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# AUTOOSCILLATIONS AND NONLINEAR ANELASTICITY*

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## Nonlinear Anelasticity

Linear anelasticity and the associated internal friction at small amplitudes has received much attention. In the vicinity of first order phase transformations nonlinear anelasticity should be pronounced for two reasons. In the martensitic transformation, $DO_3 \rightarrow 2H$, $18R_1$, $18R_2$, or 6R in Cu-Al-Ni alloys, the atom must be displaced over a finite distance to accomplish the transformation (1). Accordingly, the relationship between the stress $\sigma$ and the strain $\varepsilon$ in the vicinity of the $M_s$ temperature must be described in terms of a non-linear extension of Hooke's law. Furthermore, unusual phenomena have been observed in the course of large amplitude internal friction experiments in the same alloy (2). The displacement of atoms in the course of the transformation represents a relaxation process. Therefore, two kinds of the non-linear extensions of Hooke's law must be introduced depending on whether the relaxed or un-relaxed strain is considered. The term unrelaxed indicates that no relaxation processes have taken place. Combining the nonlinear extensions of Hooke's law for the relaxed and un-ralaxed state, the stress strain relationship characteristic of a solid in the vicinity of a martensitic transformation is given as follows (3),

$$\sigma + \tau\dot{\sigma} = (C_{r2} + C_{r3}\varepsilon + C_{r4}\varepsilon^2 + C_{r5}\varepsilon^3 + C_{r6}\varepsilon^4)\varepsilon + (C_{u2} + C_{u3}\varepsilon + C_{u4}\varepsilon^2 + C_{u5}\varepsilon^3 + C_{u6}\varepsilon^4)\tau\dot{\varepsilon}. \quad (1)$$

Here, the subscripts r and u stand for 'relaxed' and 'unrelaxed' respectively, $\tau$ is the relaxation time and the number indices indicate the order of the elastic constants C. More cautiously, these constants might be called mechanical response coefficients as the modes of deformation operating in the vicinity of $M_s$ are yet to be determined. It will be seen later why the series expansion has been terminated at the sixth order. It should be noted that for small strains Eq. (1) reduces to the well known stress-strain relationship for a linear anelastic solid (4).

## Equation of Motion

The equation of motion of a nonlinear anelastic solid can be derived from Newton's equation of motion,

$$\rho\partial^2\varepsilon / \partial t^2 = \partial^2\sigma / \partial x^2, \quad (2)$$

by use of the stress-strain relationship given by Eq. (1), where $\rho$ indicates the density and x is

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the coordinate along the appropriate direction of the specimen to be used in the experiment. It must be understood that $\varepsilon$ and $\sigma$ are corresponding strains and stresses. For the nonlinear anelastic solid characterized by Eq. (1) the equation of motion (2) reads

$$\rho\partial^2\varepsilon/\partial t^2 + \tau\partial^3\varepsilon/\partial t^3 = \partial^2[C_r(\varepsilon)\varepsilon]/\partial x^2 + \partial^3[C_u(\varepsilon)\tau\varepsilon]/\partial x^2\partial t, \tag{3}$$

where $C_r(\varepsilon)$ and $C_u(\varepsilon)$ refer to the series expansions of the relaxed and unrelaxed elastic constants shown in Eq. (1).

A vibrating reed is used for the present investigation. The complete solution of Eq. (3) for such a reed is rather complicated. For the purpose of a first orientation only the fundamental of the clamped-free reed need be considered. This fundamental may be approximated by

$$\varepsilon(x, t) = A(t)\cdot\sin(\pi x/2\ell), \tag{4}$$

where $\ell$ is the length of the reed. The combination of Eqs. (3) and (4) yields a differential equation for the amplitude A(t) of the form

$$\rho\ddot{A} + \tau\dddot{A} + (\pi/2\ell)^2[C_{r2}A + (3/4)C_{r4}A^3 + (5/16)C_{r6}A^5]$$
$$+ (\pi/2\ell)^2[C_{u2}\dot{A} + (9/4)C_{u4}A^2\dot{A} + (25/8)C_{u6}A^4\dot{A}] = E\cdot\sin\omega_1 t. \tag{5}$$

The term $E\cdot\sin(\omega_1,t)$ introduces the external excitation. An equation of this kind has already been solved approximatively to first order in a different context. A detailed description of the approximation is readily accessible (5) and will therefore not be repeated here. The solution is given by

$$A(t) = y(t)\cos[\omega_0 t + \phi(t)] + b\sin(\omega_1 t), \tag{6}$$
$$b = E/(\omega_0^2 - \omega_1^2),$$

where $\omega_0$ is the angular resonance frequency of the reed specimen and the functions y(t) and $\phi(t)$ are to be determined by yet another two differential equations,

$$\dot{y} = -1/2\tau\omega_0^2 C_{r2}^{-1} y[C_{u2} - C_{r2} + (9/16)C_{u4}y^2 + (9/8)C_{u4}b^2$$
$$+ (25/64)C_{u6}y^4 + (75/32)C_{u6}y^2b^2 + (75/64)C_{u6}b^4], \tag{7}$$

$$\dot{\phi} = 1/8C_{r2}^{-1}\omega[(9/4)C_{r4}y^2 + (9/2)C_{r4}b^2 + (25/16)C_{r6}y^4$$
$$+ (75/8)C_{r6}y^2b^2 + (75/16)C_{r6}b^4]. \tag{8}$$

## Discussion of Equations (7) and (8)

Equation (7) shows that the amplitude y is damped in most cases, i.e. $\dot{y} < 0$, since it is known that $C_{u2} > C_{r2}$ (6) and because the unrelaxed elastic constants are usually larger than zero. Equation (6) then reduces to the normally expected steady state forced vibration. In the vicinity of a martensitic transformation this is not necessarily so, however, Here, the unrelaxed mechanical elastic potential of the solid,

$$F_u(\varepsilon) = 1/2C_{u2}\varepsilon^2 + 1/4C_{u4}\varepsilon^4 + 1/6C_{u6}\varepsilon^6, \tag{9}$$

must be expected to have a metastable state. Equation (9) represents the simplest form of such a potential in which contributions of odd order have been omitted because they do not enter into Eq. (7). A metastable state occurs at $\varepsilon = \varepsilon_m$ if $C_{u4} < 2C_{u6}\varepsilon_m^2$, i.e. if the coefficient $C_{u4}$ is negative. In this case, as an inspection of Eq. (7) shows, the amplitude y can grow if the amplitude of the forced vibration, b, lies in a certain range. It thus follows that a seemingly spontaneous oscillation can arise in addition to the externally driven oscillation in a solid driven to sufficiently large amplitudes of oscillation in the vicinity of its martensitic transformation. Oscillations of this kind have been called autooscillations (7).

The characteristics of the autooscillations are obtained by integrating Eq. (7). This can be done analytically for small and large amplitudes y while a numerical solution is needed inbetween. Neglecting the term quartic in y yields for the small amplitude behaviour

$$(y/y_0)^2 = z[1 - (1 - z)\exp(1/2t\tau\omega_0^2)]^{-1}, \tag{10}$$

where $y_0$ is the amplitude of the autooscillation at time zero and the parameter z is given by

$$z = [64(C_{u2} - C_{r2}) - 32C_{u4}b^2 + 75b^4]/[(32C_{u4} - 150C_{u6}b^2)b^2]. \quad (11)$$

The limiting value of the amplitude of the autooscillation at very long times, $y_\infty$, can be obtained by neglecting the term linear in y in Eq. (7). The result is

$$(y_\infty / y_0)^2 = (36C_{u4} - 150C_{u6}b^2)/125C_{u6}. \quad (12)$$

It can be seen from Eq. (10) that the autooscillation can only occur if $z < 1$. It can also be seen that the development of the autooscillation depends very strongly on the magnitude of the forced vibration, b.

If an autooscillation is present it will manifest itself as a modulation of the forced vibration of the sample due to the superposition of the two, see Eq. (6). Since the modulator is the autooscillation this modulation will be called automodulation. The period of the automodulation, $\delta\omega$, is given by the beat frequency of the forced and the autooscillation, see Eq. (6). The difference between the frequencies of these two oscillations is determined by the amplitude dependence of the eigen-frequency of the sample, $\omega_0$. If the autooscillation is fully developed, i.e. at long times, the beat frequency can be calculated from Eqs. (6) and (8) noting that experimentally one has $\omega_1 \approx \omega_0$. This consideration yields

$$\delta\omega / \omega = 1/2[\omega(y_0) - \omega(y_\infty)]. \quad (13)$$

Equations (8) and (13) combined then give

$$\delta\omega / \omega = (9/64)(C_{r4}/C_{r2})[-(y_\infty/y_0)^2 + 1]y_0^2, \quad (14)$$

if terms of the fourth power of the amplitude are neglected.

## Experimental Observations

In order to find out if automodulation exist, large amplitude mechanical oscillations of a sample close to the martensitic transformation must be investigated. This will insure that $C_{u4} < 0$ and that therefore an automodulation may be detected. Such an experiment was performed and preliminary results will be presented next.

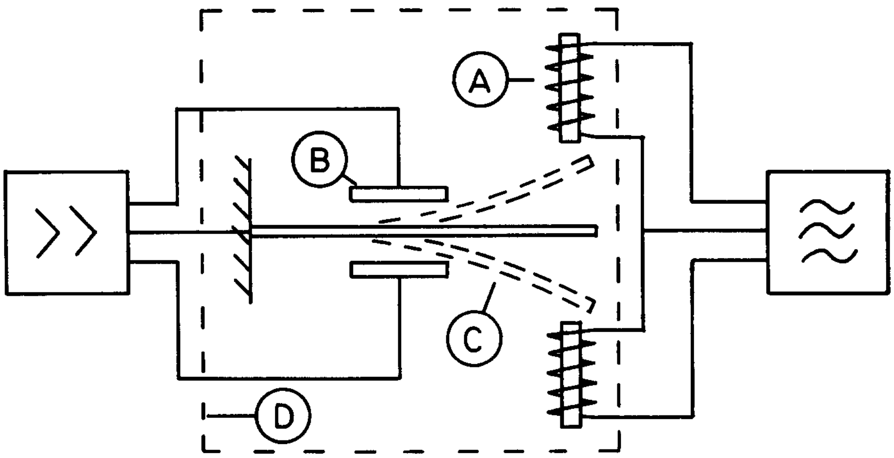


FIG. 1

Schematic of the experimental apparatus. A = electromagnetic drive, B = capacitative pickup, C = sample, D = evacuated and temperature controlled environment.

A schematic of the experimental setup used to excite flexural oscillations of a reed is shown in FIG. 1. A magnetic drive provided the power to excite large amplitude oscillations. The reed was made of a 82.9 wt% Cu, 14.1 wt% Al, 3.0 wt% Ni single crystal which had a martensite start temperature of 6°C. It was oriented such that its flexural oscillations corresponded to a (100)[110] shear mode. The present geometry resulted in an eigenfrequency of 152 $sec^{-1}$ at the temperatures of interest.

Two main observervations were made. First, it was noted that an automodulation does indeed exist close to the $M_s$ temp. if the sample was driven hard enough. Second, it was observed that a threshold amplitude of oscillation exists below which no automodulation appears. Above this threshold amplitude the growth kinetics of the automodulation depended very strongly on the amplitude as suggested by Eq. (10).

A steady state automodulation is shown in FIG. 2. The graph is redrawn from a recording of the amplitude of oscillation of the reed at resonance while the driving power was held constant. It can be seen that the amplitude of vibration is periodic in time, i.e. it is modulated. Since all conditions required for the appearance of an automodulation are fullfilled in this experiment the periodicity seen in FIG. 2 represents an automodulation. This interpretation is supported qualitatively by the observed strong sensitivity of the growth kinetics on the amplitude of the forced vibration mentioned above. Further semiquantitative support comes from the observed frequency of the automodulation, $\delta\omega \approx 5\cdot10^{-3}$ $sec^{-1}$. This frequency compares well with an estimate afforded by Eq. (15). The maxima and minima of the automodulation shown in FIG. 1 are given by $y_\infty + y_0$ and $y_\infty - y_0$. This yields $y_\infty / y_0 \approx 1.5$. The strain amplitude and

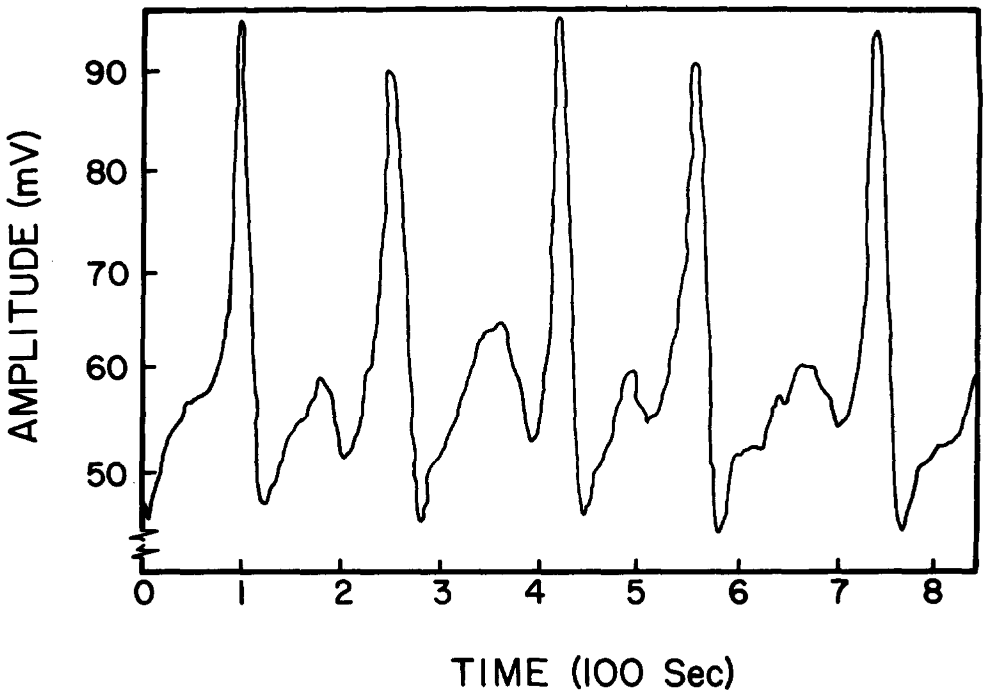


FIG. 2

Experimentally observed automodulation of flexural oscillations of a 82.9 wt%Cu, 14.1 wt%Al, 3 %Ni reed at 7°C.

eigenfrequencies were $y_0 = 3x10^{-4}$ and $\omega = 152\ sec^{-1}$ and the ratio of the fourth order elastic constant to the second order one is known to be of the order of magnitude of 100 (8). Inserting these values into Eq. (15) gives $\delta\omega \approx 2.4 \cdot 10^{-3}\ sec^{-1}$ which is close to the observed value.

The preliminary experimental results shown in FIG. 2 thus support the basic concepts presented in this letter. However, they also demonstrate the limitations of the present analysis. The waveform of the automodulation is not sinusoidal, it rather resembles a distinctly nonlinear vibration. Further, higher harmonics can be clearly recognized. Both features point out that the analytical analysis representing a first order approximation is inadequate to account for all but the essential features of the automodulation. Higher order approximations and numerical solutions are needed and will be published later together with a more complete set of experimental data.

## Summary

In conclusion it can be stated that a novel phenomenon, a mechanical autooscillation, has been described theoretically and shown to exist experimentally. These autooscillations occur when a nonlinear anelastic solid is forced into large amplitude oscillations in the vicinity of a martensitic transformation. They manifest themselves as a low frequency automodulation of the forced vibration of such a solid.

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