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Abstract

Registration by integral-square error correlation of one-dimensional discrete-time waveforms which are treated as random processes with specified autocorrelation functions is considered. An important design parameter for this class of problems is the probability of anomaly (a false dip in the correlation function) because it gives an indication of system immunity to gross registration errors. Explicit expressions for this parameter are not possible, so bounds and approximations must be derived. Two upper bounds and an approximation for the probability of anomaly are derived here. The use of these expressions is illustrated by an example. The relative utility of these performance indicators is shown for the example by comparison with actual values of the probability of anomaly obtained by computer simulation.

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I. Introduction

The problem of one-dimensional waveform registration arises often in practice in a variety of areas including correlation guidance [1]-[4], seismic processing [5], and biomedical pattern recognition [6]. A general problem statement involves the definition of a continuous-time reference waveform, denoted $s(u)$, which is modeled as a member of the ensemble of waveforms of a stationary random process and is available prior to the registration procedure. A continuous-time observation waveform is denoted $r(u)$ and is modeled as a noisy translated version of $s(u)$ with less extent. If λ denotes the relative translation between $r(u)$ and $s(u)$ as measured from an arbitrary but fixed point, then $r(u)$ takes the form

$$r(u) = s(u + \lambda) + n(u), \qquad 0 \leqslant u \leqslant U \tag{1}$$

where U is the extent of $r(u)$, and $n(u)$ is noise which is modeled as a random process which is uncorrelated with the reference waveform. The situation is as shown in Fig. 1.

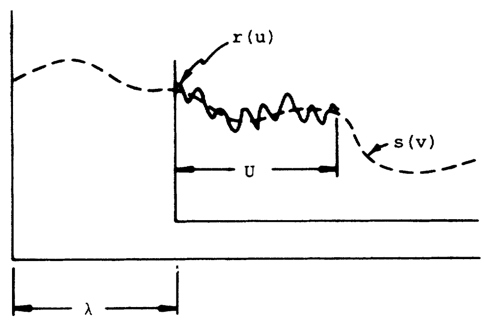


Fig. 1. Observation and reference waveforms.

The data and noise waveforms encountered in practice are commonly modeled as random in nature because they can best be described by probabilistic models. In addition, it is often reasonable to assume that the signal waveform is Gaussian and the noise waveform is zero-mean Gaussian and that both waveforms are wide-sense stationary. The statistical description of the signal and noise is then complete when the autocorrelation functions are specified. For the example included in this analysis these assumptions are used along with the exponential form of the autocorrelation function which, for a random process $x(t)$, is given by

$$R(\tau) = E[x(t)\, x(t + \tau)] = \sigma^2 \exp(-|\tau|/c). \tag{2}$$

In this expression $E[\ ]$ is the expected value, σ^2 is the variance, and c is the correlation length. This form is used to describe both the signal and noise. (Correlation length is that value of τ for which the covariance has decreased in value to 36.8 percent of the variance.) Also, the signal and noise are assumed uncorrelated.

In practice, only discrete versions of the waveforms $r(u)$

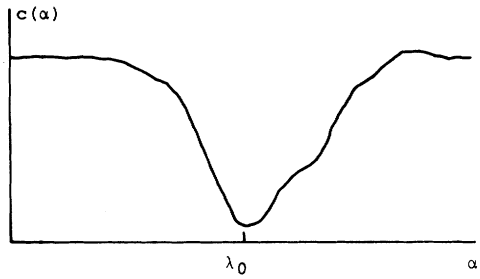


Fig. 2. Anomaly-free correlation function.

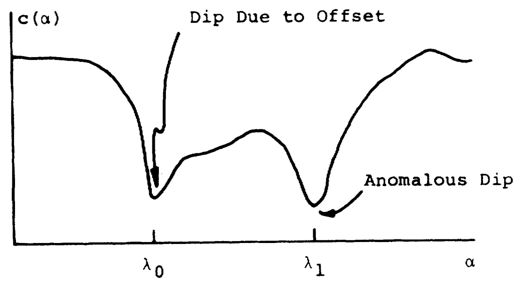
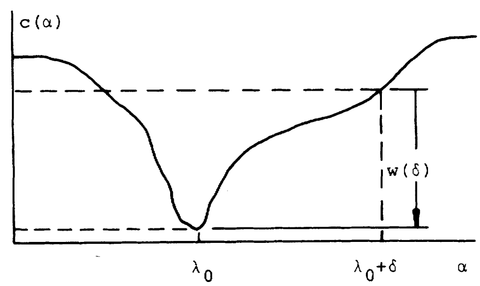


Fig. 3. Correlation function possessing anomaly.

Fig. 4. Definition of $w(\delta)$ for use in evaluating $P_A(\delta)$.



and $s(u)$ are usually available. This is due to the measurement techniques, processing, and storage being inherently digital. Thus the problem will be treated as digital in nature. The reference and observation sequences are designated $s_i(\alpha)$ and r_i which denote the values of $s(u + \alpha)$ and $r(u)$ at the ith sampling instant with sampling period T. The number of samples in the observation sequence is designated N_r. Note that $\alpha = \lambda$ when the reference and observation waveforms are aligned. The noise sequence is expressed in a similar manner as n_i. The signal and noise statistics are described by the autocorrelation functions with arguments normalized by the sampling period. These are denoted $R_s(k)$ and $R_n(k)$, respectively, and are

$$R_s(k) = E(s_i\, s_{i+k}) = \sigma_s^2\, \rho_s^{|k|} \tag{3}$$

and

$$R_n(k) = E(n_i\, n_{i+k}) = \sigma_n^2\, \rho_n^{|k|} \tag{4}$$

where σ_s^2 is the reference variance, σ_n^2 is the noise variance, and ρ_s and ρ_n are the reference and noise correlation coefficients. The correlation coefficient is related to the correlation length by $\rho = \exp(-T/c)$.

The observation and reference waveforms are said to be in registration when the actual value of offset λ is known. If this value is not known, then approximate registration can be obtained by forming an estimate of the offset, denoted $\hat{\lambda}$. This is done by comparing the observation and reference waveforms by some form of correlation and results in a continuous function of the offset, denoted $c(\alpha)$, whose minimum or maximum value determines the offset estimate $\hat{\lambda}$.

The integral-square error correlation is used in this analysis and the resulting correlation function is given by

$$c(\alpha) = \sum_{i=1}^{N_r} [s_i(\alpha) - r_i]^2. \tag{5}$$

The offset estimate is that value of α for which the function $c(\alpha)$ is a minimum, as illustrated in Fig. 2 where λ_0 is the actual value of offset. This is a least squares estimate.

Only the integral-square error correlation method is used in this analysis. It is the optimal registration method (in the maximum likelihood sense) with the assumptions that the noise is a wide-sense stationary Gaussian random process and that the reference waveform is known exactly prior to the estimation procedure which were stated above and the additional assumption that the noise samples are uncorrelated [7]. The assumption of uncorrelated noise samples is used throughout this analysis for simplicity and because uncorrelated noise samples often occur in practice. All of the results can readily be generalized to the correlated noise samples case. Note then that the integral-square error correlation method is suboptimal and the optimal estimator is of the integral-square error form but requires nonlinear weighting of the observation sequence [7].

An anomaly occurs when there is a major dip in the correlation function remote from the actual offset value which results in a gross estimation error. This situation is shown in Fig. 3 where λ_0 is the actual offset and λ_1 is the location of the anomaly.

An important performance indicator for the problem under investigation is the probability of occurrence of an anomaly. This quantity is defined using Fig. 4 where the auxiliary random variable

$$w(\delta) = c(\lambda_0) - c(\lambda_0 + \delta) \tag{6}$$

is introduced. Note that the mean value of $w(\delta)$ is less than zero since $c(\lambda_0 + \delta)$ is normally greater than $c(\lambda_0)$. The probability of anomaly at a distance δ from the actual offset is given by

$$P_A(\delta) = P[w(\delta) \geqslant 0]. \tag{7}$$

This expression is not the probability of a false match on a single trial but is specific to the offset $\lambda - \lambda_0$. It is none the less useful in specifying relative system performance. This quantity is analyzed in detail in the sections that follow.

II. Upper Bounds on the Probability of Anomaly

It is not possible to derive an exact expression for the probability of anomaly given by (7) except for the case of very simple signal and noise assumptions (e.g., uncorrelated signal samples and uncorrelated noise samples). This is because the exact probability distribution of the correlation function is not known except for simple signal and noise assumptions (e.g., for uncorrelated signal and noise samples with Gaussian distributions the correlation function has a chi-square distribution).

When the data and noise are Gaussian distributed, $w(\delta)$ is the weighted sum of chi-square random variables. This distribution is not expressible in closed form, but can be approximated. An alternative to this is to bound or approximate $P_A(\delta)$ in terms of the moments of the random variable $w(\delta)$. Upper bounds on $P_A(\delta)$ are useful because they provide an indication of the worst case system performance. If the bounds can be shown to be reasonably tight, then they can be used to provide a quantitative indication of system performance. Approximations on $P_A(\delta)$ are useful if they can be shown to be reasonably accurate because they can provide an absolute indicator of system performance and provide insight into the relation of system performance and system parameters.

Two upper bounds are considered here. The first of these results from the application of the Chebyshev inequality. This bound is quite general since it is independent of any assumption of probability distribution for the signal and noise waveforms. The second upper bound results from the application of the Chernoff bound. It provides a better bound than the first but is dependent on the assumption of a Gaussian probability distribution for the signal and noise.

A. Upper Bound on $P_A(\delta)$ from the Chebyshev Inequality

For an arbitrary random variable x with mean value μ_x and variance σ_x^2 the Chebyshev inequality is

$$P[|(x-\mu_x)| \geqslant \epsilon] \leqslant \sigma_x^2/\epsilon^2, \qquad \epsilon > 0. \tag{8}$$

If the absolute value is removed to give

$$P[(x-\mu_x) \geqslant \epsilon] \leqslant \sigma_x^2/\epsilon^2, \qquad \epsilon > 0 \tag{9}$$

then a looser bound is obtained. For $\mu_x \leqslant 0$, ϵ can be set equal to $-\mu_x$ to give the bound

$$P(x \geqslant 0) \leqslant \sigma_x^2/\mu_x^2, \qquad \mu_x \leqslant 0. \tag{10}$$

A bound on $P_A(\delta)$ can be obtained using (10) and the statistics of $w(\delta)$ (denoted μ_w and σ_w^2) since $\mu_w \leqslant 0$. The result is

$$P_A(\delta) = P[w(\delta) \geqslant 0] \leqslant \sigma_w^2/\mu_w^2. \tag{11}$$

The expression derived here and given by (11) is used often in a number of applications. It is sometimes termed the output signal-to-noise ratio because of its form. The statistics μ_w and σ_w^2 are computed in Appendix I for the case of a Gaussian distributed signal and noise.

B. Upper Bound on $P_A(\delta)$ from the Chernoff Bound

The general Chebyshev inequality can be stated as follows. Let x be a random variable and $g(x)$ be a non-negative function with a domain of the real numbers; then for $\epsilon > 0$

$$P[g(x) \geqslant \epsilon] \leqslant E[g(x)]/\epsilon. \tag{12}$$

This inequality is proved and discussed in detail in [8].

The Chernoff bound results from using the Chebyshev inequality with a particular function $g(x)$ and a particular term for ϵ. Let $g(x) = \exp(sx)$ where s is an arbitrary real number. Let $\epsilon = \exp(sA)$ where A is an arbitrary real number. Application of the Chebyshev inequality gives

$$P[\exp(sx) \geqslant \exp(sA)] \leqslant E[\exp(sx)]/\exp(sA). \tag{13}$$

Let s be positive ($s \geqslant 0$). Then the inequality $\exp(sx) \geqslant \exp(sA)$ is equivalent to the inequality $x \geqslant A$. This is because the exponential is monotonic. The expectation, $E[\exp(sx)]$, is the moment-generating function of x which will be denoted $M_x(s)$. Combining these results gives the bound

$$P(x \geqslant A) \leqslant \exp(-sA)\, M_x(s). \tag{14}$$

Recall that s is an arbitrary positive real number. As s approaches zero, the bound in (14) becomes

$$P(x \geqslant A) \leqslant \exp(-sA)\, M_x(s)\big|_{s\to 0} = 1 \tag{15}$$

and thus is not useful. Furthermore, for $A \leqslant E(x)$, it can be shown that the bound is always greater than 1 and so is not useful. In this analysis, a bound for the probability of $x \geqslant 0$ is desired. This is accomplished by setting A to zero and results in a bound which is

$$P(x \geqslant 0) \leqslant M_x(s), \qquad s > 0. \tag{16}$$

Thus a bound on $P_A(\delta)$ is derived using (16) and the moment-generating function of $w(\delta)$, denoted $M_w(s)$, to give

$$P_A(\delta) = P[w(\delta) \geqslant 0] \leqslant M_w(s), \qquad s > 0. \tag{17}$$

Note that $E[w(\delta)] \leqslant 0$, so a useful bound is obtained. Strictly speaking, the Chernoff bound is the least upper bound resulting from (17). The value of s which gives this bound can be found explicitly or by numerical evaluation. An explicit expression for $M_w(s)$ for the case of a Gaussian distributed signal and noise is derived in Appendix II.

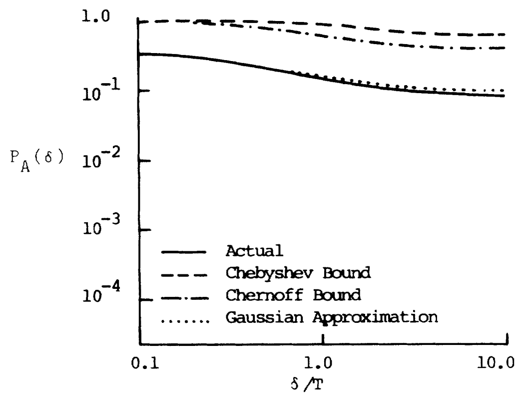


Fig. 5. Comparison of expressions for $P_A(\delta)$ with N_r = 40, ρ_s = 0.50, and signal-to-noise ratio = − 10 dB.

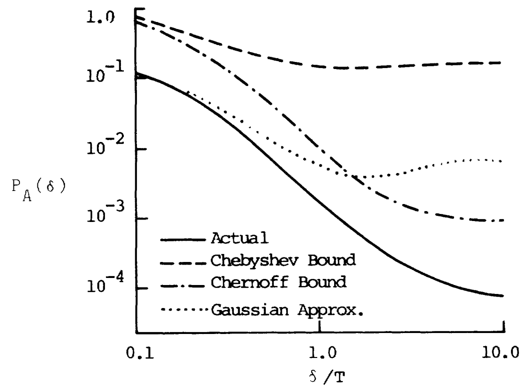
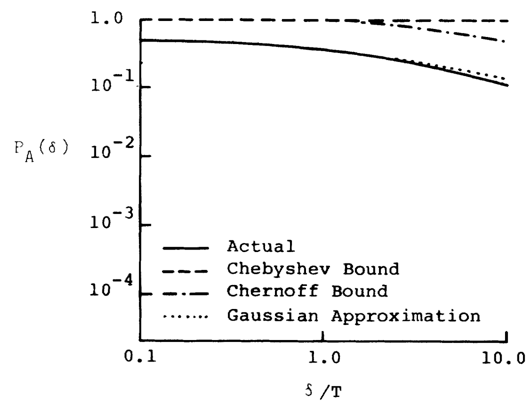


Fig. 6. Comparison of expressions for $P_A(\delta)$ with N_r = 40, ρ_s = 0.50, and signal-to-noise ratio = − 0 dB.

Fig. 7. Comparison of expressions for $P_A(\delta)$ with N_r = 40, ρ_s = 0.90, and signal-to-noise ratio = − 10 dB.



III. An Approximate Expression for the Probability of Anomaly

It is possible to obtain an approximate expression for $P_A(\delta)$ by assuming that the correlation function is Gaussian distributed. This assumption is reasonable if each value of $c(\alpha)$ is the sum of many identically distributed random variables which are not highly correlated. This permits the central limit theorem to be applied. Since $c(\alpha)$ is always positive, this assumption is most valid when the mean value of $c(\alpha)$ is sufficiently large with respect to its standard deviation because then the portion of the lower tail of the Gaussian distribution which extends below zero is negligible. The approximate expression for $P_A(\delta)$ is useful as a quantitative indicator of system performance provided its accuracy can be established. The development of this approximation follows.

The assumption of Gaussian statistics for $c(\alpha)$ means that $w(\delta)$ is also Gaussian. The technique then is to compute the mean and variance of $w(\delta)$ and then to compute $P_A(\delta)$ by using a Gaussian distribution. The calculation of the statistics μ_w and σ_w^2 for the case of a Gaussian distributed signal and noise is performed in Appendix I. The approximate expression for $P_A(\delta)$ follows by computing $P[w(\delta) \geqslant 0]$ and is

$$P_A(\delta) = P[w(\delta) \geqslant 0] \approx \tfrac{1}{2} + \mathrm{erf}(\mu_w/\sqrt{2\sigma_w^2}) \tag{18}$$

where

$$\mathrm{erf}(\alpha) = \int_0^\alpha (\sqrt{2\pi})^{-1} e^{-\omega^2/2}\, d\omega. \tag{19}$$

IV. Comparison of the Expressions for $P_A(\delta)$ and Actual Values Derived by Computer Simulation

In this section the expressions for $P_A(\delta)$ are compared with actual values derived by computer simulation for a particular signal and noise assumption. These comparisons give an indication of the utility of each of the performance indicators.

The signals used here for illustration are Gaussian with exponential autocorrelation and the noise is Gaussian with uncorrelated samples. Plots of $P_A(\delta)$ for $\rho_s = 0.50$ and $\rho_s = 0.90$, $N_r = 40$, and signal-to-noise ratios (defined by σ_s^2/σ_n^2) of −10 dB and 0 dB are shown in Figs. 5 through 8. The plots are shown as a function of normalized offset δ/T, where T is the sample interval. The results shown give an indication of the utility of the various expressions for the probability of anomaly. The actual values for the probability of anomaly were obtained by computer simulation using 100 000 replications for each value of offset. Results for higher values of signal-to-noise ratio are not shown because it is not practical to determine values for $P_A(\delta)$ by computer simulation. This is because the computation costs rise astronomically as $P_A(\delta)$ becomes small. For example, if $P_A(\delta)$ is in the range of 10^{-6}, then at least 100 million replications are required to arrive at an accurate estimate. However, the probability of anomaly is so small for these higher values of signal-to-noise ratio that they are usually of no concern anyway.

Thus only larger values of $P_A(\delta)$ are considered here. If

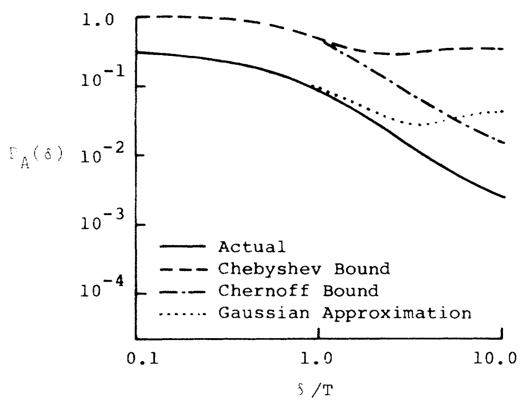


Fig. 8. Comparison of expressions for $P_A(\delta)$ with $N_r = 40$, $\rho_s = 0.90$, and signal-to-noise ratio = 0 dB.

there is interest in smaller probability of anomaly values, then the results shown indicate when extrapolation may be reasonable.

The plots reveal that the Chebyshev bound is a poor indicator of $P_A(\delta)$ in magnitude and variation with δ. Also, this bound reaches a constant value for large δ which is insentitive to system performance and thus is not a good performance indicator.

The Chernoff bound appears to be a good performance indicator. It is within an order of magnitude of the actual value of $P_A(\delta)$ for all values of δ and exhibits the same variation with δ. However, the bound is useless for small values of δ since the bound is greater than $\frac{1}{2}$ and it is known that $P_A(\delta)$ is always less than $\frac{1}{2}$ since $c(\lambda_0 + \delta) > c(\lambda_0)$ on the average.

The Gaussian approximation is a good performance indicator for small δ, but diverges for increased δ. In fact, for small δ this indicator is nearly equal to the actual value of $P_A(\delta)$. The divergence for larger values of δ is probably due to the fact that the actual distribution of $w(\delta)$ differs from a Gaussian distribution in its tail which is the primary contributor to $P_A(\delta)$ for large δ.

V. Summary

Expressions for the probability of anomaly for the problem of one-dimensional waveform registration were examined in the previous sections. This involved the derivation of three expressions: 1) an upper bound derived from the application of the Chebyshev inequality, 2) an upper bound derived from the application of the Chernoff bound, and 3) an approximation resulting from the assumption of a Gaussian distribution for the correlation function.

These expressions were compared with actual performance values derived by computer simulation for a practical example. It was shown that 1) the Chernoff bound provides a good indication of the probability of anomaly for large values of offset, 2) the Gaussian approximation gives an approximately correct value for the probability of anomaly for small values of offset, and 3) the Chebyshev bound is an unreliable performance indicator for the example.

These results reveal that it is possible to arrive at accurate expressions for the probability of anomaly for a relatively large class of problems. The improvement of the Chernoff bound and the Gaussian approximation to the probability of anomaly over the often used Chebyshev bound is significant. No attempt was made to apply these results to a physical problem. But it is reasonable to assume that these results are applicable to any physical problem that closely satisfies the assumptions listed.

Also, the methods of analysis used here are applicable to a large number of problems where a probability of the form $P(x \geqslant A)$ must be evaluated. In this case any of the three methods of evaluation given here can be used. The relative merit of the methods will depend on the specific problem.

Appendix I

Computation of μ_w and σ_w^2

When the reference position for the correlation function is defined to be the correct estimate point, then the expression for $w(\delta)$ is

$$w(\delta) = c(0) - c(\delta) \tag{20}$$

$$= \sum_{i=1}^{N_r} [s_i(0) - s_i(0) - n_i]^2 - \sum_{i=1}^{N_r} [s_i(\delta) - s_i(0) - n_i]^2 \tag{21}$$

$$= \sum_{i=1}^{N_r} n_i^2 - \sum_{i=1}^{N_r} [s_i(\delta) - s_i(0) - n_i]^2 \tag{22}$$

$$= 2 \sum_{i=1}^{N_r} [s_i(\delta) - s_i(0)]\, n_i - \sum_{i=1}^{N_r} [s_i(\delta) - s_i(0)]^2. \tag{23}$$

Define an auxiliary random variable f_i as

$$f_i = s_i(\delta) - s_i(0). \tag{24}$$

Then $w(\delta)$ is given by

$$w(\delta) = 2 \sum_{i=1}^{N_r} f_i n_i - \sum_{i=1}^{N_r} f_i n_i - \sum_{i=1}^{N_r} f_i 2. \tag{25}$$

The statistics of f_i are easily computed as

$$E(f_i) = 0 \tag{26}$$

$$R_f(i-j) = E(f_i f_j) = E\{[s_i(\delta) - s_i(0)]\,[s_j(\delta) - s_j(0)]\} \tag{27}$$

$$= 2R_s(i-j) - R_s(i-j-\delta) - R_s(i-j+\delta). \tag{28}$$

The statistics of $w(\delta)$ follow readily and the derivations are given below. For the mean value,

$$\mu_w = E[w(\delta)] = E(2 \sum_{i=1}^{N_r} f_i n_i - \sum_{i=1}^{N_r} f_i^2) \tag{29}$$

$$= -E(\sum_{i=1}^{N_r} f_i^2) = -\sum_{i=1}^{N_r} E(f_i^2) \tag{30}$$

$$= -2N_r[R_s(0) - R_s(\delta)]. \tag{31}$$

Computation of the variance begins by computing

$$E[w^2(\delta)] = E[(2\sum_{i=1}^{N_r} f_i n_i - \sum_{i=1}^{N_r} f_i^2)(2\sum_{j=1}^{N_r} f_j n_j - \sum_{i=1}^{N_r} f_j^2)] \tag{32}$$

$$= E(\sum_{i=1}^{N_r}\sum_{i=1}^{N_r} f_i^2 f_j^2) + 4E(\sum_{i=1}^{N_r}\sum_{i=1}^{N_r} f_i f_j n_i n_j) \tag{33}$$

where noncontributing terms resulting from the zero-mean assumption for the noise have been deleted. The expression in (33) can be written in terms of the data and noise statistics to give

$$E[w^2(\delta)] = \sum_{i=1}^{N_r}\sum_{i=1}^{N_r} [2R_s(\delta) - 2R_s(0)]^2 + \sum_{i=1}^{N_r}\sum_{i=1}^{N_r} 2[2R_s(i-j) - R_s(i-j-\delta) - R_s(i-j+\delta)]^2 + 4\sum_{i=1}^{N_r}\sum_{i=1}^{N_r} [2R_s(i-j) - R_s(i-j-\delta) - R_s(i-j+\delta)]\, R_n(i-j) \tag{34}$$

$$= 4N_r^2[R_s(\delta) - R_s(0)]^2 + 2N_r[2R_s(\delta) - 2R_s(0)]^2 + \sum_{\substack{i=1\\ i\neq j}}^{N_r}\sum_{\substack{j=1\\ i\neq j}}^{N_r} 2[2R_s(i-j) - R_s(i-j-\delta) - R_s(i-j+\delta)]^2 + 4\sum_{i=1}^{N_r}\sum_{j=1}^{N_r} [2R_s(i-j) - R_s(i-j- \ ) - R_s(i-j+ \ )]\, R_n(i-j). \tag{35}$$

Let the noise samples be uncorrelated with variance σ_n^2. Then the expression in (35) reduces to

$$E[w^2(\delta)] = 4N_r^2[R_s(\delta) - R_s(0)]^2 + 2N_r[2R_s(\delta) - 2R_s(0)]^2 + 8N_r[R_s(0) - R_s(\delta)]\sigma_n^2 + \sum_{i=1}^{N_r}\sum_{i=1}^{N_r} 2[2R_s(i-j) - R_s(i-j-\delta) - R_s(i-j+\delta)]^2. \tag{36}$$

Lastly, the expression in (36) reduces to

$$E[w^2(\delta)] = (4N_r^2 + 8N_r)[R_s(\delta) - R_s(0)]^2 + 8N_r[R_s(0) - R_s(\delta)]\sigma_n^2 + 4\sum_{i=1}^{N_r}(N_r - k)[2R_s(k) - R_s(k-\delta) - R_s(k+\delta)]^2. \tag{37}$$

The simplification of the double sum in (36) to the single sum in (37) follows because the argument in the sum is a function only of $k = |i - j|$.

The variance of $w(\delta)$ can now be computed as

$$\sigma_w^2 = E[w^2(\delta)] - E^2[w(\delta)] \tag{38}$$

$$= 8N_r[R_s(0) - R_s(\delta)]^2 + 8N_r[R_s(0) - R_s(\delta)]\sigma_n^2 + 4\sum_{i=1}^{N_r}(N_r - k)[2R_s(k) - R_s(k-\delta) - R_s(k+\delta)]^2. \tag{39}$$

Appendix II

Derivation of $M_w(s)$

Recall that $w(\delta)$ as a function of δ is given by

$$w(\delta) = \sum_{i=1}^{N_r} n_i^2 - \sum_{i=1}^{N_r}(f_i - n_i)^2 \tag{40}$$

$$= -\sum_{i=1}^{N_r} f_i^2 + 2\sum_{i=1}^{N_r} f_i n_i \tag{41}$$

where f_i is the auxiliary random variable defined in Appendix I as

$$f_i = s_i(\delta) - s_i(0). \tag{42}$$

In vector notation, the relation in (41) is

$$w(\delta) = -\mathbf{f}^T(\mathbf{f} - 2\mathbf{n}). \tag{43}$$

Define a new random vector **Z** as

$$\mathbf{z}^T = (\mathbf{f}^T\ \mathbf{n}^T). \tag{44}$$

Then the covariance matrix of **z** is

$$\Lambda_z = \begin{bmatrix} \Lambda_f & 0 \\ 0 & \Lambda_n \end{bmatrix} \tag{45}$$

where Λ_f and Λ_n are the covariance matrices of **f** and **n**, respectively. Since **f** and **n** are independent and Gaussian, they are jointly Gaussian; therefore, the probability density function of **z** is

$$f(\bar{z}) = (2\pi)^{-N_r} |\Lambda_z|^{-1/2} \exp(-\mathbf{z}^T \Lambda_z^{-1} \mathbf{z}). \tag{46}$$

Now $w(\delta)$ can be written as a quadratic in terms of $\mathbf{z}$ as

$$w(\delta) = -\mathbf{z}^T Y \mathbf{z} \tag{47}$$

where Y is a $2N_r \times 2N_r$ matrix given by

$$Y = \begin{bmatrix} I & -I \\ -I & 0 \end{bmatrix}. \tag{48}$$

Therefore an expression for $M_w(s)$ results which is

$$M_w(s) = \int_{-\infty}^{\infty} \exp(-s\mathbf{z}^T Y \mathbf{z})(2\pi)^{-N_r} |\Lambda_z|^{-1/2} \cdot \exp(-\tfrac{1}{2}\mathbf{z}^T \Lambda_z^{-1} \mathbf{z})\, dz. \tag{49}$$

This can be rewritten as

$$M_w(s) = \int_{-\infty}^{\infty} 2(\pi)^{-N_r} |\Lambda_z|^{-1/2} \exp[-\tfrac{1}{2}\mathbf{z}^T(\Lambda_z^{-1} + 2sY)\mathbf{z}]\, dz \tag{50}$$

$$= [|(\Lambda_z^{-1} + 2sY)^{-1}|^{1/2} / |\Lambda_z|^{1/2}] \cdot \int_{-\infty}^{\infty} (2\pi)^{-N_r} |(\Lambda_z^{-1} + 2sY)^{-1}|^{-1/2} \cdot \exp[-\tfrac{1}{2}\mathbf{z}^T(\Lambda_z^{-1} + 2sY)z]\, dz. \tag{51}$$

Assume that the matrix $(\Lambda_z^{-1} + 2sY)^{-1}$ is positive definite. Then the term in the integral is a probability density function; so the integral is equal to 1. Therefore,

$$M_w(s) = |(\Lambda_z^{-1} + 2sY)^{-1}|^{1/2} / |\Lambda_z|^{1/2}. \tag{52}$$

This expression can be further simplified to

$$M_w(s) = |\Lambda_z^{-1} + 2sY|^{1/2} |\Lambda_z|^{-1/2}. \tag{53}$$

This follows because the determinant of an inverse is the reciprocal of the determinant of the original matrix. The expression in (53) can be further simplified by noting that the determinant of the product of two matrices is the product of the determinants. Therefore,

$$M_w(s) = |I + 2sY\Lambda_z|^{-1/2}. \tag{54}$$

In terms of Λ_f and Λ_n the expression in (54) is

$$M_w(s) = \left| I + 2s \begin{bmatrix} \Lambda_f & -\Lambda_n \\ -\Lambda_f & 0 \end{bmatrix} \right|^{-1/2} \tag{55}$$

This expression is readily calculable by a simple computer program.

Note that the expression in (55) is valid only if the matrix $(\Lambda_z^{-1} + 2sY)^{-1}$ is positive definite. This fact has not been proved, but appears to be the case. It has been verified to an extent by computer evaluation of $M_w(s)$ as given by (55). In all cases examined, this function showed proper variation with s over the range of values of interest.

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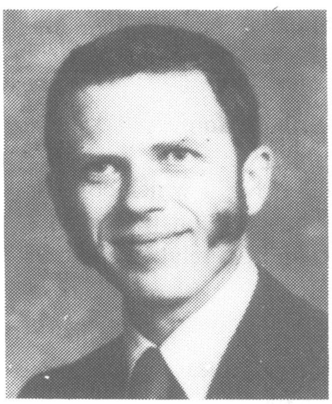

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