01 Jan 1987

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Proceedings of the 26th Conference
on Decision and Control
Los Angeles, CA • December 1987

THE STABILITY OF POLYNOMIALS UNDER CORRELATED COEFFICIENT PERTURBATIONS

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ABSTRACT

In this paper the robust stability of polynomials with repect to real parameter variations is investigated. The coefficients of the polynomial are assumed to be linear functions of several real parameters. An algorithm to calculate the maximum allowable variations of the parameters so that the roots still remain in prescribed regions of the complex plane is presented. Examples are given to illustrate the method.

I. INTRODUCTION

In many control applications the design objective is to keep the roots of a certain polynomial in prescribed regions of the complex plane. Typically, the coefficients of the polynomial are functions of one or more parameters. If the roots of the polynomial are to remain inside these prescribed regions, each parameter must have some allowable range of variations from its nominal value. As an example, consider the basic feedback control system of Figure 1, whose characteristic polynomial is given by (1). If the compensator is simply a constant k, i.e. $a_c=1$, $b_c=k$, then one must find the range of values of k so that the roots of p(s) stay in the open left-half plane.

$$p(s) = a_p(s)\, a_c(s) + b_p(s)\, b_c(s) \tag{1}$$

This can be easily accomplished using classical control techniques. Notice that in this example the coefficients of p(s) will be linear functions of k. Or, suppose that the compensator is fixed and given but, due to modeling uncertainty, the coefficients of the plant numerator $n_p(s)$ and denominator $d_p(s)$ vary from their nominal values. Now one must determine how much these coefficients can vary independently, without the roots of p(s) leaving the prescribed region. Notice that the coefficients of p(s) are linear functions of the coefficients of $d_p(s)$ and $n_p(s)$.

Consider a polynomial which has its roots in the open left half-plane for nominal values of its coefficients. The problem of finding intervals, in which the coefficients may independently vary without driving the roots across the imaginary axis, has been solved in [1] for quartics, and in [2] for polynomials of arbitrary degree. Some results and observations made in [3] allow a considerable reduction of computational effort over the method presented in [2]. Both papers make use of Kharitanov's theorem, given in [4].

In this paper it is assumed that each coefficient of the polynomial p(s) is dependent upon a number of real parameters. The object is to find bounds on the allowable variations of these parameters such that the roots of p(s) stay in a prescribed region of the complex plane.

II. DEFINITIONS AND NOTATION

Let **D** be an open region in the complex plane. The term "D-stable" implies that all the roots of the polynomial p(s) are in the region **D**. C^- denotes the open left half-plane. A point on the complex plane is denoted by $s=\sigma+j\omega$. Upper-case letters denote matrices. The standart definition of the infinity-norm of a vector x is given in (2).

$$|x|_\infty = \max_i |x_i| \quad \text{, where } x=(x_1,\ldots,x_n) \tag{2}$$

III. PROBLEM FORMULATION

The polynomial $p(s,\delta)$ is given by (3):

$$p(s,\delta) = a_o(\delta)s^n+a_1(\delta)s^{n-1}+\ldots+a_n(\delta) \tag{3}$$

where $\delta = (\delta_1, \ldots, \delta_m)$ is a vector of real perturbations, and the coefficients of $p(s,\delta)$ are linear in the elements of δ. The problem is to find the largest ϵ such that for all $|\delta_i|<\epsilon$, the roots of $p(s,\delta)$ will remain in the region **D**. The number ϵ can be considered a measure of robust **D**-stability. Since δ_i are perturbations, the problem is meaningless if $p(s,0)$ is not **D**-stable. The inequality $|\delta_i|<\epsilon$ does not imply that all parameters have to be in the same interval. Let $|\hat{\delta}_i|<\epsilon$ imply **D**-stability for (3), where $\hat{\delta}_i$ is defined by

$$\delta_i = \alpha_i\hat{\delta}_i+\beta_i \tag{4}$$

Then the **D**-stability intervals defined by

$$-\alpha_i\epsilon+\beta_i\le\delta_i\le\alpha_i\epsilon+\beta_i \quad , \quad i = 1,\ldots,m \tag{5}$$

will result for the individual perturbations, δ_i . $p(s,\delta)$ will be **D**-stable for any δ satisfying (5).

IV. STABILITY TESTS

The requirement that the poles of p(s) remain in D can be reduced to the requirement that the poles of an associated polynomial $\hat{p}(s)$ remain in C^-. The polynomial $\hat{p}(s)$ is constructed using p(s) and the definition of the specific region, D. The following regions will be considered.

Half-Planes:
Let $\mathbf{D} = \{s{:}\sigma<\sigma_o\}$. Then **D**-stability of p(s) implies C^--stability of

$$\hat{p}(s) = a_o(s+\sigma_o)^n+a_1(s+\sigma_o)^n+\ldots+a_n \tag{6}$$

Circles:
Consider circles symmetric with respect to the real axis, i.e., $\mathbf{D} = \{s{:}\ (\sigma-a)^2+\omega^2<R^2\}$ for any real a, R. Then **D**-stability of p(s) implies C^--stability of

$$\hat{p}(s) = a_o(z+a)^n+a_1(z+a)^{n-1}+\ldots+a_n \tag{7}$$

where

$$z = R\,\frac{s+1}{s-1} \tag{8}$$

Sectors:
These are regions between two rays which are symmetric with respect to the real axis and emanate from the origin. Usually the region of interest is to the left of the two rays, which partition the complex plane into two disjoint regions. $\mathbf{D} = \{s{:}\omega<a\sigma\}\bigcup\{s{:}\omega>-a\sigma\}$ for all real a. **D**-stability of p(s) implies C^--stability of

$$\hat{p}(s) = p(e^{j\theta}s) \tag{9}$$

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Again, sector stability of p(s) will imply C^--stability of $\hat{p}(s)$.

General Regions:

In [5], the regions **D** are defined via the Nyquist curve of certain rational polynomials f(s):

$$f(s) = \frac{g(s)}{h(s)} \tag{10}$$

Some regions, with certain properties depending on f(s), are defined to be strongly admissable. Strongly admissable regions include regions of the complex plane partitioned by the ellipse, hyperbola, parabola, limacon, and other curves of interest. Strongly admissable regions have the property that **D**-stability of p(s) implies C^- stability of $\hat{p}(s)$ in (11), and vice versa.

$$\hat{p}(s) = p(f(s))\cdot(h(s))^n \tag{11}$$

Composite Regions:

More complicated, even disjoint regions can be treated with a combination of the methods above. For example, these regions could be defined by a number of lines and/or sectors. It is also easy to include regions which are rotated and/or translated versions of all the above mentioned regions by replacing s with c_1s+c_2, where c_1 and c_2 are appropriate complex numbers.

V. A INFINITY NORM PROBLEM

The problem of finding absolute value bounds for the perturbations δ_i, so that the roots of the polynomial $p(s,\delta)$ remain in **D**, can be reduced to the case $D=C^-$ in view of section IV.

Definition 1: $D(\omega) = \{\delta\colon p(j\omega,\delta)=0\}$

Definition 2: $R(\epsilon) = \{\delta\colon |\delta_i|<\epsilon\}$

For all δ in $R(\epsilon)$, every root of $p(s,\delta)$ will remain in an open connected region, because the roots of a polynomial continuously depend on the coefficients. The roots will remain finite as long as $a_o(\delta)>0$ in (3). It is not difficult to show that if ϵ in Definition 2 is given by (12), the roots of $p(s,\delta)$ will remain in C^-.

$$\epsilon = \min\{\, [\min_{\omega} \min_{\delta\epsilon D(\omega)} |\delta|_\infty] \,,\, \epsilon_o \,\} \tag{12}$$

ϵ_o is the smallest number such that some member of $R(\epsilon_o)$ will make the leading coefficient of (3) nonpositive.

$p(j\omega,\delta)$ can be separated into its real and imaginary parts, $p_R(\omega,\delta)$ and $p_I(\omega,\delta)$. In order to find ϵ in (12), the following problem must be solved first:

$$\min\ |\delta|_\infty \text{ subject to } p_R(\omega,\delta)=0\ ,\ p_I(\omega,\delta)=0 \tag{13}$$

The constraint equations of (13) can be expressed as $Ax=b$ for some ω, where A is a (2xm) real matrix, b is a (2x1) real vector, and $x=\delta$.

Assuming that the column vectors of A are pairwise linearly independent, i.e. satisfy the Haar condition, it is not hard to show that any solution to (13) has the property that m-1 of the entries of x are equal in magnitude, and the remaining entry is of less or equal magnitude. If A does not satisfy the Haar condition, a new matrix $\hat{A}$ satisfying the Haar condition can be constructed from A. Let $A_1, A, \ldots, A_k$ be matrices composed from columns of A; all columns in such a A_j, $j=1,\ldots,k$, are scalar multiples of each other, i.e. $\mathrm{rank}(A_j)=1$. All other columns of A not in these submatrices are linearly independent from any other column in A. Form the matrix $\hat{A}$ using the columns of A not included in any $A_1,\ldots,A_k$, plus using k further columns obtained in the following way: From each A_j pick an arbitrary column and express all columns of A_j as scalar multiples of this chosen column. Multiply this column by the sum of the absolute values of these scalars to obtain an entry to $\hat{A}$. The columns of $\hat{A}$ thus constructed will be pairwise linearly independent. Having thus constructed $\hat{A}$, the solution of (13) can be found using Theorem 1.

Theorem 1: $r = \min_{Ax=b} |\mathbf{x}|_\infty = \min_{\hat{A}z=b} |\mathbf{z}|_\infty$

In the proof of theorem 1, given in [6], a method is described for constructing either x or z when one of them is known. The above characterization of the solution to (13) is sufficient to construct an algorithm for finding the solution. Essentially, M (2x2) systems of equations must be solved, where $M=m_1 2^{m_1-2}$, and m_1 is the number of columns in $\hat{A}$. Since we are really only interested in the number r, the minimum infinity norm in (13), a more efficient algorithm will be presented in the next section.

VI. THE SOLUTION

Let A be a (2xm) real matrix with rows a_1 and a_2, and b be a (2x1) real vector. It is desired to find

$$\min |x|_\infty\ ,\ \text{subject to } Ax = b \tag{14}$$

It will be assumed that rank(A)=2. Due to the nature of the problem, the vector b is always in the range of A. Hence when rank(A)=1, (14) can be solved almost immidiatly.

Choose c_1, c_2 so that the j^{th} element of $(c_1a_1+c_2a_2)$ is zero. Define $\hat{a}$ as the vector $c_1a_1+c_2a_2$ with the j^{th} element deleted. The solution to the problem

$$\min |z|_\infty \text{ subject to } \hat{a}z=c_1b_1+c_2b_2 \tag{15}$$

is given by (16), where $\hat{a}_i$ denotes the i^{th} entry of $\hat{a}$,

$$z_i = \frac{-\mathrm{sgn}(\hat{a}_i)(c_1b_1+c_2b_2)}{\sum_{k=1}^{m-1} |\hat{a}_k|} \qquad i=1,\ldots,m-1 \tag{16}$$

when $\hat{a}_i$ is nonzero. If $\hat{a}_i$ happens to be zero, set z_i equal to zero. Next, a vector x will be constructed using the rule

$$x_i = z_i\ ,\ i<j \quad x_{i+1} = z_i \text{ for } i\geq j \tag{17}$$

The rule given in (17) is valid for all i such that $z_i\neq 0$. Subsequently, $a_1x=b_1$ or $a_2x=b_2$ can be used to solve for those entries of x not calculated by (17). This can be done via eqs. (15) and (16).

Theorem 2: The vector x is a solution to (14) if and only if $|z|_\infty \geq |x|_\infty$, where x and z are as constructed above.

A proof of the theorem is given in [6].

Using theorem 2 and the Ascent Algorithm in [7], the following algorithm is derived in [6] to obtain a solution to (14).

Algorithm:

(a) Find c_1 and c_2 so that the j^{th} element of $[c_1a_1+c_2a_2]$ is zero. Since $\mathrm{rank}(A) = 2$, $c_1\neq 0\neq c_2$. Construct $\hat{a}$ by deleting the j^{th} element of $c_1a_1+c_2a_2$.

(b) Solve $\hat{a}z = [c_1b_1+c_2b_2]$ for $\min|z|_\infty$ and construct x via eqs. (16) and (17). If $|z|_\infty \geq |x_j|$, x is the solution to (14). In that case stop.

(c) Set $\theta_j = 0$ and $\sigma_i = \mathrm{sgn}(x_i)$, where θ_i is the i^{th} element of $c_1a_1+c_2a_2$. Then

$$\sigma_i\theta_i = |\hat{a}_i| \text{ for } i<j,\ \sigma_{i+1}\theta_{i+1} = |\hat{a}_i| \quad \text{for } i\geq j \tag{18}$$

Let $\mu = \mathrm{sgn}(x_j)$, and $\lambda_i = (-\sigma_i\mu)a_{ki}$ for $i\neq j$; a_{ki} is the i^{th} element of a_k. (k) is 1 or 2, and is restricted by the condition

$a_{kj} \neq 0$. Find the index $\alpha = i$ for which $\frac{\lambda_i}{\sigma_i \theta_i}$ is maximum. Set $j = \alpha$, and go to Step (a).

Example: Let $A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $j = 3$, $c_1=1$, $c_2=0$. Then $\hat{a} = [2,-3,1]$. Doing step (b),

$$z_1 = x_1 = -\frac{1}{6}, \; z_2 = x_2 = \frac{1}{6}, \; z_4 = x_4 = -\frac{1}{6}.$$

Using $a_2 x = b_2$, $x_3=1/2$. Hence $|z|_\infty = \frac{1}{6} < |x|_\infty = 1/2$

Doing step (c), $\mu = \mathrm{sgn}(x_3) = +1$. $\sigma_1\theta_1 = 2$, $\sigma_2\theta_2 = 3$, $\sigma_4\theta_4 = 1$.

$\lambda_1 = -\sigma_1 a_{21} = +1$, $\lambda_2 = -a_{22} = -2$, $\lambda_4 = a_{24} = 1$.

$\max\{\frac{1}{2}, \frac{-2}{2}, \frac{1}{1}\} = \frac{1}{1} = \frac{\lambda_4}{\sigma_4\theta_4}$. Hence, the new j is 4.

Now $c_1 = 1$, $c_2 = 1$. Then $\hat{a} = (1, -5, -2)$. $z_1 = x_1 = -\frac{1}{4}$, $z_2 = x_2 = \frac{1}{4}$, $z_3 = x_3 = \frac{1}{4}$. Using $a_1 x_1 = b_1$, $x_4 = \frac{1}{4}$. Hence $|x|_\infty \leq |z|_\infty$.

Algorithm 3 converges in at most m steps. Step (b) of the algorithm can be done m times, if desired, for $j = 1,...,m$. For each value of j a z and an x is obtained. One of these is the solution to (14). The following corollary determines some additional properties of this solution:

Corollary 1: Let the m possible z obtainable in Step (b) of Algorithm 1 be called $z^1,...,z^m$, and the corrosponding x constructed from them be called $x^1,...,x^m$. One of these is the solution to (14). If

$$|z^i|_\infty > |z^k|_\infty, \quad k = 1,...,m, \quad k \neq i \tag{19}$$

then x^i is the unique solution to (14).

The proof of this corollary involves the application of the monotonicity of the Ascent Algorithm in [7], and is given in [6]. There it is also proven that the above algorithm will work for a matrix A which is not Haar. In view of the corollary, a closed-form expression of the infinity norm of the solution to (14) is given by the following theorem.

Theorem 3: The norm of the solution to (14) is given by

$$\max_j \frac{|a_{1j}b_2 - a_{2j}b_1|}{\sum_{i=1}^{m} |a_{1j}a_{2i} - a_{2j}a_{1i}|} \tag{20}$$

VI. A NUMERICAL EXAMPLE

Consider the polynomial

$$p(s) = (1+\delta_0)s^2 + (4+4\delta_1)s + (4+4\delta_2) = 0 \tag{21}$$

It is desired to find ϵ, so that for all $|\delta_i| < \epsilon$, the roots of the polynomial stay to the left of the parabola $y^2 = -x-1$. From [5] we find that $f(z) = z^2+z-1$ and $h(z) = 1$ must be used for (11). Upon substitution, the coefficients of the resulting polynomial $\hat{p}(s)$ are given by

$$\alpha_0(\delta) = 1+\delta_0 \;,\; \alpha_1(\delta) = 2+2\delta_0 \;,\; \alpha_2(\delta) = 3+4\delta_1-\delta_0$$

$$\alpha_3(\delta) = 2+4\delta_1-2\delta_0 \;,\; \alpha_4(\delta) = 1+\delta_0-4\delta_1+4\delta_2$$

For all $|\delta_i| < \epsilon$, $\hat{p}(s)$ must have roots in C^-. Substituting $z = j\omega$ and setting the real and imaginary parts of $\hat{p}(j\omega)$ equal to zero, one obtains (with $t = \omega^2$):

$$(t^2+t+1)\delta_0 + (-4t-4)\delta_1 + 4\delta_2 = -t^2+3t-1 \tag{22a}$$

$$(-2t-2)\delta_0 + 4\delta_1 = 2t-2 \tag{22b}$$

The minimum infinity norm solution of (22) for each t in $[0,\infty)$ must now be found. As $t \to \infty$, $|\delta|_\infty \to 1$, since (21) reduces to $t^2\delta_0 = -t^2$ and $-2t\delta_0 = 2t$, in this case. Figure 2 shows $|\delta|_\infty$ in function of t. Table I gives $|\delta|_\infty$ for intermediate values of t. Hence for $|\delta|_\infty < 0.055$, the roots of p(s) remain to the left of the parabola.

In other words, for any set of coefficients taken from $a_0 \epsilon [0.945, 1.055]$, $a_1 \epsilon [3.78, 4.22]$, $a_2 \epsilon [3.78, 4.22]$, the roots of $(a_0 s^2 + a_1 s + a_2)$ will remain to the left of the parabola $y^2 = -x-1$.

TABLE I

Perturbation bounds with respect to a parabola

t	$\lvert\delta\rvert_\infty$	t	$\lvert\delta\rvert_\infty$
0.1	0.51	0.76	0.064
0.2	0.42	0.79	0.055
0.3	0.34	0.84	0.069
0.4	0.26	0.9	0.085
0.5	0.19	1.0	0.11
0.6	0.12	2.0	0.47
0.7	0.08	3.0	0.65

CONCLUDING REMARKS

The investigation of robust stability of polynomials with respect to coefficient variations was extended to include two new cases. First, the coefficients themselves were assumed to be linear functions of several varying real parameters. Second, a variety of regions, in which the poles of the polynomials have to remain in spite of the real parameter variations, was considered. An algorithm was given to calculate the maximum permissable real parameter variations under these generalized conditions. The algorithm easily permits the investigation of polynomials with complex coefficients. This algorithm was used in [8] to assess the effectiveness of a matrix robustness measure, the μ-measure, which was first reported in [9].

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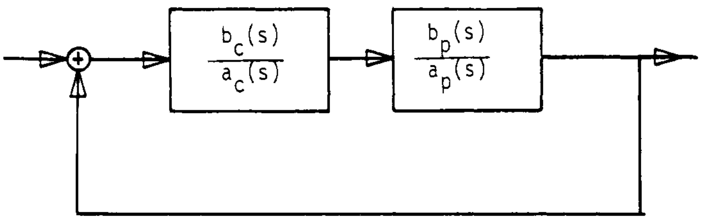

Figure 1: Unity Feedback System

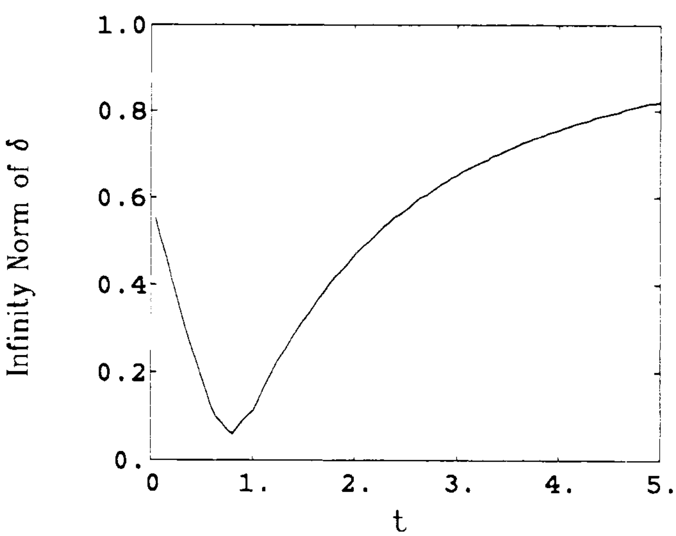

Figure 2: Norm of δ vs parameter t