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# Rubber Covered Rolls—The Nonlinear Elastic Problem[1]

*The problem of the indentation of a rubberlike layer bonded to a rigid cylinder and indented by another rigid cylinder is analyzed. The rubberlike layer is assumed to be made of a homogeneous Mooney-Rivlin material. The materially and geometrically nonlinear problem is solved by using the finite-element code developed by the author. Results computed and presented graphically include the pressure profile at the contact surface, stress distribution at the bond surface and the deformed shape of the indented surface.*



## Introduction

Traction in vehicles, the nip action in cylindrical rolls in the papermaking process and in the textile industry, and friction drives are some examples of the kind of problem studied herein. Each of these problems involves indentation, by a rigid cylinder, of the rubberlike layer bonded to a core made of a considerably harder material. Such problems have been studied analytically [1], experimentally [2], and numerically [3] by using the finite-element method. In [1] Hahn and Levinson solve the indentation problem on the assumption that the rubberlike layer is made of a Hookean material and its deformations are within the range of applicability of the linear theory. The problem is solved by using an Airy stress function and the solution is in terms of double infinite series one of which converges slowly. In the numerical study [3], Batra, et al., assume that the rubberlike layer is made of a thermorheologically simple material and its deformations are small so that the linear strain-displacement relations and a linear relation between stress and strain rate can be presumed. The experimental work [2] of Spengos is quite extensive and involves a wide range of loads, thicknesses of the rubber layer, and speed differences between the mating rollers. Other contact problems involving geometries different from the one considered here have been studied by Sve and Keer [4], Keer and Sve [5], Itou and Atsumi [6], Alblas and Kuipers [7–9], and Batra [10, 11].

A study of the results of Hahn and Levinson suggests that for moderate values of nip width, the value of the maximum principal strain is of the order of 20 percent. This observation is also confirmed by the experimental investigations of Spengos. It therefore appears that the maximum strain commonly encountered in practice is probably much higher than what is usually thought to be the range

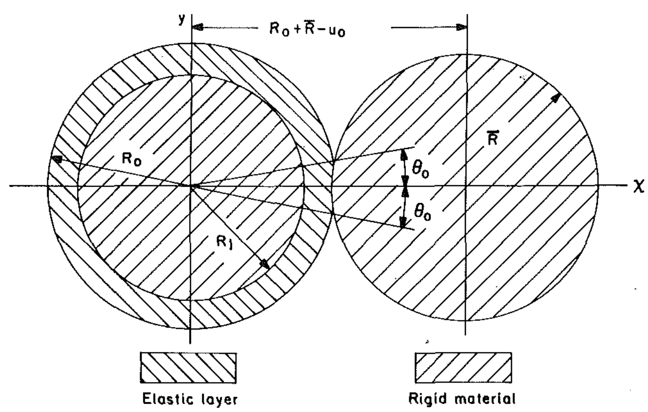


Fig. 1 System to be studied

of applicability of the linear theory. Thus one needs to develop methods to solve the problem when the deformations are large and the material of the layer is nonlinear. In this paper we assume that the material of the rubberlike layer can be modeled as a Mooney-Rivlin material and solve the large deformation problem by the finite-element method.

A schematic diagram of the system studied is shown in Fig. 1. Since the length of rolls is considerably large as compared to their diameters, we assume that plane strain state of deformation prevails. Methodologies to solve finite plane strain problems for incompressible elastic materials have been given by Oden [12] and Scharnhorst and Pian [13]. Realizing that these authors had developed computer programs tailored to solving specific problems and illustrating the principles involved, the author developed a computer code capable of solving quasi-static, mixed boundary-value finite plane strain problems for Mooney-Rivlin materials. Results obtained for two sample problems by using this program compare favorably with those obtained from the analytical solution [14]. The indentation problem considered in this paper is solved by using this basic program and the techniques

[1] Dedicated to Clifford Truesdell on the occasion of his 60th birthday.

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developed earlier [3, 10, 11] for solving contact problems numerically.

## Formulation of the Problem

We use a fixed set of rectangular Cartesian axes with origin at the center of the roll with the rubberlike cover and denote the position of a material particle in the reference configuration by $X^\alpha$ and the position of the same material particle in the current configuration by $x_i$. Thus $x_i = x_i(X^\alpha, t)$ gives the current position, at time $t$, of the material particle that occupied place $X^\alpha$ in the reference configuration. Since the core and the mating cylinder are usually made of a material considerably harder than the material of the rubberlike layer, we regard these as being rigid and study only the mechanical deformations of the rubberlike layer. Neglecting the effect of body forces such as gravity on the deformations of the roll cover, equations governing the deformations of the rubberlike layer are

$$\det F_{i\alpha} = 1, \quad F_{i\alpha} \equiv x_{i,\alpha}, \tag{1}$$
$$\rho \ddot{x}_i = T_{i\alpha,\alpha}.$$

In (1) $\rho$ is the present mass density, $T_{i\alpha}$ is the first Piola-Kirchoff stress tensor, a superimposed dot indicates material time differentiation, a comma followed by an index $\alpha$ indicates partial differentiation with respect to $X^\alpha$, $F_{i\alpha}$ is the deformation gradient, and the usual summation convention is used. Equation $(1)_1$ is the continuity equation in referential description and signifies that the mass density stays constant. The first Piola-Kirchoff stress tensor $T_{i\alpha}$ and the Cauchy stress tensor $\sigma_{ij}$ are related by

$$\sigma_{ij} = \frac{\rho}{\rho_0} T_{i\alpha} F_{j\alpha} \tag{2}$$

in which $\rho_0$ is the mass density in the reference configuration. For incompressible materials, $\rho = \rho_0$ and (2) simplies to $\sigma_{ij} = T_{i\alpha}F_{j\alpha}$. Equation (1) is to be supplemented by constitutive relation for $T_{i\alpha}$ and side conditions such as boundary conditions. Before we state these, we give the following assumptions to simplify the problem.

We assume that the material of the roll cover is homogeneous and can be modeled as a Mooney-Rivlin material, contact between the indentor and the roll cover is frictionless, and that the effect of all dynamic forces on the deformation of the roll cover is negligible. We note that the mass density of rubber is quite low (comparable to that of water). Therefore, for practical geometries and speeds in the range of 500 rpm, the effect of centrifugal force on the stress distribution is very small. Under these assumptions the indentation problem becomes quasi-static and equation (1) is replaced by

$$\det F_{i\alpha} = 1,$$
$$0 = T_{i\alpha,\alpha}$$
$$(F^{-1})_{\alpha i} T_{i\beta} = S_{\alpha\beta} = p(C^{-1})_{\alpha\beta} + 2C_1\delta_{\alpha\beta} + 2C_2(I_1\delta_{\alpha\beta} - C_{\alpha\beta}),$$
$$C_{\alpha\beta} = F_{i\alpha}F_{i\beta}, \quad I_1 = C_{\alpha\alpha}. \tag{3}$$

In these equations, $C_{\alpha\beta}$ is the right Cauchy-Green tensor, $C_1$ and $C_2$ are material constants, $p$ is the hydrostatic pressure not determined by the deformation of the roll cover but can be found from the boundary conditions, $\delta_{\alpha\beta}$ is the Kronecker delta, $I_1$ is the first invariant of the strain tensor $C_{\alpha\beta}$ and $S_{\alpha\beta}$ is the second Piola-Kirchoff stress tensor.

In practice the length of the cylindrical rollers is significantly larger than their diameters so that it is reasonable to presume that plane strain state of deformation prevails. Thus $x_3 = \delta_{3\alpha}X^\alpha$ and equation $(3)_2$ for $i = 3$ is identically satisfied. Furthermore, deformations of the roll cover are symmetrical about the line joining the centers of the rollers. Because of this symmetry, we study the deformations of the upper half of the roll cover.

Equation $(3)_1$ and the set of equations obtained by substituting $(3)_3$ into $(3)_2$ are three equations for the three unknown fields $p$, $x_1$, and $x_2$. These equations are to be solved under the following boundary conditions. At the inner surface $X_\alpha X_\alpha = R_1$,

$$u_i = x_i - \delta_{i\alpha}X^\alpha = 0, \tag{4}$$

at the outer surface $X_\alpha X_\alpha = R_0$,

$$e_i T_{i\alpha} N_\alpha = 0, \tag{5}$$

$$n_i T_{i\alpha} N_\alpha = 0, \qquad \text{if } \theta = \arc\tan\left(\frac{X_2}{X_1}\right) \ge \theta_0,$$
$$= f(\theta), \quad \text{if } \arc\tan\left(\frac{X_2}{X_1}\right) \le \theta_0, \tag{6}$$
$$f(\theta_0) = 0,$$

and at the plane through the center line of rollers,

$$u_2 = 0,$$
$$T_{12} = 0. \tag{7}$$

In equations (4)–(7), $N_\alpha$ is an outward unit normal to the surface in the reference configuration, $e_i$ is an unit tangent vector to the surface in the current configuration, and $n_i$ is an unit outward normal to the surface in the current configuration. The boundary condition (4) implies that there is perfect bonding between the core and the rubberlike layer, and the boundary conditions (5) and (6) signify that the part of the roll cover not in contact with the indentor is traction free whereas that in contact with the indentor has a normal pressure acting on it. Note that $\theta_0$ defines half nip width in the reference configuration. The boundary condition $(6)_3$ insures that a contact problem rather than a punch problem is being solved.

We note that the half nip width $\theta_0$ and the pressure $f(\theta)$ at the contact surface are unknown and are to be determined as a part of the solution. These two should assume values such that the deformed surface of the rubber like layer matches with the profile of the indentor. In practice the load $P$, given by

$$P = 2\int_0^{\theta_0} f(\theta)d\theta, \tag{8}$$

pressing the two rolls together is specified. However for ease in computation, we prescribe $\theta_0$ and find the required load. Of course one could equally well prescribe the indentation $u_0$, as is done in [10], between the two rolls and compute the necessary load. Specification of $P$ and then finding $\theta_0$ and the indentation $u_0$, though feasible, results in significantly more computing time. The indentation $u_0$ equals the distance through which the two rolls move closer when loaded and is the value of the radial displacement of that point on the outermost surface of the roll cover that lies on the center line of the rollers.

The problem as just formulated is too difficult to solve analytically, so we solve it by the finite-element method.

## Brief Description of the Finite-Element Formulation

We use the total Lagrangian formulation and the principle of stationary potential energy. That is, the potential energy

$$E = \int \left(W + \frac{p}{2}(I_3 - 1)\right)dV - \int h_\alpha u_\alpha \, dA \tag{9}$$

takes an extremum value [12, p. 253] for all admissible displacement fields that satisfy the displacement boundary condition. In (9), $\mathbf{h}$ is the surface traction acting on a unit area in the reference configuration, $W$ is the strain-energy density and $I_3 = \det \mathbf{C}$ is the third invariant of $\mathbf{C}$. For Mooney-Rivlin materials

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \quad I_2 = I_3{}^{-1}(C^{-1})_{\alpha\alpha}. \tag{10}$$

$\delta E = 0$ gives

$$\int S_{\alpha\beta}\delta E_{\alpha\beta}dV = \int h_\alpha \delta u_\alpha dA, \quad \int \delta p(I_3 - 1)dV = 0, \tag{11}$$

in which $\mathbf{E} = (\mathbf{C} - \mathbf{1})/2$.

We assume that the load $f(\theta)$ at the contact surface is applied in $M$ equal increments and denote the incremental change in the value of say $\mathbf{u}$ caused by the $(N + 1)$st load increment by $\Delta\mathbf{u}$, i.e.,

$$\mathbf{u}^{N+1} = \mathbf{u}^N + \Delta\mathbf{u}, \quad \mathbf{E}^{N+1} = \mathbf{E}^N + \Delta\mathbf{E}. \tag{12}$$

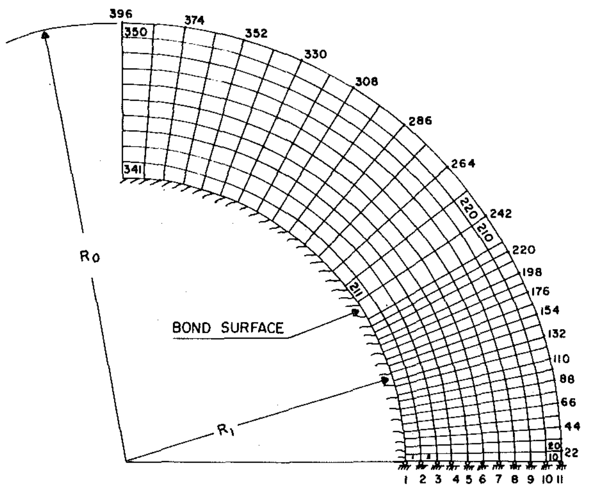


Fig. 2 Finite-element grid

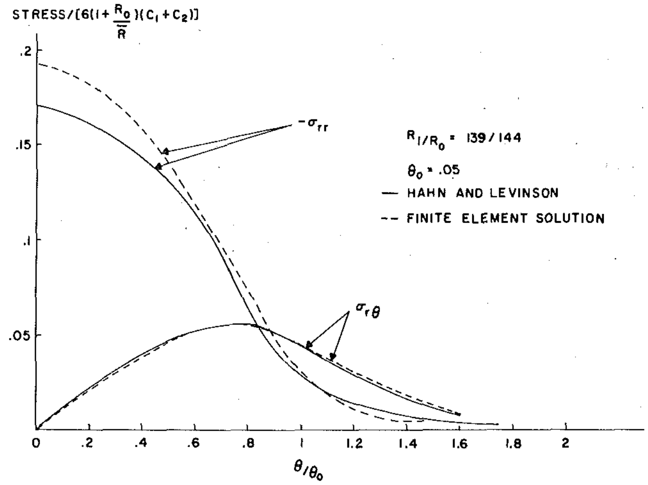


Fig. 3 Stress distribution at the bond surface; comparison of present results with those of Hahn and Levinson

From equations $(3)_4$, $(1)_2$, (4) and the definition of $\mathbf{E}$, we obtain

$$\Delta E_{\alpha\beta} = \Delta e_{\alpha\beta} + \Delta\eta_{\alpha\beta}, \quad \Delta\eta_{\alpha\beta} = \tfrac{1}{2}\Delta u_{\gamma,\alpha}\Delta u_{\gamma,\beta},$$

$$\Delta e_{\alpha\beta} = \tfrac{1}{2}(\Delta u_{\alpha,\beta} + \Delta u_{\beta,\alpha} + u^N{}_{\gamma,\alpha}\Delta u_{\gamma,\beta} + u^N{}_{\gamma,\beta}\Delta u_{\gamma,\alpha}) \quad (13)$$

We note that $\Delta I_3 = 2(C^{-1})_{\alpha\beta}\Delta E_{\alpha\beta}$. The relation between $\Delta\mathbf{S}$, $\Delta\mathbf{E}$ and $\Delta p$ is given in reference [13]. Setting $\delta E_{\alpha\beta} = \delta\Delta E_{\alpha\beta}$, $\delta u_\alpha = \delta\Delta u_\alpha$, and $\delta p = \delta\Delta p$ in (11) we obtain

$$\int (S_{\alpha\beta}{}^N + \Delta S_{\alpha\beta})\delta\Delta E_{\alpha\beta} dV = \int h_\alpha{}^{N+1}\delta\Delta u_\alpha dA, \quad (14)$$

$$\int \delta\Delta p (C^N)^{-1}{}_{\alpha\beta}\Delta E_{\alpha\beta} dV = -\tfrac{1}{2}\int \delta\Delta p(I_3{}^N - 1)dV. \quad (15)$$

We now make the assumption that the increment in the load is small so that

$$\Delta S_{\alpha\beta}\delta\Delta E_{\alpha\beta} \simeq \Delta S_{\alpha\beta}\delta\Delta e_{\alpha\beta},$$
$$(C^N)^{-1}_{\alpha\beta}\delta\Delta E_{\alpha\beta} \simeq (C^N)^{-1}_{\alpha\beta}\delta\Delta e_{\alpha\beta}, \text{etc.} \quad (16)$$

Hence an approximation to equations (14) and (15) is

$$\int \Delta S_{\alpha\beta}\delta\Delta e_{\alpha\beta} dV + \int S_{\alpha\beta}{}^N \delta\Delta\eta_{\alpha\beta} dV \simeq \int h_\alpha{}^{N+1}\delta\Delta u_\alpha dA - \int S_{\alpha\beta}{}^N \delta\Delta e_{\alpha\beta} dV \quad (17)$$

$$\int \delta\Delta p (C^N)^{-1}_{\alpha\beta}\Delta e_{\alpha\beta} dV \simeq -\tfrac{1}{2}\int \delta\Delta p (I_3{}^N - 1)dV. \quad (18)$$

We use equilibrium iterations [15], i.e., iterations within a load step, to insure that equations (17) and (18) are satisfied within a prescribed error.

A finite-element program based on equations (17) and (18) and employing 4-node isoparametric quadrilateral elements with $2 \times 2$ Gaussian integration rule has been written. The hydrostatic pressure $p$ is assumed to be constant within an element. The pressure load between two surface nodal points $a$ and $b$ is replaced by lumped nodal loads given by

$$h_i{}^a = h_i{}^b = f(\theta^*)\epsilon_{3ij}(x_j{}^b - x_j{}^a).$$

Here $\epsilon_{ijk}$ is the permutation symbol and it equals 1 or $-1$ according as $i, j, k$ form an even or an odd permutation of 1, 2, and 3 and is zero otherwise and $\theta^*$ is the value of $\theta$ for the midpoint of the line joining nodes $a$ and $b$. The loads for the $(N + 1)$th load step are calculated based upon the positions of the nodes after the $N$th load step.

The accuracy of the developed finite-element code has been established by comparing results for two sample problems with their analytical solution [14]. This program has been modified to solve the contact problem.

## Computation and Discussion of Results

In order to solve the problem by the finite-element method, we consider the quarter of the roll cover lying in the first quadrant and assume that the surface along the vertical plane is traction-free. This assumption is motivated by previous studies on this problem in which it has been found that stresses decay rather rapidly with the distance from the contact region. This assumption is verified to be true in the present study too. This portion of the roll cover is divided into quadrilateral elements as shown in Fig. 2. The mesh is finer within approximately one and a half times the contact width.

Half nip width $\theta_0$ and the form of the function $f(\theta)$ are assumed. The presumed load is divided into 30 equal steps and within each load step upto 15 equilibrium iterations [15] are performed to insure that displacements are accurate to within 1 percent of their values. The deformed surface of the roll cover is calculated and a check is made to insure that the deformed surface in the assumed contact zone matches, within a prescribed tolerance, with the circular profile of the indentor and that the nodal point just outside the assumed contact area has not penetrated into the indentor. If the second condition is not satisfied implying thereby that the nodal point outside the presumed contact width has penetrated into the indentor, either the value of $\theta_0$ is increased or the total load is decreased. However, if the second condition is satisfied but the first is not, the form of $f(\theta)$ is suitably modified until both preceeding conditions are satisfied simultaneously. The deformed surface of the roll cover is taken to match with the profile of the indentor if the distance of each nodal point on the contact surface from the indentor is within 1 percent of the indentation $u_0$. Usually, with a little experience, one can make pretty good estimates of $\theta_0$ and $f(\theta)$ so that the entire process converges in four or five iterations.

In order to insure that the modifications made in the program to solve contact problems are correct, we compare results computed from the present program with those given by Hahn and Levinson. As is clear from Fig. 3, the values of shear stress obtained by these two different methods are quite close. As for the difference in the values of the radial stress we remark that a similar difference exists between Hahn and Levinson's results and those of Betz and Levinson [16] who used the finite-element method to solve the problem. It should be added that Hahn and Levinson's solution is in the form of a double series and the computation of numerical results does involve convergence errors. However, the appreciable difference between the analytical solution and the finite-element solution can only be attributed to different methodologies.

Fig. 4 depicts the pressure profiles obtained by Spengos [3] and the present solution using the nonlinear theory. The two compare favorably. The difference between the two is possibly due to the fact that the assumption of plane strain state of deformation made in the present work is not quite valid for Spengos' experimental set up wherein the length-to-diameter ratio of rollers was of the order of one. Whereas Spengos reports that when the experimental contact widths are corrected by accounting for the finite size of the recording in-

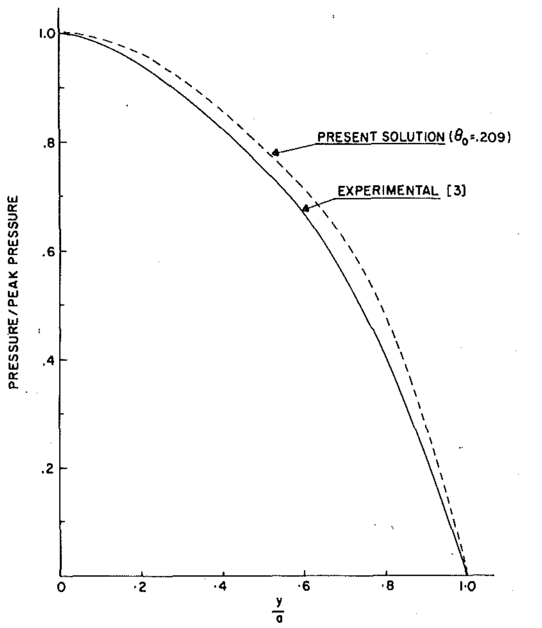


**Fig. 4 Stress distribution at the contact surface; comparison of experimental and numerical results**

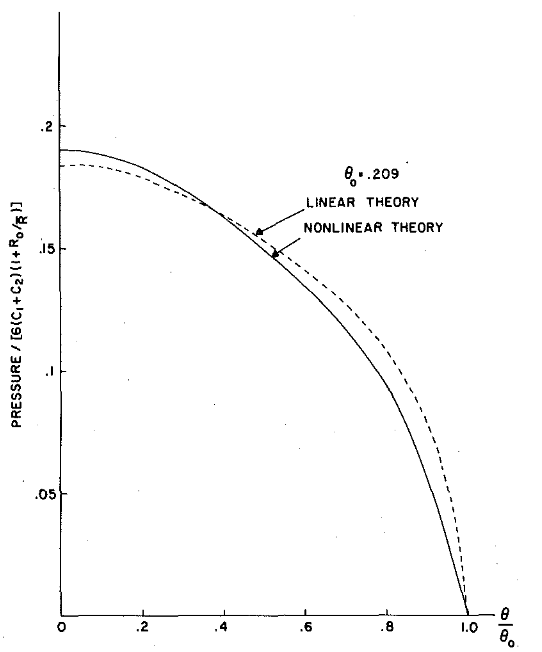


**Fig. 5 Stress distribution at the contact surface; comparison of results from linear and nonlinear theories**

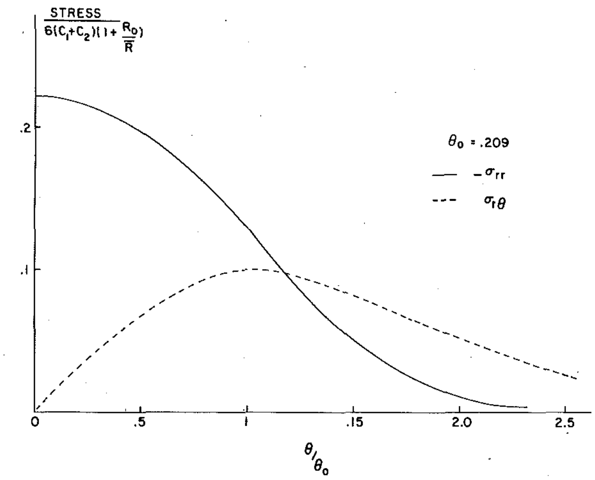


**Fig. 6 Stress distribution at the bond surface**

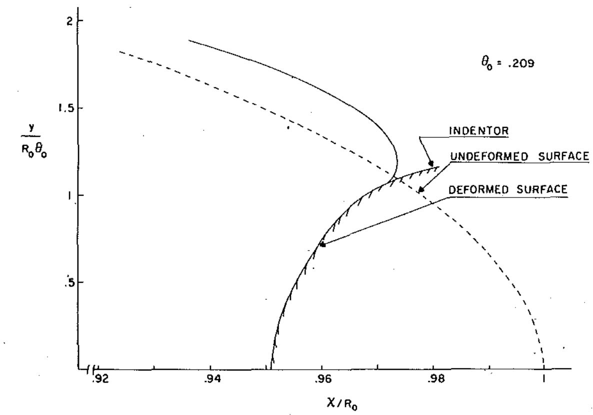


**Fig. 7 Deformed surface of the rubberlike layer**

struments, the pressure profiles for various nip widths match, we obtain slightly different pressure profiles for different contact widths. In the results presented in Figs. 4–7, the values of various geometrical parameters correspond to those for run number 30 of Spengos. (That is, $R_1$ = 47.2 mm, $R_0$ = 60.7 mm, $\bar{R}$ = 76.2 mm.) In Fig. 5 is shown the pressure profile obtained by using the linear and the nonlinear theory. In the linear theory entire load is applied in one step and no account is taken of the deformation of the surface on which the load is applied. Also the strain-displacement relation and the stress-strain relations are linear. In the nonlinear theory, the problem is solved incrementally and each increment in load is applied on the surface deformed up to the application of that load increment. We remark that in Fig. 5, the pressure profile at the contact surface represents the nondimensionalized Cauchy stress. It should be added that in the linear theory $6(C_1 + C_2)$ equals Young's modulus.

Results presented in Fig. 6 verify the assumption that stresses decay rapidly at points away from the contact zone. This insures that the assumption that the vertical surface of the quarter roll cover considered is traction-free does not introduce any significant error in the computed results.

Fig. 7 depicts the deformed surface of the roll cover. Because of symmetry, only half of the deformed surface is shown. Also due to different scales along the horizontal and vertical axes, the undeformed roll cover as well as the indentor plot as ellipses. The radius of curvature of the deformed surface changes near the point where rubber leaves the indentor.

For plane strain deformations of Mooney-Rivlin materials, one can show that [14] the values of displacement and components $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{12}$ of the Cauchy stress depend upon the material constants $C_1$ and $C_2$ only through their sum $C_1 + C_2$. Thus results presented herein are valid for all values of $C_1/C_2$ so long as the sum $C_1 + C_2$ is kept fixed. The values of the hydrostatic pressure $p$ and the stress $\sigma_{33}$ normal to the plane of deformation do depend upon $C_1/C_2$ even when $(C_1 + C_2)$ is constant.

Further extension of this work should involve the inclusion of the effects of friction at the contact surface, slipping at some points on the contact surface, and the deformations of the core and the indentor.

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Second International Conference on Computational Methods in Nonlinear Mechanics held at the University of Texas-Austin in Mar. 1979.

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