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Extension Of Vlasov's Semi-membrane Theory To Reinforced Composite Shells

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BRIEF NOTES

Figure 1 shows the spectrum for a Duffing oscillator with $g(x, \nu) = \alpha x^3$. As the excitation level increases the resonance frequency increases, and the width of the resonance peak also increases. This is in keeping with the result of Miles (1989). An increase in the response level causes an increase in the effective resonance frequency; a random response contains all amplitudes each with its effective resonance frequency and contributes to the broadening of the peak.

Figure 2 shows the spectrum for a van der Pol oscillator with $g(x, \nu) = \alpha x^3 \nu$ and $\beta = -\alpha$ in Eq. (2). In the case of sinusoidal excitation with a frequency close to one, at low excitation levels, the response has components at the excitation frequency and at the entrained free-oscillation frequency of one. As the excitation amplitude is increased beyond a critical value the free-oscillation decays. Fig. 2 shows for the random case a large peak at a frequency $\nu = 1$ for small excitation levels. The peak broadens considerably and flattens as the excitation level is increased suggesting that the free-oscillation component is also partly quenched in the random case.

Conclusion

Some results have been obtained for the spectrum of the response of nonlinear oscillators to white noise excitation. The results are obtained as an extension of previous work by the authors (Liu and Davies, 1988, 1990a) and complement earlier work by Miles (1989) and Wen (1975, 1976).

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Extension of Vlasov's Semi-membrane Theory to Reinforced Composite Shells

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Governing equations for the statics and dynamics of reinforced composite shells are developed based on Vlasov's semi-membrane shell theory. These equations have closed-form solutions

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illustrated for buckling and free vibration problems. The buckling solution converges to the known result for unstiffened isotropic shells.

Introduction

Reinforced composite shells have been a subject of a number of analytical studies. Typically, these studies were based on Donnell-type theory of shells (Block, 1968; Bogdanovich, 1986; Birman, 1988, 1990a, 1990b; Birman and Bert 1990). Donnell's shell theory is usually acceptable, if the axial or circumferential size of deformation waves is small. A comparison of Donnell, Morley, Love, and Sanders shell theories applied to unstiffened composite shells was performed by Bert and Reddy (1982). It was shown that Donnell-type theory yields results, which are in a good agreement with other theories if the radius-to-thickness ratio exceeds 20. However, it is necessary to note that Donnell's shell theory is not appropriate for long shells. In addition, this theory has been used to develop closed-form solutions only for one type of boundary condition.

Vlasov (1944) developed a theory for long isotropic cylindrical shells where stress couples M_x and M_{xy} and the transverse shear stress resultant Q_x are negligible (here, x and y are axial and circumferential coordinates, respectively).

In addition, the middle surface of the shell was assumed inextensible in the circumferential direction, i.e., $\epsilon^y = 0$ and in-surface shearing deformations were neglected ($\gamma_{xy} = 0$).

The theory based on these assumptions is called Vlasov's semi-membrane shell theory. An example of application of this theory to stability problems of isotropic cylindrical shells subject to axial compression can be found in Vol'mir's monograph (1967). Note that Vlasov's semi-membrane theory is based on Love's first approximation shell theory whose particular case it represents.

In this Note, Vlasov's theory is extended to long, reinforced composite cylindrical shells. Obviously, reinforcements should be light and closely spaced to justify the assumptions of the theory.

Governing Equations

Consider a symmetrically laminated cylindrical shell reinforced by axial and circumferential stiffeners. The strain-displacement relationships used in Vlasov's semi-membrane theory are

$$\begin{aligned} \epsilon_x &= u_{,x} & \kappa_x &= -w_{,xx} \\ \epsilon_y &= v_{,y} - \frac{w}{R} = 0 & \kappa_y &= -w_{,yy} - \frac{w}{R^2} \\ \gamma_{xy} &= u_{,y} + v_{,x} = 0 & \kappa_{xy} &= -w_{,xy} - \frac{v_{,x}}{2R} \end{aligned} \quad (1)$$

where all notations are standard and the radial deflection w is positive if directed to the center of curvature. The stresses in the shell and stiffeners can be calculated as functions of strains using Hookean relationships omitted for brevity. The axial stress resultant and the circumferential stress couple used in the analysis are

$$\begin{aligned} N_x &= A_{11}\epsilon_x + \sum_s \delta(y - y_s) E_s A_s (\epsilon_x + z_s \kappa_x) \\ M_y &= D_{12}\kappa_x + D_{22}\kappa_y + \sum_r \delta(x - x_r) E_r I_{or} \kappa_y \end{aligned} \quad (2)$$

where A_{11} is the axial extensional stiffness of the shell, D_{ij} are its bending stiffnesses, A_r and A_s are stiffener cross-sectional areas, x_r and y_s are coordinates of the stiffener centroids, E_s and E_r are the moduli of elasticity of the stiffeners, I_{or} is the moment of inertia of a ring stiffener about the shell middle

surface, and z_s is a distance between the middle surface and the centroid of an axial stiffener positive, if the stiffener is attached to the internal surface of the shell.

Equations of equilibrium or motion are obtained from Love's first-approximation shell theory:

$$\begin{aligned} N_{x,x} + N_{xy,y} &= -q_x \\ N_{xy,x} + N_{y,y} - \frac{1}{R} M_{y,y} &= -q_y \\ M_{y,y} + \frac{N_y}{R} &= -q \end{aligned} \quad (3)$$

where q_x and q_y are in-surface distributed loads and q is an outside pressure.

The compatibility equation is

$$R\epsilon_{x,yyy} + \frac{1}{R} \epsilon_{x,yy} = \kappa_{y,xx} \quad (4)$$

Combining Eqs. (3) and (4) and using (2), one obtains the following differential equation:

$$-\frac{1}{R} \Omega \Omega M_y + \frac{\bar{E}_s}{R^2} \Omega w_{,xxxx} + \frac{\bar{A}_{11}}{R} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) w_{,\alpha\alpha\alpha\alpha} = -R \Omega Q \quad (5)$$

where

$$\begin{aligned} \Omega &= \frac{\partial^4}{\partial \beta^4} + 1 \\ \alpha &= \frac{x}{R} \quad \beta = \frac{y}{R} \\ \bar{A}_{11} &= A_{11} + \sum_s \delta(y - y_s) E_s A_s \\ \bar{E}_s &= \sum_s \delta(y - y_s) E_s A_s z_s \\ Q &= -q_{,\beta\beta} - q_{x,\alpha\alpha} + q_{y,\beta} \end{aligned} \quad (6)$$

The substitution of M_y from (2) into (5) and the exclusion of the operator $(\partial^2/\partial \beta^2 + 1)$ yield

$$\bar{D}_{22} \Omega \Omega w + D_{12} \Omega w_{,\alpha\alpha\beta\beta} + \bar{E}_s R w_{,\alpha\alpha\alpha\alpha\beta\beta} + \bar{A}_{11} R^2 w_{,\alpha\alpha\alpha\alpha} = -R^4 Q_{,\beta\beta} \quad (7)$$

where

$$\bar{D}_{22} = D_{22} + \sum_r \delta(x - x_r) E_r I_{or} \quad (8)$$

In a particular case of an isotropic shell, Eq. (7) reduces to that presented by Vol'mir (1967).

Note that the present theory is applicable in the case of light, closely spaced stiffeners. This justifies the application of the smeared stiffeners technique. Therefore,

$$\begin{aligned} \delta(y - y_s) &= 1/l_s \\ \delta(x - x_r) &= 1/l_r \end{aligned}$$

where l_s and l_r are the spacings of the corresponding stiffeners.

Buckling Problem. If the shell is subject to an axial loading N_1 , $q = -N_1 w_{,xx}$, $q_y = -N_1 v_{,xx}$, and $q_x = 0$. Substituting these expressions into Q given by (6) and using (1), one obtains

$$Q = -\frac{N_1}{R^2} (w_{,\alpha\alpha} - w_{,\alpha\alpha\beta\beta}) \quad (9)$$

Suppose that

$$w = W \sin n\beta, \quad (10)$$

n being an integer. Then (7) yields an ordinary differential equation for W :

$$\begin{aligned} (\bar{A}_{11} R^2 - \bar{E}_s R n^2) W_{,\alpha\alpha\alpha\alpha} + n^2 [(1 - n^2) n^2 D_{12} \\ + (n^2 + 1) N_1 R^2] W_{,\alpha\alpha} + \bar{D}_{22} n^4 (1 - n^2)^2 W = 0. \end{aligned} \quad (11)$$

The integral of (11) includes four constants of integration. If

the ends of the shell are clamped, i.e., $w = w_{,\alpha} = 0$, the substitution of the integral of (11) into the boundary conditions and the nonzero requirement for constants of integration yield the buckling equation. Another type of boundary condition can be formulated, if the shell is supported by equally spaced elastic bulkheads. Then for each span of the shell the boundary conditions are $w_{,\alpha} = 0$, $w(L, 0) = \pm g Q_x(L, 0)$ where g is a bulkhead radial compliance and Q_x is the transverse shear stress resultant. Notably, although Q_x , M_x , and M_{xy} were neglected to develop the governing equation, in reality they exist, although negligible compared to Q_y and M_y . Therefore, Q_x can be expressed in terms of w using Eqs. (1) and

$$\begin{aligned} Q_x &= M_{x,x} + M_{xy,y} \\ M_x &= D_{11} \kappa_x + D_{12} \kappa_y + \sum_s \delta(y - y_s) E_s (A_s z_s \epsilon_x + I_{os} \kappa_x) \\ M_{xy} &= D_{66} \kappa_{xy} \\ u &= U \sin n\beta \quad U = -\frac{1}{n} V_{,x} \\ v &= V \cos n\beta \quad V_{,x} = -2RnW_{,x}. \end{aligned} \quad (12)$$

The expression for the twisting stress couple can be extended to include torsional stiffnesses of reinforcements without significant complication of the analysis.

If the ends are simply supported and unrestricted against axial movements ($N_x = 0$),

$$W = f \sin \lambda \alpha \quad \lambda = \frac{m\pi R}{L} \quad (13)$$

where m is an integer satisfies the boundary conditions. Critical loads obtained from (11) are

$$N_{1cr} = \frac{(\bar{A}_{11} R^2 - \bar{E}_s R n^2) \lambda^4 + n^4 (n^2 - 1) D_{12} \lambda^2 + n^4 (n^2 - 1)^2 \bar{D}_{22}}{n^2 (n^2 + 1) R^2 \lambda^2} \quad (14)$$

The buckling load corresponding to a chosen value of n is obtained from (14) where $\lambda = \bar{\lambda}$ obtained by minimization of N_{1cr} with respect to λ . If the shell is unstiffened and the material is isotropic, these results converge to the solution obtained by Vol'mir (1967).

Vibration Problem. In this problem, $q = -\rho w_{,tt}$, $q_x = -\rho u_{,tt}$, and $q_y = -\rho v_{,tt}$, ρ being the mass per unit area

$$\rho = \bar{\rho} + \sum_s \delta(y - y_s) \rho_s A_s + \sum_r \delta(x - x_r) \rho_r A_r \quad (15)$$

In (15), $\bar{\rho}$ is the mass per unit area of the unstiffened shell and ρ_s , ρ_r are mass densities of stiffener materials. Using

$$w = W e^{i\omega t} \sin n\beta, \quad (16)$$

and smeared stiffeners technique, one obtains a dynamic counterpart of (11):

$$\begin{aligned} (\bar{A}_{11} R^2 - \bar{E}_s R n^2) W_{,\alpha\alpha\alpha\alpha} + [D_{12} n^4 (1 - n^2) + \rho R^4 \omega^2] W_{,\alpha\alpha} \\ + [\bar{D}_{22} n^4 (1 - n^2)^2 - \rho R^4 n^2 (n^2 + 1) \omega^2] W = 0. \end{aligned} \quad (17)$$

If the shell is simply supported and (13) can be used, the corresponding squared frequency is

$$\omega^2 = \frac{(\bar{A}_{11} R^2 - \bar{E}_s R n^2) \lambda^4 + n^4 (n^2 - 1) D_{12} \lambda^2 + n^4 (n^2 - 1)^2 \bar{D}_{22}}{\rho R^4 [\lambda^2 + n^2 (n^2 + 1)]} \quad (18)$$

The integral of (17) can also be subject to other boundary conditions discussed above yielding the frequency equation for these cases.

Concluding Remarks

Important conclusions can be obtained from (14) and (18). Ring stiffeners always increase buckling loads and natural fre-

quencies of semi-membrane cylindrical shells. Axial stiffeners have the same effect in all practically important situations.

Limitations of the semi-membrane theory, i.e., shell and stiffener geometries and material characteristics appropriate for its application can be established by comparison of results (14), (18) with available solutions. It would be preferable to use Love's first-approximation theory for the comparison, since Vlasov's theory represents its particular case. An extensive parametric analysis necessary to formulate these limitation exceeds the scope of this Note.

Vlasov's semi-membrane theory of isotropic shells represents a particular case of the theory developed here. The advantage of the present theory is that it can be used to obtain closed-form solutions for various boundary condition which are not available using other theories of shells.

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Stability of Flow Between Two Rotating Cylinders in the Presence of a Constant Heat Flux at the Outer Cylinder and Radial Temperature Gradient: Narrow Gap Problem

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Introduction

The study of the effects of constant heat flux at the inner cylinder on the stability of flow of a viscous incompressible fluid between two rotating concentric cylinders was presented by Takhar et al. (1988) in the case of a narrow gap. Instead of constant heat flux at the inner cylinder, if there is a constant heat flux at the outer cylinder, how is the stability of flow affected? This question is studied in this paper. All the earlier references on this topic are referred in Takhar et al. (1988).

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Mathematical Analysis For a three-dimensional, axisymmetric, and incompressible viscous flow, and neglecting viscous dissipative heat, the steady state solutions can be shown to be

$$u = w = 0, V = Ar + \frac{B}{r}$$

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, B = \frac{R_1^2 R_2^2 (\Omega_1 - \Omega_2)}{R_2^2 - R_1^2}$$

$$\bar{\theta} = T - T_1 = \frac{qR_2}{K} \ln \frac{r}{R_1} \quad (1)$$

where R_1, R_2 are the radii of the inner and the outer cylinders, respectively. For the velocity field, the usual no-slip boundary conditions are assumed and for the temperature field. For constant heat flux at the outer cylinder and the inner cylinder at temperature T_1 , the boundary conditions are assumed as follows:

$$T = T_1 \text{ at } r = R_1 \text{ and } \frac{dT}{dr} = \frac{q}{K} \text{ at } r = R_2. \quad (2)$$

Here, (u, v, w) are the velocity components in the (r, θ, z) directions, K is the thermal conductivity, and q is the constant heat flux at the outer cylinder.

Following the usual procedure for deriving the differential equations for the marginal state of stability, we can show that these differential equations are as follows for a narrow gap:

$$(D^2 - a^2)^2 u = -a^2 Ta [g(x)v + N \cdot (g(x))^2 \theta] \quad (3)$$

$$(D^2 - a^2)v = u \quad (4)$$

$$(D^2 - a^2)\theta = u \quad (5)$$

with following boundary conditions:

$$u = Du = V = \theta = 0 \text{ at } x = 0$$

$$u = Du = V = D\theta = 0 \text{ at } x = 1. \quad (6)$$

The nondimensional quantities are defined as follows:

$$d = R_2 - R_1, x = \frac{r - R_1}{d}, D = \frac{d}{dx}$$

$$a = \lambda d, \mu = \Omega_2 / \Omega_1, g(x) = 1 - (1 - \mu)x$$

$$Pr = \nu / K, u = \frac{\nu}{2Ad^2} \bar{u}, \theta = \frac{2A}{\Omega_1} \frac{1}{Pr} \left(\frac{K}{qR_2} \right) T$$

$$Ta = -\frac{4A\Omega_1 d^4}{\nu^2}, Ra = \frac{Pr\Omega_1^2 d^4 \left(\frac{qR_2}{K} \right) \alpha}{\nu_2}$$

$$N = -\frac{Pr\alpha(qR_2/K)\Omega_2}{4A} = \frac{Ra}{Ta} \quad (7)$$

Here, Ra is the Rayleigh number, Ta is the Taylor number, Pr is the Prandtl number, and N is the ratio of Ra and Ta . The only difference between the present set of Eqs. (3)-(6) and those of Eqs. (12)-(15) of Takhar et al. (1988) is that the sign of N in Eq. (3) is positive in the present case and here the boundary conditions on θ are interchanged. Thus, we have a two-point boundary value problem defined by Eqs. (3)-(5) with boundary conditions (6) for determining the eigenvalues a_c, Ta_c for given values of μ and N . Here, a_c, Ta_c are the critical values of the wave number a and the Taylor number Ta . Ta_c helps us determine the speeds of the two cylinders in relative motion at which the transition in the fluid-flow takes place from its initial state to its final unstable state with the corresponding a_c which then determines the spacing of the vortices in the axial direction.

Results and Discussion These values of a_c and Ta_c are listed in Table 1 and in order to get the physical insight into the problem, we show the variation of Ta_c in Figs. 1-2. To compare the effect of constant heat flux at the outer cylinder, CHF_0 ,