Thermal Dynamic Problems Of Reinforced Composite Cylinders

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Thermal dynamic problems of circular cylindrical composite shells reinforced in the axial and circumferential directions and subject to variations of temperature are considered. Nonlinear governing equations are formulated based on the extension of Donnell shell theory. These equations are used to determine the response of geometrically nonlinear and linear shells to a thermal loading represented by the Heaviside step function (thermal shock). The solution of the nonlinear problem obtained by the assumption that displacements are single-term functions of coordinates is discussed. The analysis of the linear problem illustrates different types of response to thermal shock. The condition of thermally-induced buckling of shells is formulated. Numerical analysis results in conclusions regarding the behavior of shells subject to thermal shock if the temperature is uniformly distributed throughout the shell and stiffeners.

Introduction

Reinforced cylindrical shells represent one of the principal elements of aerospace and ship structures, pressure vessels, etc. Increasing application of composite materials in design makes investigation of static and dynamic behavior of composite shells very important. Studies of reinforced composite cylindrical shells were published by Thielemann (1960), Block (1968), Bogdanovich and Koshkina (1983, 1984), Bogdanovich (1986), and Birman (1988a,b, 1990).

Composite shells used in practical applications are often subject to uniform or nonuniform thermal fields. The research of effects of temperature on response of isotropic structures has been pursued for a long time; see the book of Boley and Weiner (1962) which outlines the principle elements of this research and the paper of Tauchert (1986) concentrating on plated structures. The response of composite material structures subject to thermal effects has been also investigated. In particular, thermal problems of composite cylinders were considered by Birger (1971), Tauchert (1980), and Hyer and Cooper (1986). However, the behavior of reinforced composite shells in thermal fields has not been considered.

In this paper geometrically nonlinear dynamic equations for reinforced circular cylindrical shells subject to nonuniform elevated temperature are formulated. These equations represent the generalization of the Donnell shell theory to composite material cylinders. The solution of geometrically nonlinear and linear problems of reinforced composite cylinders subject to instantaneous increase of temperature (thermal shock) is discussed. A method of experimental prediction of the temperature resulting in thermally-induced buckling is proposed. This method can be used for nondestructive evaluation of shells subject to thermal shock if the boundary conditions are not specified. The response of a graphite/epoxy shell is illustrated in numerical analysis which also elucidates the importance of ring stiffeners for prevention of thermally-induced buckling.

Governing Equations

Consider a cylindrical shell reinforced by axial and ring stiffeners as shown in Fig. 1. The shell is composed of many identical layers symmetrically arranged about its middle surface. In this case the behavior of the real shell can be modeled by an equivalent orthotropic shell. The geometry of the shell is such that the Donnell-type theory is applicable. The stiffeners are supposed to work independently; i.e., axial stiffeners do not affect deformations of ring stiffeners and visa versa. Such an assumption is often used in the studies of reinforced shell structures. An example of a structure where this assumption is justified is a shell with riveted reinforcements in both axial and circumferential directions which are not connected to each other (Timashev, 1974). The torsional stiffness of the stiffeners is neglected according to the conclusion of Birman (1988b).

The shell is subject to a nonuniform thermal field $T(x, y, z, t)$, x and y being the axial and circumferential coordinates; z

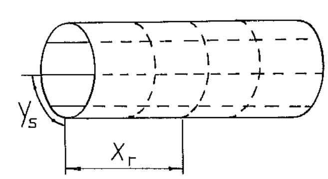

Fig. 1 Shell geometry. Internal stiffeners are indicated by broken lines; x_r and y_s are the coordinates of stiffeners.

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is the radial coordinate positive in the inward direction and t time. The webs and the flanges of the stiffeners are supposed to be thin so that temperature is uniform in the thickness directions of the elements of each profile.

The strain-displacement relationships in Donnell's theory of geometrically nonlinear shells are assumed to be

$$
\begin{aligned}
&\epsilon_x^0 = u_{,x} + 1/2 w_{,x}^2 \\
&\epsilon_y^0 = v_{,y} + 1/2 w_{,y}^2 - \frac{w}{R} \\
&\gamma_{xy}^0 = u_{,y} + v_{,x} + w_{,x} w_{,y} \\
&\kappa_x = -w_{,xx} \quad \kappa_y = -w_{,yy} \quad \kappa_{xy} = -2w_{,xy} \\
&\epsilon_x = \epsilon_x^0 + z\kappa_x \\
&\epsilon_y = \epsilon_y^0 + z\kappa_y \\
&\gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy}
\end{aligned} \tag{1}
$$

where ϵ_x^0, ϵ_y^0, γ_{xy}^0 denote strains in the middle surface, κ_x, κ_y, and κ_{xy} are the changes of the middle surface curvature and twist, and ϵ_x, ϵ_y, γ_{xy} are strains at the distance z from the middle surface. Axial, circumferential, and radial displacements of the middle surface are denoted by u, v, and w, respectively, w being positive in the inward direction, and R is the radius of the middle surface.

The linear thermoelastic orthotropic constitutive relations for the kth layer of the shell are

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \epsilon_x - \alpha_1 T \\ \epsilon_y - \alpha_2 T \\ \gamma_{xy} - \alpha_6 T \end{Bmatrix}_k \tag{2}
$$

where σ_1 and σ_2 are the axial and circumferential stresses, σ_6 is the shearing stress, and Q_{ij} are transformed reduced stiffness of the layer. The thermal expansion coefficients of the layer are related to the principal coefficients in the fiber and transverse directions (α_L, α_T), and to the lamination angle θ_k by

$$
\begin{aligned}
&\alpha_1 = \alpha_L \cos^2\theta_k + \alpha_T \sin^2\theta_k \\
&\alpha_2 = \alpha_L \sin^2\theta_k + \alpha_T \cos^2\theta_k \\
&\alpha_6 = (\alpha_L - \alpha_T)\sin\theta_k \cos\theta_k .
\end{aligned} \tag{3}
$$

If the shell is manufactured from an orthotopic material with the fiber direction parallel to the x-axis, $\alpha_1 = \alpha_L$, $\alpha_2 = \alpha_T$, $\alpha_6 = 0$ in equation (2). The stresses in the stiffeners obtained by the assumption that the stiffeners in the axial and circumferential directions do not affect each other directly are:

$$
\begin{aligned}
&\sigma_1^s = E_s(\epsilon_x - \alpha_s T) \\
&\sigma_2^r = E_r(\epsilon_y - \alpha_r T)
\end{aligned} \tag{4}
$$

where E_s and E_r are the moduli of elasticity of the axial (s) and ring (r) stiffener materials in the respective stiffener length directions, α_s and α_r are the corresponding thermal expansion coefficients.

The stress resultants and stress couples can be obtained by integration of the stresses given by equation (2) and (4) with respect to the thickness (radial) coordinate:

$$
\begin{aligned}
N_1 &= A_{11}\epsilon_x{}^0 + A_{12}\epsilon_y{}^0 - N_1{}^T + \sum_s \delta(y - y_s) E_s A_s (\epsilon_x^0 + z_s \kappa_x) \\
N_2 &= A_{12}\epsilon_x^0 + A_{22}\epsilon_0^y - N_2^T + \sum_r \delta(x - x_r) E_r A_r (\epsilon_y^0 + z_r \kappa_y) \\
N_6 &= A_{66}\gamma_{xy}{}^0 - N_6^T \\
M_1 &= D_{11}\kappa_x + D_{12}\kappa_y - M_1^T + \sum_s \delta(y - y_s) E_s (A_s z_s \epsilon_x^0 + I_{os}\kappa_x) \\
M_2 &= D_{12}\kappa_x + D_{22}\kappa_y - M_2{}^T + \sum_r \delta(x - x_r) E_r (A_r z_r \epsilon_y^0 + I_{or}\kappa_y) \\
M_6 &= D_{66}\kappa_{xy} - M_6^T .
\end{aligned} \tag{5}
$$

In equations (5), $\delta(\ldots)$ is the Dirac delta function, y_s and x_r are the coordinates of the axial and ring stiffeners, respectively, A_{ij} and D_{ij} are extensional and bending stiffnesses of the shell, z_s and z_r are the distances from the centroids of the corresponding stiffeners to the middle surface positive for inside reinforcement, A_s and A_r are the cross-sectional areas of the respective stiffeners, and I_{os}, I_{or} are their moments of inertia with respect to the middle surface. The thermal terms in equation (5) are:

$$
\begin{aligned}
N_1^T &= \int_{-h/2}^{h/2} (Q_{11}\alpha_1 + Q_{12}\alpha_2 + Q_{16}\alpha_6) T dz + \sum_s \delta(y - y_s) E_s \alpha_s \int_z \beta_s(z) T_s dz \\
N_2^T &= \int_{-h/2}^{h/2} (Q_{12}\alpha_1 + Q_{22}\alpha_2 + Q_{26}\alpha_6) T dz + \sum_s \delta(x - x_r) E_r \alpha_r \int_z \beta_r(z) T_r dz \\
N_6^T &= \int_{-h/2}^{h/2} (Q_{16}\alpha_1 + Q_{26}\alpha_2 + Q_{66}\alpha_6) T dz \\
M_1^T &= \int_{-h/2}^{h/2} (Q_{11}\alpha_1 + Q_{12}\alpha_2 + Q_{16}\alpha_6) T z dz + \sum_s \delta(y - y_s) E_s \alpha_s \int_z \beta_s(z) T_s z dz \\
M_2^T &= \int_{-h/2}^{h/2} (Q_{12}\alpha_1 + Q_{22}\alpha_2 + Q_{26}\alpha_6) T z dz + \sum_r \delta(x - x_r) E_r \alpha_r \int_z \beta_r(z) T_r z dz \\
M_6^T &= \int_{-h/2}^{h/2} (Q_{16}\alpha_1 + Q_{26}\alpha_2 + Q_{66}\alpha_6) T z dz
\end{aligned} \tag{6}
$$

where T_s and T_r denote temperature distributions in the stiffeners, $\beta_s(z)$, $\beta_r(z)$ are the widths of the stiffeners, and h is the thickness of the shell.

If the temperature is constant throughout the stiffener cross-sections, the corresponding terms in equation (6) are simplified as follows:

$$
\begin{aligned}
&\sum_s \delta(y - y_s) E_s \alpha_s \int_z \beta_s(z) T_s dz \rightarrow \sum_s \delta(y - y_s) E_s A_s \alpha_s T_s \\
&\sum_r \delta(x - x_r) E_r \alpha_r \int_z \beta_r(z) T_r dz \rightarrow \sum_r \delta(x - x_r) E_r A_r \alpha_r T_r \\
&\sum_s \delta(y - y_s) E_s \alpha_s \int_z \beta_s(z) T_s z dz \rightarrow \sum_s \delta(y - y_s) E_s F_s \alpha_s T_s \\
&\sum_s \delta(x - x_r) E_r \alpha_r \int_z \beta_r(z) T_r z dz \rightarrow \sum_r \delta(x - x_r) E_r F_r \alpha_r T_r
\end{aligned} \tag{7}
$$

where F_s and F_r are the first moments of the respective stiffeners about the shell middle surface. Note that the term N_6^T is equal to zero if temperature is independent on the z-coordinate.

Equations of motion, in terms of stress resultants and stress couples, are

$$
\begin{aligned}
&N_{1,x} + N_{6,y} = 0 \\
&N_{6,x} + N_{2,y} = 0 \\
&M_{1,xx} + 2M_{6,xy} + M_{2,yy} \frac{N_2}{R} + (N_1 w_{,x} + N_6 w_{,y})_{,x} \\
&\qquad + (N_6 w_{,x} + N_2 w_{,y})_{,y} = \rho_{eq} h w_{,tt}
\end{aligned} \tag{8}
$$

where

$$
\rho_{eq} h = \rho h + \sum_s \delta(y - y_s)\rho_s A_s + \sum_r \delta(x - x_r)\rho_r A_r . \tag{9}
$$

In equation (9), ρ, ρ_s, and ρ_r are the mass densities of the materials of the shell, axial, and ring stiffeners, respectively.

The substitution of equations (5) and (6) into (8) and neglect of the in-surface inertias yield the following equations of motion in terms of displacements:

$$[A_{11}+\sum_s \delta(y-y_s)E_sA_s]u_{,xx}+A_{66}u_{,yy}+(A_{12}+A_{66})v_{,xy}$$
$$-A_{12}w_{,x}/R-\sum_s \delta(y-y_s)E_sA_sz_sw_{,xxx}$$
$$+[A_{11}+\sum_s \delta(y-y_s)E_sA_s]w_{,x}w_{,xx}$$
$$+(A_{12}+A_{66})w_{,y}w_{,xy}+A_{66}w_{,x}w_{,yy}=N^T_{1,x}+N^T_{6,y}$$
$$(A_{12}+A_{66})u_{,xy}+[A_{22}+\sum_r \delta(x-x_r)E_rA_r]v_{,yy}+A_{66}v_{,xx}$$
$$-[A_{22}+\sum_r \delta(x-x_r)E_rA_r]w_{,y}/R-\sum_r \delta(x-x_r)E_rA_rz_rw_{,yyy}$$
$$+[A_{22}+\sum_r \delta(x-x_r)E_rA_r]w_{,y}w_{,yy}$$
$$+(A_{12}+A_{66})w_{,x}w_{,xy}+A_{66}w_{,xx}w_{,y}=N^T_{6,x}+N^T_{2,y}$$
$$-D_{11}w_{,xxxx}-2(D_{12}+2D_{66})w_{,xxyy}-D_{22}w_{,yyyy}$$
$$+\sum_s \delta(y-y_s)E_s\{A_sz_s[u_{,xxx}+w_{,x}w_{,xxx}$$
$$+w^2_{,xx}]-I_{os}w_{,xxxx}\}+\sum_r \delta(x-x_r)E_r\{A_rz_r[v_{,yyy}$$
$$-w_{,yy}/R+w_{,y}w_{,yyy}+w^2_{,yy}]$$
$$-I_{or}w_{,yyyy}\}+A_{12}(u_{,x}+1/2w^2_{,x})/R$$
$$+[A_{22}+\sum_r \delta(x-x_r)E_rA_r](v_{,y}-w/R+1/2w^2_{,y})/R$$
$$-\sum_r \delta(x-x_r)E_rA_rz_rw_{,yy}/R+\{\{[A_{11}+\sum_s \delta(y-y_s)E_sA_s]$$
$$(u_{,x}+1/2w^2_{,x})+A_{12}(v_{,y}-w/R+1/2w^2_{,y})$$
$$-\sum_s \delta(y-y_s)E_sA_sz_sw_{,xx}\}w_{,x}+A_{66}(u_{,y}+v_{,x}+w_{,x}w_{,y})w_{,y}\}_{,x}$$
$$+\{A_{66}(u_{,y}+v_{,x}+w_{,x}w_{,y})w_{,x}$$
$$+\{A_{12}(u_{,x}+1/2w^2_{,x})+[A_{22}$$
$$+\sum_r \delta(x-x_r)E_rA_r](v_{,y}-w/R+1/2w^2_{,y})$$
$$-\sum_r \delta(x-x_r)E_rA_rz_rw_{yy}\}w_{,y}\}_{,y}=M^T_{1,xx}+2M^T_{6,xy}$$
$$+M^T_{2,yy}+N^T_2/R+(N^T_1w_{,x}$$
$$+N^T_6w_{,y})_{,x}+(N^T_6w_{,x}+N^T_2w_{,y})_{,y}+\rho_{eq}hw_{,tt}. \qquad (10)$$

As follows from equations (10), the shell will experience radial deflections when subject to a nonuniform temperature even if other loads are absent. This conclusion applies to both linear and nonlinear problems. Even if the distribution of temperature is uniform, the shell will bend in the radial direction; this follows from the presence of the term N^T_2/R on the right side of the last equation (10).

Thermal Shock-Geometrically Nonlinear Problem

If the temperature increases very rapidly so that the thermal ramp occurs over a small time compared to the fundamental period of the structure, it is convenient to treat these temperature increases as instantaneous. Consider the case where the temperature is represented by the Heaviside step function, i.e., it increases to a certain level and remains constant after that. Shells subject to such thermal fields may experience dynamic deformations. Suppose that displacements u, v, and w can be expressed as single-term functions of the in-surface coordinates:

$$u=U(t)f_1(x,y)$$
$$v=V(t)f_2(x,y)$$
$$w=W(t)f_3(x,y) \qquad (11)$$

where $f_i(x, y)$ satisfy boundary and periodicity conditions. Then the substitution of equations (11) into (10) and the application of the Galerkin procedure yield a set of nonlinear equations for the functions $U(t)$, $V(t)$, and $W(t)$. The first two of these functions can be expressed in terms of $W(t)$ from the first two equations of the set. Then the remaining equation, which corresponds to the last equation in equations (10), can be represented in terms of $W(t)$ only:

$$W_{,tt}+(\beta_1+\beta_1^1T_1)W+\beta_2W^2+\beta_3W^3+\beta_0T_1=0, \qquad (12)$$

where β_i and β_1^1 are coefficients and T_1 is a characteristic of the thermal field. An example of the coefficients in equation (12) is shown in the Appendix. As follows from equation (12), the shell may exhibit two types of behavior. If $\beta_1+\beta_1^1T_1>0$ the shell will oscillate periodically while if $\beta_1+\beta_1^1T_1<0$ the behavior is aperiodic, and is referred to as a thermal instability phenomenon.

In the following the analysis is concerned with limited variations of temperature so that the resulting motion is oscillatory. The exact solution of equation (12) can be sought in terms of elliptic integrals and functions similar to the approach used in the problems of free vibrations of perfect and imperfect bars (Woinowsky-Krieger, 1950; Burgreen, 1950; Elishakoff et al., 1985; Birman 1986). However, note that in these references the solution was simplified for the case where $\beta_2=0$.

Multiplication of equation (12) by dW and the subsequent integration yield

$$W_{,t}^{\,2}=-(\beta_1+\beta_1^1T_1)W^2$$
$$-2/3\beta_2W^3-1/2\beta_3W^4-2\beta_0T_1+C \qquad (13)$$

where C is a constant found from the condition that $W_{,t}=0$ at $W=W_{max}$. Then substituting the value of C into equation (13), separating variables and integrating, one obtains

$$t-t_0=\int_{\bar{W}0}^{\bar{W}}\frac{d\bar{W}}{\sqrt{(\beta_1+\beta_1^1T_1)(1-\bar{W}^2)+2/3\beta_2W_{max}(1-\bar{W}^3)+1/2\beta_3W^2_{max}(1-\bar{W}^4)+2(\beta_0/W_{max})T_1(1-\bar{W})}} \qquad (14)$$

where

$$\bar{W}=W/W_{max} \text{ and } \bar{W}_0=\bar{W}(t_0). \qquad (15)$$

The right side of equation (14) can often be represented as a tabulated elliptic integral. Then $\bar{W}$ can be found as an elliptic function of time. The value W_{max} should be specified using the initial conditions.

Note that the scenario discussed previously involves laborious transformations. Moreover, the value of W_{max} is necessary for evaluation of the integral in equation (14). However, in the solution just described, W_{max} can be specified only after $\bar{W}(t)$ has already been determined. Hence, the solution must be iterative.

Therefore, an approximate solution of equation (12) may be quite attractive. For example, the difference between the exact solution for the frequencies of free vibration of imperfect elastic bars and a timewise, two-term Galerkin approximation was shown to be less than 5 percent (Elishakoff et al. 1985). The approach similar to that used by Elishakoff et al. (1985) is used in this paper. Assuming

$$W=W_0+W_1\cos\omega t \qquad (16)$$

and applying the Galerkin procedure to equation (12), one obtains

$$-\omega^2 W_1 + (\beta_1+\beta_1{}^1 T_1) W_1 + 2\beta_2 W_0 W_1 + 3\beta_3 W_0^2 W_1 + 3/4\beta_3 W_1^3 = 0$$

$$(\beta_1+\beta_1^1 T_1) W_0 + \beta_2 W_0^2 + \beta_3 W_0^3 + 1/2\beta_2 W_1^2 + 3/2\beta_3 W_0 W_1^2 + \beta_0 T_1 = 0. \qquad (17)$$

Equations (17) include three unknowns: W_0, W_1 and ω. If the shell is at rest at the instant of thermal shock,

$$W(0)=0 \qquad W_{,t}(0)=0. \qquad (18)$$

The substitution of equation (16) in the first equation (18) yields $W_0 = -W_1$, the second condition (18) is identically satisfied. Then the relationship between W_1 and T_1 as well as the corresponding frequency of oscillations can be determined from equation (17).

Thermal Shock-Geometrically Linear Problem

If displacements remain small so that nonlinear effects can be neglected, the equation of motion equation (12) is reduced to

$$W_{,tt} + (\beta_1+\beta_1{}^1 T_1) W + \beta_0 T_1 = 0 \qquad (19)$$

with the obvious solution

$$W = A\sin\tilde{\omega}t + B\cos\tilde{\omega}t - \frac{\beta_0 T_1}{\beta_1+\beta_1{}^1 T_1} \qquad (20)$$

where $\tilde{\omega}=\sqrt{\beta_1+\beta_1{}^1 T_1}$.
Initial conditions (18) yield

$$A=0,\ B=\frac{\beta_0 T_1}{\beta_1+\beta_1{}^1 T_1} \qquad (21)$$

Note that β_1 is proportional to the linear buckling load of the shell subject to axial compression, say $\beta_1 = N_{cr}\psi$, where N_{cr} is the buckling load and ψ is a coefficient. Then introducing a nondimensional time parameter τ so that $(\ldots)_{,tt} = (\ldots)_{,\tau\tau}\omega_0^2$ where $\omega_0=\sqrt{\beta_1}$ is the natural frequency of the shell at the room temperature, one obtains from equation (19):

$$\tilde{W}_{,\tau\tau} + \left(1+\frac{\beta_1{}^1 T_i}{\psi N_{cr}}\right)\tilde{W} = -\frac{\beta_0 T_1}{\beta_1 h} \qquad (22)$$

where $\tilde{W} = W/h$.

Equation (22) includes only nondimensional parameters. Denote

$$\frac{\beta_1{}^1 T_1}{\psi N_{cr}} = \lambda_1 \bar{N}_1{}^T \qquad \frac{\beta_0 T_1}{\beta_1 h} = \lambda_2 \bar{N}_1{}^T \qquad (23)$$

where λ_1 and λ_2 are coefficients and $\bar{N}_1{}^T = N_1{}^T/N_{cr}$.

Then equation (22) yields

$$\tilde{W}_{,\tau\tau} + (1+\lambda_1\bar{N}_1{}^T)\tilde{W} = -\lambda_2\bar{N}_1{}^T. \qquad (24)$$

The solution of equation (24) subject to zero initial condition is

$$\tilde{W} = \frac{\lambda_2\bar{N}_1{}^T}{1+\lambda_1\bar{N}_1{}^T}\left(\cos\sqrt{1+\lambda_1\bar{N}_1{}^T}\tau - 1\right). \qquad (25)$$

The validity of this solution is limited to the values of $\bar{N}_1{}^T$ which satisfy the inequality $1+\lambda_1\bar{N}_1{}^T > 0$. The convenience of equation (25) is that it enables us to determine the amplitude of motion as a function of a dimensionless thermal load $\bar{N}_1{}^T$ and two nondimensional parameters which can be easily evaluated in particular cases. For example, if the ends of the shell are simply supported so that

$$w = \sum_{mn} W_{mn}(t)\sin\frac{m\pi x}{L}\sin\frac{ny}{R}, \qquad (26)$$

and the thermal field is uniform, i.e., $T(x,y,z) = T_r = T_s = T$,

$$\rho_{eq} h\psi = \left(\frac{m\pi}{L}\right)^2 \qquad \rho_{eq} h\beta_0 T_1 = \frac{N_2{}^T}{R}$$

$$\rho_{eq} h\beta_1{}^1 T_1 = -\left(\frac{m\pi}{L}\right)^2\left[1+\left(\frac{\bar{L}n}{m\pi}\right)^2 n_2{}^T\right] N_1{}^T \qquad (27)$$

where

$$\bar{L} = \frac{L}{R} \qquad n_2{}^T = \frac{N_2{}^T}{N_1{}^T}. \qquad (28)$$

Then

$$\lambda_1 = -\left[1+\left(\frac{\bar{L}n}{m\pi}\right)^2 n_2{}^T\right]$$

$$\lambda_2 = \frac{n_2{}^T\bar{L}}{(m\pi)^2\bar{h}} \qquad (29)$$

where $\bar{h} = \frac{h}{L}$.

Note that the relationship $\tilde{W}(\bar{N}_1{}^T)$ can be obtained even without specifying the in-surface boundary conditions. The change of the response from steady-state oscillations to a periodic motion, which can be conveniently called thermally-induced buckling, occurs if

$$\bar{N}_1{}^T > \frac{1}{1+\left(\frac{\bar{L}n}{m\pi}\right)^2 n_2{}^T}. \qquad (30)$$

Therefore, even if in-surface boundary conditions are not known but the axial buckling load the corresponding buckling mode shape are determined from experiments, the value of temperature corresponding to thermal buckling can be cal-

Table 1 Critical temperatures and nondimensional buckling loads of shells

Type of Reinforcement	Thermally induced buckling: Critical temperature T K	Mode shape m	n	Buckling due to axial loading: Critical load $\|N_{cr}/E_t h\|$	Mode shape m	n
Stiffeners in both directions	2110	1	2	0.41	3	3
Ring Stiffeners	1152	12	3	6.95×10^{-2}	12	3
Axial Stiffeners	60	1	4	6.01×10^{-2}	1	4
Unstiffened shell	56	1	4	4.60×10^{-2}	11	2

Table 2 Comparison of the amplitudes of harmonics of oscillations of the shell reinforced in both directions $|100\ W_{mn}/h|$; $T=100$ K

m	n					
	2	3	4	5	6	7
1	50.30	16.51	5.87	2.36	1.16	0.62
2	15.38	13.69	5.24	2.37	1.14	0.63
3	7.09	8.00	4.41	2.05	1.09	0.60
4	3.55	4.59	2.94	1.67	0.92	0.55
5	2.02	2.36	1.92	1.19	0.75	0.46

Table 3 Comparison of the amplitudes of harmonics of oscillations of the shell reinforced in both directions $|100\ W_{mn}/h|$; $T=200$ K

m	n					
	2	3	4	5	6	7
1	105.86	31.97	11.98	4.66	2.35	1.23
2	30.04	28.38	10.27	4.80	2.36	1.27
3	14.44	15.56	9.01	4.04	2.20	1.20
4	7.01	9.38	5.77	3.39	1.82	1.12
5	4.09	4.65	3.89	2.34	1.51	0.92

culated. Similar conclusions can be obtained using nondestructive vibration testing with the critical load related to the natural frequency by $N_{cr}=\frac{\omega_0^2}{\psi}$.

Numerical Analysis

The calculations were performed for a simply-supported multilayered cylindrical shell symmetrically laminated about the middle surface with the angle ±30 deg. Both the shell and the stiffeners were manufactured from graphite/epoxy AS/3501 which has the following properties at room temperature (Tsai and Hahn, 1980):

$$E_L=138 \text{ GPa},\ E_T=8.96 \text{ GPa},\ G_{LT}=7.1 \text{ GPa},\ \nu_{LT}=0.30,$$
$$\alpha_L=0.18\ \mu\text{m/m/K},\ \alpha_T=22.50\ \mu\text{m/m/K}.$$

It is well known that mechanical properties are functions of temperature. In the problem of thermal shock, temperature increases very rapidly and remains constant thereafter. In the following examples all properties have been taken as indicated previously. This is justified if the temperature increase is moderate and the thermal field is uniform as was assumed in the numerical analysis.

The geometry of the shell was $L=2\ m$, R = 0.5 m, h = 0.01 m, the internal reinforcements in both directions were 0.05-m high and 0.01-m wide with the spacing 0.1 m (the spacing of the axial reinforcements should be 0.1013 m to provide 31 stiffeners). The fiber directions coincided with the axes of the corresponding stiffeners.

Critical temperatures corresponding to thermally-induced buckling, i.e., to the change of dynamic response character of the shell were calculated for the shells whose axial buckling loads have been determined in the previous paper (Birman, 1988b). The results of calculations are summarized in Table 1. As follows from this table the mode shape of thermally-induced buckling does not necessarily coincide with the mode shape of buckling due to axial loading, although they can be close. The presence of ring stiffeners appears to have a very significant effect on the response to the thermal shock. The shells, even reinforced in the axial direction, experience thermally-induced buckling at very small elevations of temperature. On the contrary, ring-reinforced shells exhibit stable oscillatory response at very high temperatures. Note that thermally-induced buckling does not necessarily mean failure of the shell. However, it results in large deflections and the response has to be analyzed using nonlinear theory.

The results obtained for high temperatures are unreliable since the properties of material will be different at such temperatures. However, the conclusion that can be drawn from Table 1, i.e., that the shells will exhibit steady-state oscillations even if the temperature "jumps" to several hundred degrees, is very important.

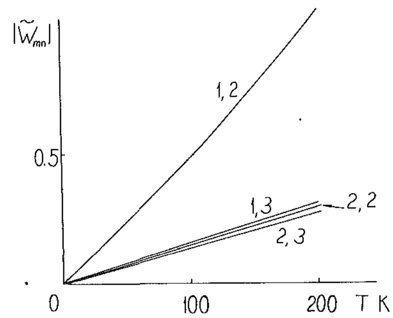

Fig. 2 Increase of the amplitudes of harmonics of oscillations of the shell reinforced in both directions with temperature. The numbers m and n are shown at the corresponding curves.

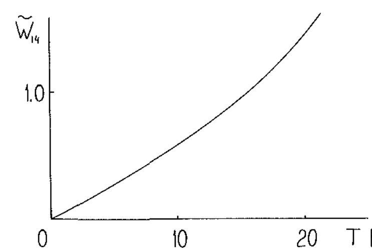

Fig. 3 Increase of the amplitude of the dominant harmonic of oscillations of the shell reinforced in the axial direction only with temperature

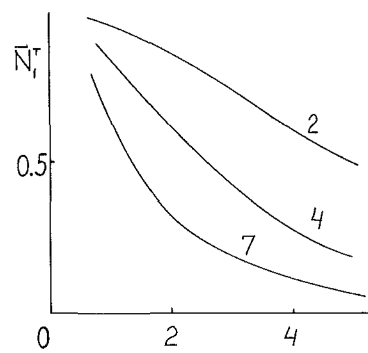

Fig. 4 Nondimensional critical thermal loads of reinforced shells; $n_2^T=0.1$

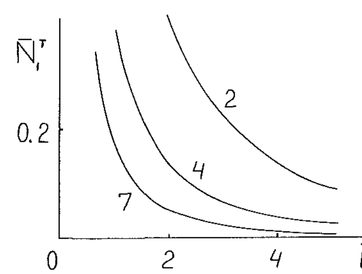

Fig. 5 Nondimensional critical thermal loads of reinforced shells; $n_2^T=1.0$

The comparison of the amplitudes of harmonics in the series (26) representing radial oscillations of the shell is given in Tables 2 and 3 for different temperatures. The comparison of these two tables illustrates that the same harmonics remain dominant at various temperatures. The increase of the amplitudes of these harmonics with temperature is shown in Fig. 2. The rapid increase of the amplitude of the dominant harmonic is shown in Fig. 3 for the axially-reinforced shell. Finally, variations of the nondimensional critical thermal load with the ratio $\bar{L} = L/R$ are shown in Figs. 4 and 5. These figures illustrate that the ratio of the thermal term N_1^T corresponding to thermally-induced buckling to the critical axial load associated with the same mode shape decreases for longer shells.

Conclusions

The analysis presented in the paper illustrates that an elevated temperature results in bending deflections of reinforced circular cylindrical shells manufactured from composite materials. Naturally, this conclusion remains valid for isotropic shells as well. If a shell is subject to an instantaneous increase of temperature (thermal shock) it can exhibit steady-state oscillations. However, if the temperature exceeds a certain critical level, associated with thermally-induced buckling, the character of response changes and the deflections can increase dramatically.

Shells reinforced in the circumferential direction have much higher temperature corresponding to thermally-induced buckling than shells without such reinforcements. On the contrary, axial stiffening does not show a significant effect on the thermally-induced buckling temperature. However, it should be emphasized that the conclusions regarding relative effectiveness of stiffeners were obtained for a particular material and boundary conditions; shells manufactured from other composites may behave differently. The mode shape of thermally-induced buckling does not always coincide with the mode shape of buckling due to axial loading. However, in the examples considered here these mode shapes were remarkably similar for reinforced shells. The thermally-induced buckling of shells subjected to uniform temperature can be predicted based on the static buckling loads in axial compression and/or the natural frequencies.

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APPENDIX

Coefficients of Equation (12)

The substitution of equations (11) into equations (10) and Galerkin procedure yield a set of three algebraic equations for $U(t)$, $V(t)$, and $W(t)$. These equations can be represented in a nondimensional form:

$$a_1\ddot{\bar{U}} + a_2\bar{V} + a_3\bar{W} + a_4\bar{W}^2 = n_1T_1$$
$$b_1\bar{U} + b_2\bar{V} + b_3\bar{W} + b_4\bar{W}^2 = n_2T_1$$
$$c_1\bar{W} + c_2\bar{U} + c_3\bar{V} + c_4\bar{W}^2 + c_5\bar{W}^3 + c_6\bar{U}\bar{W} + c_7\bar{V}\bar{W} = n_3t_1 + n_4\bar{W}T_1 + \bar{\rho}\,\bar{W}_{,\tau\tau} \quad \text{(A1)}$$

where

$$\{\bar{U},\ \bar{V},\ \bar{W}\} = \{U(t),\ V(t),\ W(t)\}/h. \quad \text{(A2)}$$

T_1 is a dimensionless characteristic of the thermal field, $T_1 = \alpha T_1^0$, where α is an arbitrary thermal expansion coefficient and T_1^0 is temperature. A nondimensional time scale τ is introduced by

$$\tau = \omega t, \quad \text{(A3)}$$

ω being a normalizing frequency which can be chosen arbitrarily. Accordingly,

$$\bar{\rho} = \frac{\rho_{eq}h^2\omega^2}{E} \quad \text{(A4)}$$

where E is a reference modulus of elasticity.

The values of $\bar{U}$ and $\bar{V}$ can be evaluated from the first two equations (A1). The substitution of these values into the third equation (A1) yields a nondimensional version of equation (12):

$$\bar{W}_{,\tau\tau} + (\bar{\beta}_1 + \bar{\beta}_1{}^1T_1)\bar{W} + \bar{\beta}_2\bar{W}^2 + \bar{\beta}_3\bar{W}^3 + \bar{\beta}_0T_1 = 0 \quad \text{(A5)}$$

where

$$\begin{aligned}
\bar{\rho}\bar{\beta}_0 &= n_3 - (c_2s_3 + c_3s_6)\\
\bar{\rho}\bar{\beta}_1 &= -(c_1 + c_1s_1 + c_3s_4)\\
\bar{\rho}\bar{\beta}_1{}^1 &= n_4 - (c_6s_3 + c_7s_6)\\
\bar{\rho}\bar{\beta}_2 &= -(c_1s_2 + c_3s_5 + c_4 + c_6s_1 + c_7s_4)\\
\bar{\rho}\bar{\beta}_3 &= -(c_6s_2 + c_7s_5 + c_5).
\end{aligned} \quad \text{(A6)}$$

In equation (A6)

$$\begin{aligned}
s_1 &= (b_3a_2 - b_2a_3)/s \qquad & s_2 &= (a_2b_4 - a_4b_2)/s\\
s_3 &= (b_2n_1 - a_2n_2)/s \qquad & s_4 &= (a_3b_1 - a_1b_3)/s\\
s_5 &= (a_4b_1 - a_1b_4)/s \qquad & s_6 &= (a_1n_2 - b_1n_1)/s\\
s &= a_1b_2 - a_2b_1. & &
\end{aligned} \quad \text{(A7)}$$

The coefficients of equation (12) can be expressed in terms of the coefficients of equation (A5) as follows:

$$\beta_0=\bar{\beta}_0\omega^2 h$$
$$\{\beta_1, \beta_1{}^1\}=\{\bar{\beta}_1, \bar{\beta}_1{}^1\}\omega^2$$
$$\beta_2=\bar{\beta}_2\omega^2/h$$
$$\beta_3=\bar{\beta}_3\omega^2/h^2. \quad \text{(A8)}$$

Particular Case: Thermal Shock of a Plate Reinforced in One Direction

Consider a simply-supported plate reinforced in the x-direction. The tangential movements of the edges are restricted but the in-plane displacements in the directions normal to the edges are permitted. Therefore, the boundary conditions are:

$$x=0,\ x=a:\ w=v=M_1=N_1=0$$
$$y=0,\ y=b:\ w=u=M_2=N_2=0 \quad \text{(A9)}$$

where a and b are the lengths of the edges in the x and y directions, respectively.

The thermal field is represented by

$$T(x, y, t)=T_1{}^0(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \text{(A10)}$$

m and n being integers.

Thermal terms corresponding to the thermal field equation (A10) can be written as

$$\{N_1{}^T, N_2{}^T, N_6{}^T, M_1{}^T, M_2{}^T, M_6{}^T\}T_1$$
$$=\{\bar{n}_1{}^T, \bar{n}_2{}^T, \bar{n}_6{}^T, \bar{m}_1{}^T, \bar{m}_2{}^T, \bar{m}_6{}^T\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad \text{(A11)}$$

where $\bar{n}_1{}^T$, $\bar{n}_2{}^T$, . . . are coefficients which can be easily evaluated using equation (6).

Boundary conditions (A9) can be satisfied if the displacements are given by

$$u = U(t)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$v = V(t)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$
$$w = W(t)\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{b}. \quad \text{(A12)}$$

The substitution of equations (A11) and (A12) into the equations of motion (10), the Galerkin procedure, and some transformations yield the equation (A5). The coefficients in equations of the set of equations (A1) used to evaluate $\bar{\beta}_1$ and $\bar{\beta}_1{}^1$ are:

$$a_1 = -(m\pi h_a)^2\left[\bar{A}_{11}+\left(\frac{\lambda n}{m}\right)^2\bar{A}_{66}\right] - 2(m\pi h_a)^2 h_a\lambda\sum_s \bar{E}_s\bar{A}_s\sin^2\frac{n\pi y_s}{b}$$

$$a_2 = -mn\pi^2 h_a{}^2\lambda(\bar{A}_{12}+\bar{A}_{66})$$

$$a_3 = 2(m\pi h_a)^3 h_a\lambda\sum_s \bar{E}_s\bar{A}_s\bar{z}_s\sin^2\frac{n\pi y_s}{b}$$

$$a_4 = \frac{16}{9mn\pi^2}[-2(m\pi h_a)^3\bar{A}_{11} + mn^2\pi^3 h_a{}^3\lambda^2(\bar{A}_{12}-\bar{A}_{66})]f(m)f(n) - \frac{8h_a\lambda}{3m\pi}(m\pi h_a)^3\sum_s \bar{E}_s\bar{A}_s\sin^3\frac{n\pi y_s}{b}f(m)$$

$$n_1 = m\pi h_a\frac{\bar{n}_1{}^T}{Eh}$$

$$n_2 = n\pi h_a\lambda\frac{\bar{n}_2{}^T}{Eh}$$

$$c_1 = -[(m\pi h_a)^4\bar{D}_{11}+2(m\pi h_a)^2(n\pi h_a\lambda)^2(\bar{D}_{12}+2\bar{D}_{66}) + (n\pi h_a\lambda)^4\bar{D}_{22}] - 2\sum_s (m\pi h_a)^4\bar{E}_s\bar{I}_{0s}\sin^2\frac{n\pi y_s}{b}$$

$$c_2 = (m\pi h_a)^3\sum_s \bar{E}_s\bar{A}_s\bar{z}_s h_a\lambda\sin^2\frac{n\pi y_s}{b}$$

$$c_3 = 0$$

$$c_4 = 0$$

$$c_5 = -\left[\frac{9}{32}(m\pi h_a)^4\bar{A}_{11}+\frac{1}{8}(m\pi h_a)^2(n\pi h_a\lambda)^2\left(\frac{\bar{A}_{12}}{2}+\bar{A}_{66}\right) + \frac{9}{32}(n\pi h_a\lambda)^4\bar{A}_{22}\right] - \frac{1}{4}(m\pi h_a)^4 h_a\lambda\sum_s \bar{E}_s\bar{A}_s\sin^2\frac{n\pi y_s}{b}$$

$$c_6 = \frac{32(m\pi h_a)^3}{9mn\pi^2}\left[\bar{A}_{11} + \left(\frac{n\lambda}{m}\right)^2(\bar{A}_{12}-\bar{A}_{66})\right]f(m)f(n) - \frac{8}{3}(m\pi h_a)^3\frac{h_a\lambda}{m\pi}\sum_s \bar{E}_s\bar{A}_s\sin^3\frac{n\pi y_s}{b}f(m)$$

$$c_7 = \frac{32(n\pi h_a\lambda)^3}{9mn\pi^2}\left[\bar{A}_{22} + \left(\frac{m}{n\lambda}\right)^2(\bar{A}_{12}-\bar{A}_{66})\right]f(m)f(n)$$

$$n_3 = -(m\pi h_a)^2\left[\frac{\bar{m}_1{}^T}{Eh^2}+\left(\frac{n\lambda}{m}\right)^2\frac{\bar{m}_2{}^T}{Eh^2}\right]$$

$$n_4 = -\frac{32}{9}(m\pi h_a)^2\left[\frac{\bar{n}_1{}^T}{Eh}+\left(\frac{n\lambda}{m}\right)^2\frac{\bar{n}_2{}^T}{Eh}\right]f(m)f(n). \quad \text{(A13)}$$

The coefficients b_i can be obtained from a_i by the following rotation of symbols:

$$m\Leftrightarrow n,\ \bar{A}_{11}\rightarrow\bar{A}_{22},\ \sum_s=0,\ h_a\rightarrow h_a\lambda,\ h_a\lambda\rightarrow h_a.$$

Additional notation used in equation (A13) is:

$$\bar{A}_{ij} = A_{ij}/Eh \qquad \bar{D}_{ij}={}_{ij}/Eh^3$$
$$h_a = h/a \qquad \lambda=a/b$$
$$\bar{E}_s = E_s/E \qquad \bar{A}_s=A_s/h^2$$
$$\bar{z}_s = z_s/h \qquad \bar{I}_{0s}=I_{0s}/bh^3$$
$$f(i)=\begin{cases}0 \text{ if } i=\text{even}\\ 1 \text{ if } i=\text{odd}.\end{cases} \quad \text{(A14)}$$