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Sequential Probability Ratio Tests For The Shape Parameter Of A Nonhomogeneous Poisson Process

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Reader Aids—
Purpose: Widen state of art
Special math needed for explanations: Statistics
Special math needed to use results: Same
Results useful to: Practicing statisticians

Abstract—**Sequential probability ratio tests for the shape parameter of one or more nonhomogeneous Poisson processes, with power intensity functions, are provided. The tests can be performed when the scale parameter is an unknown nuisance parameter; the effective loss of not knowing the scale parameter is one observation per process. The resulting tests can be expressed in terms of the maximum likelihood estimators of the shape parameters for the usual fixed sample procedure. A further advantage of the present approach is that the scale parameters for different processes, in the multiple sample procedures, need not be equal. Approximations for the operating characteristic function and the average sample number are provided.**

1. INTRODUCTION

Suppose T_1, T_2, T_3, ... denote successive occurrence times (e.g. failures) of some phenomenon. A very useful model for phenomena whose intensity can change with time is the non-homogeneous Poisson process (NHPP) with intensity $\nu(t) = (\gamma/\theta)(t/\theta)^{\gamma-1}$, which is often unwisely called a Weibull process. Applications are discussed by Crow [5] and Lee & Lee [10]. For example, T_1, T_2, ... can represent the failure times of a repairable system. The parameter γ reflects the degree and type of change taking place in the system over time. '$\gamma > 1$' indicates the system is wearing out or degrading and '$\gamma < 1$' indicates the system is improving under the repair and maintenance program.

Recent work on the development of statistical procedures for the NHPP include papers by Bain & Engelhardt [3, 7], Crow [6], Lee & Lee [10], and Finkelstein [8]. The main objective of this paper is to derive a sequential probability ratio test (SPRT) for the shape parameter γ with the scale parameter θ an unknown nuisance parameter. Point estimates and *s*-confidence intervals for γ, in the more familiar fixed sample framework, are discussed by Bain [2, p 313].

It is also desirable, in some situations, to test hypotheses about γ. Suppose, for example, a repairable system is under development. A developmental program might consist of testing to identify deficiencies, a redesign effort to correct the deficiencies, and further testing to verify these corrections and identify new problem areas. It would be advantageous to track the reliability growth of the system, by means of the failure data collected during development testing, so that the program could be revised, if necessary, in order to attain the system reliability objectives. An important part of this would involve testing hypotheses about γ in order to make decisions concerning the effectiveness of the developmental program in improving the system. For example, rejection of the hypothesis H_0: $\gamma < 1$ would indicate that the program has not been effective.

Since the data often occur naturally in a sequential fashion, it will be useful to have sequential procedures available for this model. 1-sample and M-sample sequential tests are developed below for the parameter γ, with θ an unknown nuisance parameter.

Notation

NHPP	nonhomogeneous Poisson process
SPRT	sequential probability ratio test
ASN	average sample number
$\nu(t)$	$(\gamma/\theta)(t/\theta)^{\gamma-1}$, the intensity function
γ, θ	shape and scale parameters
T_i	occurrence time *i* for a single process, $i = 1, 2, \ldots$
$T_{j,k}$	occurrence time *j* for process *k*, $j = 1, 2, \ldots$, $k = 1, 2, \ldots, M$
$E\{\cdot;\gamma\}$	*s*-expectation when γ is true
$Q(\gamma)$	operating characteristic (OC) function
α	probability of a type I error
β	probability of a type II error
$\chi^2_p(2(n-1))$	quantile *p* of the chi-square Cdf with $2(n-1)$ degrees-of-freedom
a, b	Wald's approximate critical values, $a \equiv \ln[(1-\beta)/\alpha]$, $b \equiv \ln[\beta/(1-\alpha)]$

2. SEQUENTIAL TESTS FOR γ

It is well-known, in the area of hypothesis testing, that a decision can often be reached with fewer observations required, using a SPRT, than is possible with an ordinary fixed-sample test. The tests below are illustrated by example in section 3 and derived in section 4.

2.1 *1-Sample SPRT for* γ

A SPRT for the hypothesis H_0: $\gamma \leqslant \gamma_0$ against the alternative H_1: $\gamma \geqslant \gamma_1$, $0 < \gamma_0 < \gamma_1 < \infty$, can be based on the statistic

$$\hat{\gamma}_n \equiv n/\sum_{i=1}^{n-1} \ln(T_n/T_i) \tag{1}$$

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which is the usual (fixed sample) maximum likelihood estimator of γ.

For type I and type II errors of size α and β, this test is:

1. accept H_0 if $n/\hat{\gamma}_n \geqslant [b + (n-1)\ln(\gamma_0/\gamma_1)]/(\gamma_0 - \gamma_1)$,

2. reject H_0 if $n/\hat{\gamma}_n \leqslant [a + (n-1)\ln(\gamma_0/\gamma_1)]/(\gamma_0 - \gamma_1)$, (2)

and continue otherwise; when (using Wald's approximations) $b \equiv \ln[\beta/(1-\alpha)]$ and $a \equiv \ln[(1-\beta)/\alpha]$.

The Wald approximation for the OC function is:

$$Q(\gamma) \approx [e^{ah(\gamma)} - 1]/[e^{ah(\gamma)} - e^{bh(\gamma)}], \tag{3}$$

where $h(\gamma)$ is the solution of $\phi(h;\gamma) = 1$ with

$$\phi(h;\gamma) = \gamma(\gamma_1/\gamma_0)^h/[\gamma + (\gamma_1 - \gamma_0)h]. \tag{4}$$

The number of observations, N, required to make a decision, using a SPRT, is a random variable. It is shown in section 4 that the procedure defined by (2) will terminate in a finite number of steps. Furthermore, using the Wald approximation, the following formula for the average sample number (ASN) is also derived:

$$E\{N;\gamma\} \approx 1 + [bQ(\gamma) + a(1 - Q(\gamma))]/D(\gamma) \tag{5}$$

$$D(\gamma) \equiv \ln(\gamma_1/\gamma_0) + (\gamma_0 - \gamma_1)/\gamma. \tag{6}$$

Evaluating (5) at γ_0 and γ_1, and using (6), it follows that:

$$E\{N;\gamma_0\} \approx 1 + [(1-\alpha)\ln(\beta/(1-\alpha)) + \alpha\ln((1-\beta)/\alpha)]/D(\gamma_0), \tag{7}$$

$$E\{N;\gamma_1\} \approx 1 + [\beta\ln(\beta/(1-\alpha)) + (1-\beta)\ln((1-\beta)/\alpha)]/D(\gamma_1). \tag{8}$$

The sample number required for the corresponding optimum fixed sample test is the value of n which satisfies

$$\chi^2_{1-\beta}(2(n-1))/\chi^2_\alpha(2(n-1)) = \gamma_1/\gamma_0. \tag{9}$$

2.2 *M-Sample SPRT with Common* γ

Suppose that observations $T_{j,k}$ are becoming available from M NHPP's which have a common value of γ but possibly different values of θ. Define—

$$\hat{\gamma}_{n_k,k} \equiv n_k \Big/ \sum_{j=1}^{n_k-1} \ln(T_{n_k,k}/T_{j,k}) \tag{10}$$

$$n^* \equiv \sum_{k=1}^{M} (n_k - 1), \tag{11}$$

$$Z_{n^*} \equiv \sum_{k=1}^{M} n_k/\hat{\gamma}_{n_k,k}. \tag{12}$$

The SPRT of H_0: $\gamma \leqslant \gamma_0$ against H_1: $\gamma \geqslant \gamma_1$ is:

1. accept H_0 if $Z_{n^*} \geqslant [b + n^*\ln(\gamma_0/\gamma_1)]/(\gamma_0 - \gamma_1)$,

2. reject H_0 if $Z_{n^*} \leqslant [a + n^*\ln(\gamma_0/\gamma_1)]/(\gamma_0 - \gamma_1)$, (13)

and continue otherwise. If the test continues, Z_{n^*} can be recomputed as soon as an additional observation is available from any one of the NHPP's.

2.3 *2-Sample Test of Discrimination for* γ

Suppose pairs of observations $(T_{n,1}, T_{n,2})$ are considered sequentially from two NHPP's with parameters γ_k, θ_k, $k = 1, 2$. The problem is to test H_0: $\gamma_1 - \gamma_2 \leqslant -\psi_0$ against H_1: $\gamma_1 - \gamma_2 \geqslant \psi_0$ for some specified $\psi_0 > 0$ with risks not exceeding (α, β).

Let $\hat{\gamma}_{n,k}$, $k = 1, 2$, be defined by (10) with $n_1 = n_2 = n$. The SPRT in this case is:

1. accept H_0 if $\psi_0[(n/\hat{\gamma}_{n,2}) - (n/\hat{\gamma}_{n,1})] \leqslant b$,

2. reject H_0 if $\psi_0[(n/\hat{\gamma}_{n,2}) - (n/\hat{\gamma}_{n,1})] \geqslant a$, (14)

and continue otherwise.

The approximate OC function and ASN of pairs are:

$$Q(\gamma_1,\gamma_2) \approx [e^{a(\gamma_2-\gamma_1)/\psi_0} - 1]/[e^{a(\gamma_2-\gamma_1)/\psi_0} - e^{b(\gamma_2-\gamma_1)/\psi_0}], \tag{15}$$

$$E\{N;\gamma_1,\gamma_2\} \approx 1 + [a - (a-b)\,Q(\gamma_1,\gamma_2)] \big/[(\gamma_2-\gamma_1)\psi_0/\gamma_0\gamma_1]. \tag{16}$$

3. NUMERICAL EXAMPLES

The 1-sample and M-sample SPRT's will be illustrated by applying them to the following simulated life data from $M = 3$ systems with true common $\gamma = 0.5$ and $\theta = 2.778$:

System 1: 4.3, 4.4, 10.2, 23.5, 23.8, 26.4, 74.0, 77.1, 92.1, 197.2
System 2: 0.1, 5.6, 18.6, 19.5, 24.2, 26.7, 45.1, 45.8, 75.7, 79.7, 98.6, 120.1, 161.8, 180.6, 190.8
System 3: 8.4, 32.5, 44.7, 48.4, 50.6, 73.6, 98.7, 112.2, 129.8, 136.0, 195.8

The data, which are found in Crow [5] and Bain [2], were actually obtained using type I censoring at time 200 with a fixed-sample plan, but for illustrative purposes we will regard them as sequential data.

3.1 *Application of the 1-Sample SPRT*

Suppose it is desired, on the basis of data from system 1, to test H_0: $\gamma \leqslant 0.25$ against H_1: $\gamma \geqslant 0.50$ with $\alpha = \beta = 0.10$. The sequential procedure as described by (2) would accept H_0 if $n/\hat{\gamma}_n \geqslant 8.80 + (n-1)\,2.77$, reject H_0 if $n/\hat{\gamma}_n \leqslant -8.80 + (n-1)\,2.77$, and continue otherwise. The decision would be to reject H_0 when failure #6, 26.4, is observed since $n/\hat{\gamma}_n = 4.78 < 5.05 = -8.80 + (n-1)\,2.77$

when $n = 6$. The ASNs can be approximated using (7) and (8). Specifically, they are $E\{N; \gamma_0\} \approx 6.73$ and $E\{N; \gamma_1\} \approx 10.10$. Using (9) the optimum fixed sample test requires $n = 15$ observations. Suppose it is desired to find the power for $\gamma = 0.6$. The OC can be approximated by (3). Since in general, it is not possible to solve $\phi(h; \gamma) = 1$ explicitly, the solution must be found by numerical methods. A reasonable approach is to compute both $\phi(h; \gamma)$ and $Q(\gamma)$ as functions of h for a fixed γ. In the present example, with $\gamma = 0.6$, $\phi(h; \gamma) = 1.00$ for $h = -1.62$ and the corresponding value of $Q(\gamma)$ is $Q(0.6) = 0.03$. The approximate power of the test for $\gamma = 0.6$ is $1 - Q(0.6) = 0.97$.

3.2 *Application of the M-Sample SPRT*

The same hypotheses can also be tested using the M-sample test described in section 2.2. In fact, the same interval would apply with $n - 1$ replaced by $n^* = \Sigma_{k=1}^{3}(n_k - 1)$, and $n/\hat{\gamma}_n$ replaced by $Z_{n^*} = \Sigma_{k=1}^{3} n_k/\hat{\gamma}_{n_k,k}$. For the data from the three systems, and $\alpha = \beta = 0.10$, the decision would be to reject H_0 when failure #11, 24.2, is observed since $Z_{n^*} = 11.69 < 13.36 = -8.80 + n^*2.77$ when $n^* = 8$. Although this procedure required more observed failures than the 1-sample test, the duration of the test, prior to rejecting H_0, was shorter. In particular, the duration based on system 1 above was 26.4, while the duration based on all three systems was 24.2. A given system will not contribute to n^* or Z_{n^*} until at least two failures have been observed for that system. This occurred in the present example since the second failure from system 3 did not occur until after the test was terminated. The computational formulas for n^* and Z_{n^*} will take this into account if we follow the convention of excluding any terms for which $n_k - 1 < 1$.

3.3 *Application of the 2-Sample Test of Discrimination*

The sequential test of section 2.3 will be illustrated by applying it to the results of fatigue failure tests conducted by Butler & Rees [4] to determine the suitability of various metals for aircraft construction. In one phase of the study, titanium and steel specimens were tested for crack initiation due to fatigue. Each specimen was subjected to stresses in varying amounts and patterns similar to those occurring in flight. These stress patterns were repeated until a crack was detected, the number of load cycles (which can be thought of as a laboratory measure of flight time) until a crack detection was recorded, the crack was repaired, the test was resumed until another crack was detected, the total number of load cycles to this failure was recorded, etc. Because of a possible threshold effect, the measurement of time was started at the time of the initial failure. The data which we shall consider are:

Test 1: (titanium)	2356, 3498, 5038, 9501, 11500, 12000, 12000, 12000, 12000, 13000, 13000, 13500, 14000, 17000, 17500, 17500, 18500, 20505, 20505
Test 2: (steel)	2452, 3750, 3788, 3788, 4000, 4000, 4350, 4350, 4750, 5000, 5750, 5750, 5750, 6000, 6000, 6000, 6075, 6400, 6798, 7400, 7500

Suppose it is desired to test sequentially the hypothesis H_0: $\gamma_1 - \gamma_2 \leqslant -1$ against the alternative H_1: $\gamma_1 - \gamma_2 \geqslant 1$ with risks $\alpha = \beta = .01$. The decision would be to accept H_0 when pair #17, (18500, 6075), is observed since $(n/\hat{\gamma}_{n,2}) - (n/\hat{\gamma}_{n,1}) = -4.915 < -4.595 = b$ when $n = 17$.

The ASN of pairs depends on the values γ_1 and γ_2 individually, and not just on their difference. For example, if $\gamma_1 = 1$ and $\gamma_2 = 2$, $E\{N; \gamma_1, \gamma_2\} = 10.01$ and if $\gamma_1 = 2$ and $\gamma_2 = 3$ $E\{N; \gamma_1, \gamma_2\} \doteq 28.02$ which follow from (16) with $\alpha = \beta = .01$.

4. DERIVATIONS

4.1 *Derivation of the 1-Sample SPRT*

The joint pdf of the first n failure times from a NHPP with intensity $\nu(t) = (\gamma/\theta)(t/\theta)^{\gamma-1}$ is:

$$f(t_1, \ldots, t_n) = \left(\frac{\gamma}{\theta}\right)^n \left[\prod_{i=1}^{n} (t_i/\theta)^{\gamma-1}\right] \exp[-(t_n/\theta)^\gamma],$$
$$0 < t_1 < \ldots < t_n < \infty. \quad (17)$$

Let

$$U_{i,n} \equiv \ell n(T_n/T_{n-i}),\ i = 1, 2, \ldots, n. \quad (18)$$

The $U_{i,n}$ are distributed as order statistics for a sample of size $n - 1$ from an exponential distribution. Specifically, the joint pdf is:

$$f(u_{1,n}, \ldots, u_{n-1,n}; \gamma) = (n-1)!\, \gamma^{n-1} \exp\left[-\gamma \sum_{i=1}^{n-1} u_{i,n}\right],$$
$$0 < u_{1,n} < \ldots < u_{n-1,n} < \infty, \quad (19)$$

$$\sum_{i=1}^{n-1} U_{i,n} = \sum_{i=1}^{n-1} \ell n(T_n/T_i) = n/\hat{\gamma}_n. \quad (20)$$

$\hat{\gamma}_n$ is the usual (fixed sample) maximum likelihood estimator of γ. The $U_{i,n}$'s correspond to a maximal invariant which eliminates the dependence on the scale parameter, but which also reduces the size of the data set from n to $n - 1$.

Thus a SPRT of H_0: $\gamma = \gamma_0$ against H_1: $\gamma = \gamma_1$, or H_0: $\gamma \leqslant \gamma_0$ against H_1: $\gamma \geqslant \gamma_1$, $0 < \gamma_0 < \gamma_1 < \infty$, can be based on (19), and the unknown nuisance parameter θ has been eliminated with an effective loss of only one observation. Let

$$\lambda_{n-1} \equiv \frac{f(u_{1,n}, \ldots, u_{n-1,n}; \gamma_1)}{f(u_{1,n}, \ldots, u_{n-1,n}; \gamma_0)} = (\gamma_1/\gamma_0)^{n-1} \exp\left[-(\gamma_1 - \gamma_0) \sum_{i=1}^{n-1} u_{i,n}\right], \quad (21)$$

$$Z_{n-1} \equiv \ell n\ \lambda_{n-1}.$$

The form of the SPRT is

i. accept H_0 if $Z_{n-1} \leqslant b$,

ii. reject H_0 if $Z_{n-1} \geqslant a$,

iii. continue if $b < Z_{n-1} < a$. (22)

Because of (20), it follows that the test (2) is equivalent to (22).

The observations are not s-independent; however, as discussed by Ghosh [9], the test can equivalently be expressed in terms of $z_1 \equiv \ln \lambda_1$ and $z_j \equiv \ln(\lambda_j/\lambda_{j-1})$, $j > 1$, since

$$Z_{n-1} = \sum_{j=1}^{n-1} z_j.$$

Note that:

$$\sum_{i=1}^{n-2} U_{i,n-1} = (n-2) \ln T_{n-1} - \sum_{i=1}^{n-2} \ln T_i$$
$$= \sum_{i=1}^{n-1} U_{i,n} + (n-1) \ln(T_{n-1}/T_n), \qquad (23)$$

It follows that:

$$z_j = \ln(\gamma_1/\gamma_0) + (\gamma_0 - \gamma_1)j \ln(T_{j+1}/T_j),$$
$$j = 1, \ldots, n-1. \qquad (24)$$

Transforming from (17) shows that the z_j's are fortuitously i.i.d. with:

$$2j\gamma \ln(T_{j+1}/T_j) = 2\gamma[z_j - \ln(\gamma_1/\gamma_0)]/(\gamma_0 - \gamma_1) \sim \chi^2(2). \qquad (25)$$

Thus the basic properties of the test can be stated easily in terms of the z_j's following the notation in Ghosh [9].

It follows immediately that the SPRT is closed; that is, it terminates with probability one, and that:

$$E\{z_j;\gamma\} = \ln(\gamma_1/\gamma_0) + (\gamma_0 - \gamma_1)/\gamma, \qquad (26)$$

$$\phi(h;\gamma) \equiv E\{e^{hz_j};\gamma\} = \gamma(\gamma_1/\gamma_0)^h/[\gamma + (\gamma_1 - \gamma_0)h]. \qquad (27)$$

It follows that the Wald approximation for the OC function is given by (3). The Wald approximation for the ASN is then obtained from:

$$E\{N-1;\gamma\} \approx \frac{bQ(\gamma) + a[1 - Q(\gamma)]}{E\{z_j;\gamma\}}. \qquad (28)$$

Approximation (5) follows from (26) and (28).

If one were to consider a SPRT of γ for the exponential distribution with pdf:

$$f(x) \equiv \gamma\, e^{-\gamma x},\ 0 < x < \infty,$$

then

$$z_j \equiv \ln(\lambda_j/\lambda_{j-1}) = \ln(\gamma_1/\gamma_0) + (\gamma_0 - \gamma_1)x_j,\ j = 1, \ldots, n, \qquad (29)$$

where $2\gamma\, x_j \sim \chi^2(2)$. This is equivalent to (24) and (25) with $n-1$ replaced by n and $j \ln(T_{j+1}/T_j)$ replaced by x_j. Thus results for the exponential distribution can be applied directly to the test considered here simply by replacing n with $n-1$, if they are stated in terms of the z_j given by (29). Results for the exponential distribution stated in terms of the x_j also apply under the transformation $x_j = j \ln(T_{j+1}/T_j)$.

Albert [1] has studied the exact properties of the SPRT for the exponential parameter and these are reviewed by Ghosh [9]. They provide exact expressions for the critical values and exact bounds for $E\{N;\gamma\}$, however the Wald approximations are more convenient and should be adequate for most applications.

Example [9, p. 157]. Suppose $\gamma_0 = 0.010$, $\gamma_1 = 0.015$, $\alpha = \beta = 0.05$. Then the Wald approximations give $b = -2.944$, $a = 2.944$. From (2), accept H_0 if $n/\hat{\gamma}_n \geqslant 588.89 + (n-1)81.09$, and reject H_0 if $n/\hat{\gamma}_n \leqslant -588.89 + (n-1)81.09$. From (7) and (8) $E\{N-1;\gamma_0\} \approx 28.03$, and $E\{N-1;\gamma_1\} \approx 36.73$. The exact test is to accept H_0 if $n/\hat{\gamma}_n \geqslant 507.80 + (n-1)81.09$, reject H_0 if $n/\hat{\gamma}_n \leqslant -561.29 + (n-1)81.09$, and $28.93 \leqslant E\{N-1;\gamma_0\} \leqslant 29.06$, $34.97 \leqslant E\{N-1;\gamma_1\} \leqslant 38.17$. From (9) the optimum fixed sample test requires $n - 1 = 66$.

4.2 *Derivation of the M-Sample SPRT*

Suppose that observations $T_{j,k}$ are becoming available from NHPPs which have a common value of γ but possibly different values of θ. Define:

$$x_{j,k} \equiv j \ln(T_{j+1,k}/T_{j,k}),\ j = 1, \ldots, n_k - 1;\ k = 1, \ldots, M.$$

Then the $x_{j,k}$ represent $n^* \equiv \Sigma_{k=1}^{M} (n_k - 1)$ s-independent exponential variables with parameter γ. In this case:

$$Z_{n^*} = \sum_{k=1}^{M} \sum_{j=1}^{n_k-1} x_{j,k} = \sum_{k=1}^{M} n_k/\hat{\gamma}_{n_k,k}, \qquad (30)$$

and the SPRT of H_0: $\gamma \leqslant \gamma_0$ against H_1: $\gamma \geqslant \gamma_1$ is defined by (13). Results such as (28) apply directly to determining ASN, in the M-sample case, with $N-1$ replaced by N^*.

4.3 *Derivation of the 2-Sample Test of Discrimination*

Consider two NHPPs with parameters γ_k, θ_k, $k = 1, 2$. The problem is to test H_0: $\gamma_1 - \gamma_2 \leqslant -\psi_0$ against H_1: $\gamma_1 - \gamma_2 \geqslant \psi_0$ for some specified $\psi_0 > 0$ with risks not exceeding (α, β). It will be shown that such a test can be carried out according to (14).

2-Sample tests of this type are considered by Ghosh [9, p 240] for the more general Darmois-Koopman form:

$$f_k(x;\gamma_k) \equiv C(\gamma_k)e^{B(\gamma_k)T_k(x)}\, g_k(x). \qquad (31)$$

The problem is considered first for hypotheses of the form H_0: $\gamma_1 = \gamma'$, $\gamma_2 = \gamma''$ versus H_1: $\gamma_1 = \gamma''$, $\gamma_2 = \gamma'$ with $\gamma' < \gamma''$. In the present application, pairs of observations $(T_{n,1}, T_{n,2})$ are considered sequentially from two NHPPs. As in section 4.1, the transformation $x_{j,k} \equiv j \ln(T_{j+1,k}/T_{j,k})$, $j = 1, 2, \ldots, n-1$; $k = 1, 2$, relates the problem to a test of exponential parameters based on sequential samples of size $n - 1$. In this case, the Darmois-Koopman form applies with:

$$C(\gamma) = B(\gamma) = \gamma,\ T(x) = -x;\ g(x) = 1 \text{ if } x > 0,$$

and 0 otherwise.

Let $\psi(\gamma_1, \gamma_2) \equiv B(\gamma_1) - B(\gamma_2) = \gamma_1 - \gamma_2$ and $\psi_0 \equiv \psi(\gamma'', \gamma')$. It follows that:

$$\lambda_j = [T(x_{1,j}) - T(x_{2,j})]\,\psi_0 = (x_{2,j} - x_{1,j})\,\psi_0,$$

$$Z_{n-1} \equiv \sum_{j=1}^{n-1} \lambda_j = \Big(\sum_{j=1}^{n-1} x_{2,j} - \sum_{j=1}^{n-1} x_{1,j}\Big)\,\psi_0 \qquad (32)$$

$$= \psi_0[(n/\hat{\gamma}_{n,2}) - (n/\hat{\gamma}_{n,1})].$$

The test defined by (14) also solves the more realistic problem of testing H_0: $\psi(\gamma_1, \gamma_2) = \gamma_1 - \gamma_2 \leqslant -\psi_0$ against H_1: $\psi(\gamma_1, \gamma_2) = \gamma_1 - \gamma_2 \geqslant \psi_0$.

Although the tests provided appear to correspond to upper 1-sided alternatives, sequential tests of this type treat H_0 and H_1 symmetrically. Consequently, the roles of H_0 and H_1 can be interchanged provided the values of α and β are also interchanged.

5. ACKNOWLEDGMENT

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The Editorial Board welcomes your comments on material published in this Transactions (e.g., technical papers, editorials, and book reviews) as well as on topics of interest to Transactions readers. If the correspondence concerns technical matters it will be sent to the original authors and/or other capable people for their comments. The correspondent need not make any changes because of those "referee" comments, but is permitted to do so. Correspondence items are labeled as "not refereed" because the comments are not binding on the correspondent.

Comments on technical errors, with a reply by the original author, will usually be published as a short note. Interaction between the correspondent and the original author is encouraged in order to reduce the area of disagreement as much as feasible.

Material which is judged not suitable as a refereed paper can sometimes be abbreviated and published as correspondence.

Correspondence will be edited as to length and some rather broad limits of propriety—at the sole discretion of the Editor.

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