Comments On The Raindrop Problem

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I. INTRODUCTION

In his paper on the falling raindrop, Krane[1] raises some interesting questions which the following comments should answer and clarify. Specifically, our comments deal with Krane's Eqs. (24) and (26), which describe two special cases in the limit of $g = 0$, and Eqs. (32) and (33), which describe the general problem. We give solutions to these problems and clarify the nature of the constant acceleration solutions. The notation of Krane will be used and additional notation will be introduced as needed.

II. $g = 0$ LIMIT

(1) In deriving Krane's Eq. (24) for the case in which $g = 0$ and the mass accretion rate of the drop $dm/dt = km^{2/3}v$, he states that "This case does not lend itself to direct integration." Direct integration is possible, since

$$\frac{dm}{dt} = km^{2/3}v = km^{2/3}\frac{dx}{dt} \tag{1}$$

can be integrated to give

$$3(m^{1/3} - m_0^{1/3}) = kx$$

or

$$m = (kx/3 + m_0^{1/3})^3, \tag{2}$$

which when substituted in the momentum conservation equation

$$m\frac{dx}{dt} = m_0 v_0 \tag{3}$$

yields by integration

$$m_0 v_0 t = (3/4k)[(kx/3 + m_0^{1/3})^4 - m_0^{4/3}], \tag{4}$$

which together with Eqs. (2) and (3) give

$$v = v_0[1 + (4k/3)m_0^{-1/3}v_0 t]^{-3/4}, \tag{5}$$

which is Krane's Eq. (24).

(2) For the case $g = 0$ and

$$\frac{dm}{dt} = km^{2/3}xv, \tag{6}$$

Krane's solution as given by his Eq. (26) is correct only asymptotically (for large t). For small t, Krane's Eq. (26) gives that the variation of v with t is linear when it is actually quadratic. Equation (6) gives the time derivatives (denoted by overdots) $\dot{m}(0) = 0$, $\ddot{m}(0) = km_0^{2/3}v_0^2$ and leads to the Taylor series expansion

$$m(t) \approx m_0 + km_0^{2/3}v_0^2 t^2/2, \tag{7}$$

which when substituted into Eq. (3) gives

$$v \approx v_0(1 - m_0^{-1/3}v_0^2 t^2/2). \tag{8}$$

We give now the solution of Eqs. (3) and (6). Since $v = dx/dt$, Eq. (6) gives immediately by integration,

$$3(m^{1/3} - m_0^{1/3}) = kx^2/2. \tag{9}$$

Equations (3) and (9) are used to eliminate v and x, respectively, from Eq. (6) and obtain

$$\int_{m_0}^{m} \frac{m^{1/3}dm}{(m^{1/3} - m_0^{1/3})^{1/2}} = \sqrt{6k}\, m_0 v_0 t, \tag{10}$$

which can be integrated by the substitution $y = m^{1/3}$ to give

$$35\sqrt{k/6}\, m_0 v_0 t = (y - y_0)^{1/2}(5y^3 + 6y_0 y^2 + 8y_0^2 y + 16y_0^3). \tag{11}$$

For small t Eq. (11) gives

$$(y - y_0)^{1/2} \approx \sqrt{k/6}\, v_0 t, \tag{12}$$

which reproduces Eqs. (7) and (8). For large t,

$$y^{7/2} \sim 7\sqrt{k/6}\, m_0 v_0 t, \tag{13}$$

$$m \sim (7\sqrt{k/6}\, m_0 v_0 t)^{6/7}, \tag{14}$$

and

$$v \sim v_0(7\sqrt{k/6}\, m_0^{-1/6} v_0 t)^{-6/7}. \tag{15}$$

Krane's Eq. (26) for large t agrees with Eq. (15), but is incorrect for other values of t.

The distance x can be obtained from Eq. (9) once m is determined; alternatively one can use Eq. (9) to eliminate m from Eq. (3) and integrate the resulting equation to obtain

$$m_0 v_0 t = \left(\frac{k}{6}\right)^3 \frac{x^7}{7} + \frac{3}{5}\left(\frac{k}{6}\right)^2 \times m_0^{1/3}x^5 + \frac{k}{6}m_0^{2/3}x^3 + m_0 x, \tag{16}$$

which can be iterated for small and large t. In general, for a given t, Eqs. (11) and (16) have to be solved numerically for m and x, and one can show easily that there is only one solution in each case, as is expected physically.

III. GENERAL CASE

(1) We now give a solution to the raindrop problem under general conditions. Assume the mass accretion rate of the drop to be given by

$$\dot{m} = km^{\alpha}v, \quad \alpha \leqslant 1, \tag{17}$$

which is more general than the two special cases $\alpha = 0$ and $\alpha = 2/3$ considered by Krane. The equation of motion

$$\frac{d}{dt}(mv) = \dot{m}v + m\dot{v} = mg \tag{18}$$

is to be solved simultaneously with Eq. (17) subject to the initial conditions $m = m_0$, $x = 0$, and $v = v_0$ at $t = 0$.

In Eq. (18) substitute $\dot{m}/km^{\alpha}$ for v from Eq. (17) to obtain

$$\frac{d^2\chi}{dt^2} = (2 - \alpha)kg\chi^{1/(2-\alpha)}, \quad \chi = m^{2-\alpha}, \tag{19}$$

which is now solved by the substitutions $\dot{\chi} = p$ and $\ddot{\chi} = p\, dp/d\chi$ to give

$$p^2 - p_0^2 = 2(2-\alpha)^2 kg \times (\chi^{(3-\alpha)/(2-\alpha)} - \chi_0^{(3-\alpha)/(2-\alpha)})/(3-\alpha), \tag{20}$$

$$t = \int_{m_0}^{m} \frac{m^{1-\alpha}dm}{\{[2kg/(3-\alpha)](m^{3-\alpha} - m_0^{3-\alpha}) + m_0^2 v_0^2 k^2\}^{1/2}}, \tag{21}$$

which give a table for m and v as a function of time. [Note that $p = (2-\alpha)kmv$, which is proportional to the linear momentum.] By integration, Eq. (17) gives x in terms of m, namely,

$$(1-\alpha)kx = m^{1-\alpha} - m_0^{1-\alpha}. \tag{22}$$

The acceleration $\dot{v}$ is determined by Eq. (18), which can be written as

$$m^{1-\alpha} = kv^2/(g - \dot{v}). \tag{23}$$

In general, the integration in Eq. (21) has to be performed numerically.[2]

(2) An alternative approach to the problem is to study a differential equation in v. By differentiating Eq. (23) with respect to time and substituting for $\dot{m}$ from Eq. (17) we obtain

$$\ddot{v} = (g - \dot{v})[(1-\alpha)g - (3-\alpha)\dot{v}]/v, \tag{24}$$

which is identical to Krane's Eqs. (32) and (33) for $\alpha = 0$ and $\alpha = 2/3$, respectively. Equation (24), like Eq. (19), can be integrated by the substitutions $\dot{v} = w$ and $\ddot{v} = w\, dw/dv$ and gives

$$|v/v_0| = \left|\frac{w-g}{w_0-g}\right|^{1/2} \times \left|\frac{(3-\alpha)w_0 - (1-\alpha)g}{(3-\alpha)w - (1-\alpha)g}\right|^{(1-\alpha)/2(3-\alpha)}. \tag{25}$$

Equations (25), (23), and (26) give tables for $\dot{v}$, v, m, and x. The time corresponding to these variables is obtained from the integral $t = \int dw/\dot{w}$, where $\dot{w}(\ddot{v})$ is expressed in terms of w by using Eqs. (24) and (25), namely,

$$t = \pm \int_{w_0}^{w} \frac{v_0\, dw}{(w-g)[(3-\alpha)w - (1-\alpha)g]} \times \left|\frac{w-g}{w_0-g}\right|^{1/2} \left|\frac{(3-\alpha)w_0 - (1-\alpha)g}{(3-\alpha)w - (1-\alpha)g}\right|^{(1-\alpha)/2(3-\alpha)}, \tag{26}$$

which completes the solution to the problem by this method.[3,4] Actually, the integral in Eq. (26) can be transformed to that of Eq. (21) by the substitution given by Eq. (23) in which v is eliminated by using Eq. (25).

(3) We conclude by making a few remarks about the nature of the solution. If

$$km_0^{\alpha-1}v_0^2 = 2g/(3-\alpha), \tag{27}$$

the integration in Eq. (21) is trivial and we obtain the constant acceleration solution

$$\dot{v} = (1-\alpha)g/(3-\alpha), \tag{28}$$

which is root of the quadratic (in $\dot{v}$) appearing in Eq. (24). Indeed, if $\dot{v}_0 = (1-\alpha)g/(3-\alpha)$, then Eq. (24) shows that $\ddot{v}_0$ and all the higher derivatives are zero and that[5] $\dot{v} = \dot{v}_0$. The integral in Eq. (26) diverges as $w \to (1-\alpha)g/(3-\alpha)$, reflecting the fact that for arbitrary initial conditions $\dot{v}$ approaches $(1-\alpha)g/(3-\alpha)$ asymptotically.

Krane has demonstrated that the constant acceleration solutions are abundant for this problem. A systematic method for obtaining them is to solve the mass accretion equation together with

$$\dot{m}v/m = c, \tag{29}$$

where c is a constant, and if a solution can be found the constant acceleration solution is

$$\dot{v} = g - c, \tag{30}$$

as follows from Eq. (18). To illustrate for this particular case, $\dot{m}v/m = km^{\alpha-1}v^2 = c$, which can be used to eliminate m from Eq. (22) and obtain the relation $v^2 - v_0^2 = c(1-\alpha)x$, which is characteristic of a motion with constant acceleration $\dot{v} = c(1-\alpha)/2$. Equation (30) now gives $c = 2g/(3-\alpha)$. Equations (27) and (28) have been reproduced with little effort. All the examples in Krane's paper can be worked out by this method.

[1] K. S. Krane, Am. J. Phys. **49**, 113 (1981).

[2] The solution to the simple case $\alpha = 1$ is easily obtained from Eqs. (17) and (19), namely,

$$m = m_0 e^{kx} = m_0(\cosh\sqrt{kg}\,t + v_0\sqrt{k/g}\sinh\sqrt{kg}\,t),$$

which shows that the drop has the limiting velocity $\sqrt{g/k}$ and that the velocity is constant if $v_0 = \sqrt{g/k}$. This is similar to the case $\dot{m} = km$ which is discussed by Krane.

[3] The $\pm$ sign and the absolute values are easily handled by studying Eq. (24). If, for example, $v_0 > 0$ and $w_0 > (1-\alpha)g/(3-\alpha)$, the plus sign is taken, $-w+g$ and $-w_0+g$ are positive, $w < w_0$, and the asymptotic limit $w = (1-\alpha)g/(3-\alpha)$ is approached from above. In the integral $v_0/(g-w_0)^{1/2}$ can be replaced by $(m_0^{1-\alpha}/k)^{1/2}$ from Eq. (23) which makes the integral well behaved as $v_0 \to 0$ and $w_0 \to g$.

[4] The interested reader might apply this method to the simple case for which $\dot{m} = km^{\beta}$, $\beta < 1$, and for which Krane gives solutions for $\beta = 0$, 2/3, 1. The differential equation is $\ddot{v} = (g-\dot{v})[(1-\beta)g - (2-\beta)\dot{v}]/v$ which is similar to Eq. (24); but the integral corresponding to that of Eq. (26) is elementary as the integrand contains only a power of the factor $(1-\beta)g - (2-\beta)\dot{v}$. For arbitrary initial conditions $\dot{v} \sim (1-\beta)g/(2-\beta)$ asymptotically, and for the condition $km_0^{\beta-1}v_0 = g/(2-\beta)$, the solution is $\dot{v} = (1-\beta)g/(2-\beta)$.

[5] The initial conditions must satisfy Eq. (18) which implies Eq. (27).