implementation of control algorithms in real time under conditions of limited computing resources and time.

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