01 Jan 1984

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Recommended Citation

L. J. Bain et al., "Approximate Tolerance Limits And Confidence Limits On Reliability For The Gamma Distribution," *IEEE Transactions on Reliability*, vol. R thru 33, no. 2, pp. 184 - 187, Institute of Electrical and Electronics Engineers, Jan 1984.
The definitive version is available at https://doi.org/10.1109/TR.1984.5221771

Approximate Tolerance Limits and Confidence Limits on Reliability for the Gamma Distribution

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Key Words—**Gamma distribution, Tolerance limits, Percentile, Approximate methods.**

Reader Aids—
Purpose: Widen state of art
Special math needed for explanations: Statistics
Special math needed to use results: Same
Results useful to: Practicing statisticians

Summary & Conclusions—**No exact method is known for determining tolerance limits or *s*-confidence limits for reliability for the gamma distribution when both parameters are unknown. Perhaps the simplest approximate method is to determine a tolerance limit assuming the shape parameter known and then replace the shape parameter with its ML estimate to obtain approximate limits. Simulated values of the true probability levels, achieved by this method, indicate that this method is not suitable, contrary to what has been anticipated. A second approach is to consider the corresponding tolerance limits assuming the distribution mean known and the shape parameter unknown, and then replace the distribution mean by the sample mean. This approach gives useful results for many practical cases. Simulated values of the true probability levels achieved are presented for some typical cases and limiting values are provided. This method appears satisfactory for all values of the shape parameter, for the common *s*-confidence levels, and moderate sample sizes.**

1. INTRODUCTION

The gamma distribution is a very useful model for reliability and life-testing as well as many other areas. It has been difficult to develop statistical techniques for the model, partly because the parameters are not of the more convenient location-scale type. The maximum likelihood estimate of the shape parameter can be easily obtained with a computer, or with convenient tables provided by Choi & Wette [4] and Wilk, Gnanadesikan, Huyett [11]. Greenwood & Durand [7] give a very accurate rational approximation. Tests of hypotheses for the shape parameter with the scale parameter unknown can be based on the maximum likelihood estimate of the shape parameter or equivalently on the ratio of the arithmetic mean to the geometric mean. Approximate distributional results for such a test are provided by Bain & Engelhardt [1]. Optimum tests for the scale parameter with the shape parameter unknown are given by Engelhardt & Bain [6]. Grice & Bain [8] provide approximate tests or *s*-confidence limits for the mean of the gamma distribution when both parameters are unknown. Very few other results have been developed when both parameters are unknown.

Notation

κ, θ	shape and scale parameters
MLE	maximum likelihood estimate
$\hat{}$	implies the MLE
$\bar{x}$, $\tilde{x}$	sample arithmetic and geometric mean
csqf(X^2; ν)	Cdf of chisquare with ν degrees of freedom
gamf($\cdot$; β)	Cdf of standard gamma distribution with shape parameter β
$G(x; \theta, \kappa)$	Cdf of gamma distribution, gamf(x/θ; κ)
$R(t; \theta, \kappa)$	reliability at time t, $1 - G(t; \theta, \kappa)$
x_α	lower α fractile of gamma distribution
$X^2_\alpha(\nu)$	lower α fractile for chisquare distribution with ν degrees of freedom
LCL	lower *s*-confidence limit
$L(U(\kappa, \theta); \gamma)$	γ LCL for $U(\kappa, \theta)$, an arbitrary function of κ and θ
κ_L	$L(\kappa; \gamma)$
$\psi(x)$	digamma function
gauf(z)	standard *s*-normal Cdf
Z_α	lower α fractile of gauf(z)

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

2. MAXIMUM LIKELIHOOD SOLUTION

The gamma distribution is defined by the pdf:

$$g(x) \equiv x^{\kappa-1}\exp(-x/\theta)/\theta^{\kappa}\Gamma(\kappa), x > 0 \tag{1}$$

The parameters κ and θ are referred to respectively as shape and scale parameters. The MLEs of the parameters are the solutions of —

$$\hat{\theta} = \bar{x}/\hat{\kappa} \tag{2}$$

$$\ln \hat{\kappa} - \psi(\hat{\kappa}) = \ln(\bar{x}/\tilde{x}).$$

The MLEs can be easily obtained with a computer, or by using tabulations of Choi & Wette [4] which give $\hat{\kappa}$ as a function of $M \equiv \ln(\bar{x}/\tilde{x})$, or the tables of Wilk, et al. [11] which give $\hat{\kappa}$ as a function of $[1 - (\bar{x}/\tilde{x})^{-1}]^{-1}$. Greenwood & Durand [7] give the following very good rational approximation for $\hat{\kappa}$, $\hat{\kappa} = (0.5000876 + 0.1648852M - 0.0544274M^2)/M$, $0 < M \leq 0.5772$,

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$$\hat{\kappa} = \frac{8.898919 + 9.059950M + 0.9775373M^2}{M(17.79728 + 11.968477M + M^2)}, \quad 0.5772 < M \le 17,$$

$$\hat{\kappa} = 1/M, \qquad 17 < M. \tag{3}$$

3. APPROXIMATE METHODS

Suppose X is distributed according to (1). Since —

$$\frac{2X}{\theta} \sim X^2(2\kappa), \tag{4}$$

x_α can be expressed as:

$$x_\alpha = \theta X_\alpha^2(2\kappa)/2, \tag{5}$$

$$= \mu X_\alpha^2(2\kappa)/2\kappa, \tag{6}$$

where $\mu \equiv E\{X\} = \kappa\theta$.

A lower γ LCL for x_α, $L(x_\alpha; \gamma)$, is also referred to as a lower γ probability tolerance limit for proportion $1 - \alpha$ since

$$\Pr\{R(L(x_\alpha; \gamma); \theta, \kappa) \ge 1 - \alpha\} = \gamma. \tag{7}$$

There is also a direct connection between tolerance limits and s-confidence limits on reliability. For a tolerance limit the proportion $1 - \alpha$ is fixed and the limit $L(x_\alpha; \gamma)$ is a random variable. For a LCL on $R(t)$, the time t is fixed and the proportion above t is a random variable. That is, a LCL for $R(t)$, say $L(R(t); \gamma)$, is obtained by setting $t = L(x_\alpha; \gamma)$ and solving for $L(R(t); \gamma) = 1 - \alpha$.

Two approaches are discussed for obtaining approximate limits when both parameters are assumed unknown.

3.1. *Method Based on Estimation of* κ

Since —

$$\frac{2n\bar{X}}{\theta} \sim X^2(2n\kappa), \tag{8}$$

a γ LCL for θ is easily expressed, when κ is known, as:

$$L(\theta; \gamma) = 2n\bar{x}/X_\gamma^2(2n\kappa). \tag{9}$$

For κ known this gives from (5) the LCL —

$$L(x_\alpha; \gamma) = n\bar{x}\, X_\alpha^2(2\kappa)/X_\gamma^2(2n\kappa). \tag{10}$$

When κ is unknown, a reasonable inclination of a practitioner is to estimate κ and use —

$$L_1(x_\alpha; \gamma) \equiv n\bar{x}X_\alpha^2(2\hat{\kappa})/X_\gamma^2(2n\hat{\kappa}) \tag{11}$$

as an approximate LCL for the percentile. Grice & Bain [8] followed this approach in developing s-confidence limits for the mean, μ, which is also a function of both unknown parameters. Although the true s-confidence levels did differ somewhat from the nominal levels in that case, they were nearly independent of κ, and it was possible to adjust the methods so that the achieved level was close to the nominal level. Similarly Crow [5] estimates κ when considering s-confidence limits for the ratios of means in 2-sample problems, and the method appears to work very well in that case.

Unfortunately, this simple approach is not very satisfactory for the tolerance limit or reliability problem, at least for moderate sample sizes. To indicate the magnitude of error involved, table 1 shows the actual achieved γ levels for specified values of γ = 0.90, 0.95, 0.99; for α = 0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.90, 0.95, 0.99; n = 20. The achieved values are not at all close to the specified s-confidence levels for small α values, which correspond to lower tolerance limits or LCL's on reliability. This method is more suitable for large α values. The true levels are still substantially below the prescribed levels, but they are relatively constant over κ; with further study it might be possible to modify this method to be useful for large α. This approach was not followed since the lower tail is more important for reliability.

TABLE 1

Simulated values of $\Pr\{L_1(x_\alpha; \gamma) \le x_\alpha\}$, for prescribed s-confidence levels γ for n = 20.

		α								
γ	κ	.01	.05	.10	.2	.3	.4	.90	.95	.99
	0.5	.47	.48	.54	.61	.67	.73	.86	.85	.82
.90	1.0	.48	.55	.59	.67	.73	.78	.83	.80	.76
	2.0	.56	.62	.68	.73	.80	.84	.83	.82	.78
	0.5	.48	.50	.58	.67	.73	.79	.92	.91	.89
.95	1.0	.50	.58	.63	.73	.80	.84	.89	.87	.84
	2.0	.59	.67	.72	.79	.85	.88	.90	.88	.85
	0.5	.51	.55	.65	.74	.82	.88	.98	.97	.95
.99	1.0	.56	.64	.71	.82	.88	.92	.97	.96	.94
	2.0	.65	.75	.80	.87	.92	.95	.96	.95	.93

3.2. *Method Based on Estimation of* μ

An examination of tabled values provided by Beyer [3, p 295] reveals that for small α, $X_\alpha^2(\nu)/\nu$ is an increasing function of ν. An alternate approach is to estimate μ as $\bar{x}$ in (6) and replace κ by $\kappa_L = L(\kappa; \gamma)$, a LCL for κ. This gives the approximate LCL for x_α:

$$L_2(x_\alpha; \gamma) = \bar{x}X_\alpha^2(2\kappa_L)/2\kappa_L, \tag{12}$$

where an approximate LCL, κ_L, is the solution of:

$$\ln(\bar{x}/\tilde{x}) = X_{1-\gamma}^2(\nu(\kappa_L, n))/2n\kappa_L c(\kappa_L, n). \tag{13}$$

TABLE 2
Simulated values of $\Pr\{L_2(x_\alpha; \gamma) \le x_\alpha\}$ for prescribed confidence levels γ.

		$\gamma = .90$				$\gamma = .95$				$\gamma = .99$			
		$\varkappa$				$\varkappa$				$\varkappa$			
α	n	.5	1.0	2.0	∞	.5	1.0	2.0	∞	.5	1.0	2.0	∞
	10	.86	.91	.92	.88	.87	.95	.95	.94	.99	.99	.99	.99
.01	20	.89	.90	.90	.87	.94	.95	.95	.93	.97	.99	.99	.98
	∞	.90	.89	.89		.95	.95	.94		.99	.99	.99	
	10	.88	.91	.91	.86	.94	.95	.95	.92	.99	.99	.99	.98
.05	20	.89	.90	.90	.85	.94	.95	.94	.91	.98	.99	.99	.97
	∞	.89	.89	.88		.94	.94	.93		.99	.99	.98	
	10	.89	.91	.91	.84	.94	.95	.95	.91	.99	.99	.99	.97
.10	20	.90	.90	.89	.83	.94	.95	.94	.89	.98	.99	.99	.96
	∞	.89	.88	.85		.94	.93	.91		.99	.98	.97	
	10	.63	.89	.88	.80	.60	.94	.93	.87	.64	.97	.98	.96
.20	20	.80	.89	.87	.78	.77	.94	.93	.84	.60	.98	.98	.93
	∞	.88	.85	.76		.93	.91	.82		.98	.97	.90	

Tabulations and approximations for $\nu(\varkappa, n)$ and $c(\varkappa, n)$ are provided by Bain & Engelhardt [1]. For larger values of $\varkappa_L$, say $\varkappa_L > 2$, $\varkappa_L \approx X^2_{1-\gamma}(n-1)/2n \ln(\bar{x}/\tilde{x})$.

The corresponding γ LCL for reliability at time t is obtained by setting $L_2(x_\alpha; \gamma) = t$: to give

$$L(R(t); \gamma) = 1 - \text{csqf}(2\varkappa_L t/\bar{x}; 2\varkappa_L). \tag{14}$$

For prescribed values $\gamma = 0.90, 0.95, 0.99$ the actual probability levels achieved by $L_2(x_\alpha; \gamma)$ were determined by Monte Carlo simulation for $\alpha = 0.01, 0.05, 0.1, 0.2$; $\varkappa = 0.5, 1.0, 2.0$; $n = 10, 20$. These results are presented in table 2. Method 2 provides a very suitable lower tolerance limit for the commonly used proportions $1 - \alpha = 0.99, 0.95, 0.90$. The corresponding LCL on reliability is reasonable if the LCL is as high as 0.8 or so, which includes most practical reliability problems. The true *s*-confidence levels are somewhat below the stated levels γ for large values of $\varkappa$ and large values of n; however, they are sufficiently close in most cases for general practical applications.

The simulated values in tables 1 and 2 are intended for illustrative purposes only, and they may not be accurate, in every case, to two decimal places.

4. MONTE CARLO METHOD

The simulation was similar to that of Grice & Bain [8]. Since $Y \equiv \bar{X}/\mu \sim G(y; 1/n\varkappa, n\varkappa)$, $\bar{X}$, and $\hat{\varkappa}$ are *s*-independent, and $\varkappa_L$ is a function only of $\hat{\varkappa}$, then —

$$\Pr\{L_2(x_\alpha; \gamma) \le x_\alpha\} = \Pr\{\bar{X} X^2_\alpha(2\varkappa_L)/2\varkappa_L \le \mu X^2_\alpha(2\varkappa)/2\varkappa\}$$

$$= \Pr\left\{ \frac{\bar{X}}{\mu} \le \frac{2\varkappa_L\, X^2_\alpha(2\varkappa)}{2\varkappa\, X^2_\alpha(2\varkappa_L)} \right\}$$

$$= E\{G(\varkappa_L X^2_\alpha(2\varkappa)/\varkappa X^2_\alpha(2\varkappa_L); 1/n\varkappa, n\varkappa)\}. \tag{15}$$

For each combination of n, α, γ, $\varkappa$, 1000 samples were generated; the quantity inside the braces was computed, and the values were averaged.

The IMSL [9] subroutine GGAMR was used to generate the pseudo-random numbers, and IMSL subroutine Z REAL 1 was used to solve (13) for $\varkappa_L$. The approximate formulas for $\nu(\varkappa, n)$ and $c(\varkappa, n)$ provided by Bain & Engelhardt [1, p 949] were used.

5. NUMERICAL EXAMPLE

Lieblein & Zelen [10] provided the following sample of size $n = 23$ to illustrate the use of a Weibull model for the endurance, in millions of revolutions, for deep-groove ball bearings. Bain & Engelhardt [2] show that a gamma model is preferable for these data.

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84
51.96	54.12	55.56	67.80	68.64	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

The results are: $\bar{x} = 72.22$, $\tilde{x} = 63.46$, $\hat{\varkappa} = 4.025$. An approximate 95% LCL for $\varkappa$ is $\varkappa_L \approx X^2_{.05}(n-1)/2n \ln(\bar{x}/\tilde{x}) = 2.07$.

An approximate 95% lower tolerance limit for proportion 0.95 is, from (12), $L_2(x_{.05}, 0.95) = 72.22(0.77)/4.14 = 13.5$. Interpolating on $1/\varkappa$ in table 2 for $n = 20$ and $\hat{\varkappa} = 4$ indicates that the true *s*-confidence level is approximately 0.925.

An approximate 95% LCL for the reliability of the ball bearings at 13 million revolutions, based on (14), is $L(R(13), 0.95) = 1 - \text{csqf}(0.745; 4.14) = 0.953$.

6. ASYMPTOTIC RESULTS

It is possible to obtain limiting values of (15) either as $\varkappa \to \infty$ for fixed n, or as $n \to \infty$ for fixed $\varkappa$. This is due to the fact that $G(t, 1, n\varkappa)$ can be uniformly approximated by $\text{gauf}[(t - n\varkappa)/\sqrt{n\varkappa}\,]$ for either large $\varkappa$ or large n. Furthermore, $G_\alpha(1, \varkappa) \approx \varkappa + Z_\alpha\sqrt{\varkappa}$ for large $\varkappa$.

The limiting value of (15) as $\varkappa \to \infty$ for fixed n is —

$$E\{\text{gauf}(\sqrt{n}\, Z_\alpha[1 - (V/X^2_{1-\alpha}(n-1))^{1/2}])\} \tag{16}$$

where $V \sim X^2(n-1)$ is the limit, in distribution, of $V_\varkappa \equiv 2n\varkappa \ln(\bar{x}/\tilde{x})$. The limiting value of (15) as $n \to \infty$ for fixed $\varkappa$ is —

$$E\{\text{gauf}(C_\alpha(\varkappa)(Z - Z_\gamma))\} \tag{17}$$

where Z is standard *s*-normal, and —

$$C_\alpha(\varkappa) \equiv \left[1 - \varkappa \frac{d}{d\varkappa} G_\alpha(1, \varkappa)/G_\alpha(1, \varkappa)\right] / [\varkappa\psi'(\varkappa) - 1]. \tag{18}$$

The results of (16)-(18) were used to obtain, by numerical integration, the limiting values in table 2.

ACKNOWLEDGMENT

Research sponsored by the Air Force Office of Scientific Research, Air Force System Command, USAF, under Grant Number AFOSR 84-0164. The U.S. Government is authorized to reproduce and distribute reprints for Govermental purposes notwithstanding any copyright notation thereon.

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Manuscript TR83-02 received 1983 January 10; revised 1983 November 18. ★★★

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