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Embedding Discrete Flows on R in a Continuous Flow

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The problem of determining when a given discrete flow on a topological space is embeddable in some continuous flow was mentioned by G. R. Sell ("Topological Dynamics and Ordinary Differential Equations," Van Nostrand, New York, 1971) in his book on topological dynamics. In this book, the theory of generalized dynamical systems is exploited in the qualitative study of differential equations. Even more complicated is the problem of simultaneously embedding two or more discrete flows in a single continuous flow. We examine both of these problems when the underlying topological space is the space R of the real numbers.

1. DEFINITIONS AND SOME CONSEQUENCES

Let G be a topological group and let X be a topological space. Then, a *q-flow* on X is a map $\pi\colon G \times X \to X$ satisfying the following conditions:

(A1) $\pi(0, x) = x$

(A2) $\pi(s + t, x) = \pi(s, \pi(t, x))$

(A3) π is separately continuous.

A q-flow π such that π is continuous is called a *flow*.

Let $\pi\colon G \times X \to X$ be a q-flow. For each $t \in G$ and $x \in X$, define $\pi^t\colon X \to X$ and $\pi_x\colon G \to X$ by $\pi^t(x) = \pi(t, x) = \pi_x(t)$. Then each π^t is a surjective homeomorphism and each π_x is a continuous map. The set $\{\pi^t\colon t \in G\}$ is known as the *transition set* of homeomorphisms of the q-flow. A homeomorphism f on X is said to be *generated by* the q-flow π if f belongs to the transition set of π. A point $x \in X$ is called a *periodic point* of π provided there exists some nonzero element t in G such that $\pi(t, x) = x$. A point $x \in X$ such that $\pi(t, x) = x$ for all $t \in G$ is called a *critical point* of the q-flow.

A flow $\pi\colon G \times X \to X$, such that G is the additive group of integers with the discrete topology (resp. the additive group of real numbers with the usual topology), is called a *discrete flow* (resp. a *continuous flow*) on X.

It is easily seen that π is a discrete flow on a space X if and only if there is a

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homeomorphism F on X such that $\pi(n, x) = F^n(x)$. The homeomorphism F, thus, determines the discrete flow completely and is called the *defining homeomorphism* of the discrete flow.

A collection $\{\pi_\alpha\}$ of discrete flows on a topological space X is said to be *embeddable* together in a continuous flow π on X provided the defining homeomorphism of each π_α is generated by π. Throughout this paper, we will consider only surjective homeomorphisms. We note that a surjection between two intervals in R is a homeomorphism if and only if it is monotone.

2. Flows on R

Throughout this section, we let D be an arbitrary but fixed dense subgroup of R and we let $\pi: D \times R \to R$ be a q-flow. Here, we will prove that π is the restriction of a continuous flow on R. This result appears to be folklore but it does not seem to be recorded elsewhere. Of course, if D is R itself then this result can be immediately derived from known theorems [1] which require the local compactness of the group and the space. Our proof here is based upon several properties of π related to the order structure on R. Some of these properties are important, on their own, and are established in the following sequence of lemmas.

Lemma 2.1. *Each homeomorphism generated by π is monotonically increasing.*

Proof. We first note that for any homeomorphism f on R the homeomorphism $f^2 = f \circ f$ is always monotonically increasing. This is so because f is monotone.

Let $t \in D$ be given. Take x, y in R such that $x < y$. Take a sequence (s_n) in D converging to $t/2$. Since $\pi^{2s_n} = \pi^{s_n} \circ \pi^{s_n}$ is monotonically increasing, $\pi(2s_n, x) < \pi(2s_n, y)$. From this and the continuity of π_x and π_y, it follows that $\pi(t, x) \leqslant \pi(t, y)$. But since π^t is a homeomorphism and $x \neq y$, $\pi(t, x) \neq \pi(t, y)$. Consequently $\pi(t, x) < \pi(t, y)$, and the lemma is proved.

Lemma 2.2. *The map π is a flow.*

Proof. We will show that π is jointly continuous at $(0, x)$ for any x in R. Then the joint continuity of π at any (t, x) in $D \times R$ would follow from the separate continuity and axiom (A2).

So let $x_0 \in R$ and let a basic neighborhood $G = \{x \in R: | x - x_0 | < \epsilon\}$ of x_0 be given. We will show that there is a neighborhood H of 0 in D and a neighborhood K of x_0 such that if $t \in H$ and $x \in K$ then $\pi(t, x)$ is in G. Let $a = x_0 - \epsilon/2$, $b = x_0 + \epsilon/2$, and $K = \{x: a < x < b\}$. Since π_a and π_b are both continuous, we can find a number $\delta > 0$ such that if $t \in D$ and $| t | < \delta$ then $| \pi(t, a) - a | < \epsilon/2$ and $| \pi(t, b) - b | < \epsilon/2$. Now let $H = \{t \in D: | t | < \delta\}$. For $x \in K$ we have $a < x < b$, and so, by Lemma 2.1, $\pi(t, a) < \pi(t, x) < \pi(t, b)$ for any real number t. Moreover if $t \in H$ then $x_0 - \epsilon < \pi(t, a)$ and $\pi(t, b) < x_0 + \epsilon$. It follows now that if $x \in K$ and $t \in H$ then $\pi(t, x)$ is in G. The proof of the lemma is now complete.

LEMMA 2.3. *Suppose a given x in R is not a critical point of π. Then the function $f(t) = \pi(t, x) - x$ has the same sign for all positive t in D and the opposite sign for all negative t in D.*

Proof. Clearly $f(t)$ is continuous and does not vanish identically. So, using (A2), we can find a positive t_1 in D such that $f(t_1)$ is not zero. By continuity of f, we find a neighborhood N of t_1 in D such that for every t in N, $f(t)$ has the same sign as $f(t_1)$. Take some u in D such that $0 < 2u < t_1$ and $2u \in N$. We claim that $f(u)$ has the same sign as $f(t_1)$. If possible, suppose $f(u) \leqslant 0 < f(t_1)$. Then $\pi(u, x) \leqslant x < \pi(t_1, x)$. Recalling that π^u is increasing we conclude that $\pi(2u, x) \leqslant \pi(u, x) \leqslant x < \pi(t_1, x)$. From this, it follows that $f(t_1)$ is positive and $f(2u)$ is nonpositive. This contradicts the fact that u was so chosen that $2u \in N$. Similarly, if we suppose that $f(u) \geqslant 0 > f(t_1)$, we will again arrive at a contradiction. Thus we have shown that $f(u)$ and $f(t_1)$ have the same sign. A repeated application of this argument shows that each neighborhood of zero contains a positive p in D such that $f(p)$ and $f(t_1)$ have the same sign. Furthermore, as π^p is increasing, $f(p)$ and $f(2p)$ also have the same sign. It is now clear that the set $\{t \in D: f(t)$ has the same sign as $f(t_1)\}$, is dense in the set $D^+ = \{t \in D: t > 0\}$. From this, and the continuity of $f(t)$, it follows that $f(t)$ is either nonnegative throughout D^+ or else it is nonpositive throughout D^+. Now take any $s \in D^+$. Then, there is some $r \in D^+$, $r < s$, such that $f(r)$ and $f(t_1)$ have the same sign. Suppose $f(t_1)$ and $f(r)$ are both positive. Then $\pi(r, x) > x$ and so by applying π^{s-r} we get $\pi(s, x) > \pi(s - r, x)$. As $f(t_1)$ is positive and $s - r$ is positive, so $\pi(s - r, x) \geqslant 0$. Thus we have shown that if $f(t_1)$ is positive, so is $f(s)$ for any positive s in D. A similar argument will show that if $f(t_1)$ is negative, so is $f(s)$ for all positive s in D. Consequently for all positive t in D, $f(t)$ has the same sign. Finally we note that $\pi(t, x) > x$ if and only if $x > \pi(-t, x)$. Hence the lemma is proved.

COROLLARY 1. *The only periodic points of π are the critical ones.*

We partition R into three subsets A_π, B_π, and C_π, defined by $A_\pi = \{x \in R: \pi(t, x) > x$ for all t in $D^+\}$, $B_\pi = \{x \in R: \pi(t, x) < x$ for all t in $D^+\}$, and $C_\pi = \{x \in R: x$ is a critical point of $\pi\}$.

LEMMA 2.4. (i) *For each $x \in A_\pi$, π_x is monotonically increasing on D.*

(ii) *For each $x \in B_\pi$, π_x is monotonically decreasing on D.*

Proof. Take $x \in A_\pi$ and s, t in D such that $s < t$. Then $t - s > 0$ and so $\pi(t - s, x) > x$. Applying π^s gives $\pi(t, x) > \pi(s, x)$. The proof of (ii) is similar.

LEMMA 2.5. *Let (u_n) be an increasing sequence and (v_n) be a decreasing sequence in D and let (u_n) and (v_n) both converge to some given t in R. Then for any x in R the sequences $(\pi(u_n, x))$ and $(\pi(v_n, x))$ have the same limit in R.*

Proof. Suppose $x \in A_\pi$. Let a and b be the limits of $(\pi(u_n, x))$ and $(\pi(v_n, x))$, respectively. Then $a \leqslant b$. Let $\epsilon > 0$ be given. Since $(v_n - u_n)$ converges to zero, we can choose a positive integer N so that $\pi_a(v_n - u_n) < a + \epsilon$ for any $n > N$. So for $n > N$, we have

$$\begin{aligned}\pi(v_n, x) &= \pi(v_n - u_n, \pi(u_n, x)) \\ &= \pi(v_n - u_n, y_n), \qquad \text{where} \quad y_n = \pi(u_n, x) < a, \\ &< \pi(v_n - u_n, a) \\ &< a + \epsilon.\end{aligned}$$

So we have $a \leqslant b \leqslant a + \epsilon$. Since ϵ is arbitrary, we must have $a = b$. The case $x \in B_\pi$ can be handled similarly.

COROLLARY 2. *For each $x \in R$, the closure of the range of π_x is a connected set.*

LEMMA 2.6. *Let (t_n) be a sequence in D converging to some t in R. Then for each x in R, the sequence $(\pi(t_n, x))$ is convergent.*

The proof of this lemma is based upon Lemmas 2.4 and 2.5 and will be omitted.

We define a map $\pi^*: R \times R \to R$ as follows: For any $t \in R$, take a sequence (t_n) in D converging to t, and define $\pi^*(t, x) = \lim(\pi(t_n, x))$, $x \in R$. Note that π^* is well defined and $\pi^*(s, x) = \pi(s, x)$ for any $s \in D$, $x \in R$. In particular $\pi^*(0, x) = x$.

THEOREM 2.7. *The map $\pi^*: R \times R \to R$ is a continuous flow.*

Proof. It is easily seen that for each $x \in A_\pi \cup B_\pi$ the map π_x^* defined by $\pi_x^*(t) = \pi^*(t, x)$ is monotone and has connected range. Consequently π_x^* is continuous for all x in R. It is a simple excercise to prove that for each t in R the map $(\pi^*)^t$ defined by $(\pi^*)^t(x) = \pi^*(t, x)$ is monotonically increasing and for any t, s in R, $\pi^*(t + s, x) = \pi^*(t, \pi^*(s, x))$. Consequently π^* is a q-flow and therefore, by Lemma 2.2, it is a continuous flow on R.

3. EMBEDDING A SINGLE DISCRETE FLOW

In this section we prove that a single discrete flow on R can be embedded in some continuous flow if and only if its defining homeomorphism is monotonically increasing.

Let a homeomorphism f on an interval $I = \{x: a \leqslant x \leqslant b\}$ of the real line R be given and suppose f satisfies the condition $f(x) = x$ if and only if $x = a$ or $x = b$. For each real number t, we define a homeomorphism f^t on I in a special way. To do so we let g be f if $f(x) \geqslant x$ for all x in I; otherwise we let g be f^{-1}. We first define a homeomorphism g^t on I, for each real number t, and then we let f^t be g^t or g^{-t} according as f is g or g^{-1}. The homeomorphisms g^t, $t \in R$ are defined as follows:

(i) Let g^0 be the identity map on I. For a positive integer n, define $g^n = g \circ g^{n-1}$; and let g^{-n} be the inverse map of g^n.

Now take some c such that $a < c < b$ and fix it. We will first define g^t at this fixed element c of $[a, b]$ and then we will define $g^t(x)$ for arbitrary x in I.

(ii) For $0 \leqslant t \leqslant 1$, define $g^t(c) = c + t[g(c) - c]$.

(iii) For an arbitrary real number t, we write $t = n + s$, where $0 \leqslant s < 1$ and n is an integer; and we define $g^t(c) = g^n[g^s(c)]$.

(iv) Now take an arbitrary x such that $a < x < b$. Then there is a unique integer n and a unique p, $c \leqslant p < g(c)$, such that $g^n(p) = x$. Now there is a unique real number s, $0 \leqslant s < 1$, such that $g^s(c) = p$. In fact s is determined by the equation $p = c + s[g(c) - c]$. For an arbitrary real number t, we define $g^t(x) = g^{t+n+s}(c)$.

(v) Finally we let $g^t(a) = a$ and $g^t(b) = b$, for all t.

LEMMA 3.1. *For any two real numbers u, v, $g^u \circ g^v = g^{u+v}$.*

Proof. We first show that $g^{u+v}(c) = g^u g^v(c)$. To do so, we write $v = n + s$, where n is an integer and $0 \leqslant s < 1$. Then

$$\begin{aligned} g^u g^v(c) &= g^u g^{n+s}(c) \\ &= g^u[g^n g^s(c)] && \text{by (iii)} \\ &= g^{u+n+s}(c) && \text{by (iv)} \\ &= g^{u+v}(c). \end{aligned}$$

Now take x such that $a < x < b$ and find an integer m and a number r, $0 \leqslant r < 1$, such that $x = g^{m+r}(c)$. Then

$$\begin{aligned} g^u g^v(x) &= g^u g^v[g^{m+r}(c)] \\ &= g^{u+v+m+r}(c) \\ &= g^{u+v} g^{m+r}(c) \\ &= g^{u+v}(x). \end{aligned}$$

LEMMA 3.2. *For each t, g^t is a bijection of I.*

Proof. $g^t g^{-t} = g^0 = g^{-t} g^t$.

LEMMA 3.3. *For any $t > 0$ and for any x such that $a < x < b$, $g^t(x) > x$.*

Proof. We first prove this assertion for $t < \frac{1}{2}$. It is evident from the definition of g^t that if $c \leqslant x < g(c)$ then $g^t(x) > x$. For an arbitrary x such that $a < x < b$, we first write $x = g^n(p)$, where n is an integer and $c \leqslant p < g(c)$. Note that g^n is monotonically increasing and $g^t(p) > p$. Consequently $g^n g^t(p) > g^n(p)$; i.e., $g^t(x) > x$. Finally, if $t \geqslant \frac{1}{2}$, then we write it as a sum of finitely many positive numbers each less than $\frac{1}{2}$. That completes the proof.

LEMMA 3.4. *If $a < x < b$ and $s < t$ then $g^s(x) < g^t(x)$.*

Proof. As $t - s > 0$, it follows from Lemma 3.3 that $g^{t-s}[g^s(x)] > g^s(x)$; i.e., $g^t(x) > g^s(x)$.

LEMMA 3.5. *If $a \leqslant x < y \leqslant b$, then for any real number t, $g^t(x) < g^t(y)$.*

Proof. We need consider only the case $a < x < y < b$. Note that $A_x = \{g^s(x): s \in R\} = (a, b)$. So there is a positive number s, such that $g^s(x) = y$. As $s + t > t$, so $g^{s+t}(x) > g^t(x)$; i.e., $g^t(y) > g^t(x)$.

LEMMA 3.6. *For any x in (a, b), the function $\phi_x: R \to (a, b)$ defined by $\phi_x(t) = g^t(x)$, is an increasing surjection. In particular, ϕ_x is continuous.*

Proof. Use Lemma 3.4 and the proof of Lemma 3.5.

LEMMA 3.7. *For each t in R, the map g^t is a monotonically increasing homeomorphism on I.*

Proof. Use Lemmas 3.2 and 3.5.

THEOREM 3.8. *A discrete flow on R can be embedded in a continuous flow if and only if its defining homeomorphism is monotonically increasing.*

Proof. Let f be the defining homeomorphism of the given discrete flow. If the discrete flow is embeddable in a continuous flow π, then $f = \pi^t$ for some real number t, and therefore, by Lemma 2.1, f must be increasing.

To prove the "if" part, we suppose that f is a monotonically increasing homeomorphism on R. Let $G = \{x \in R: f(x) \neq x\}$. Then G is an open subset of R, and therefore, it can be expressed as a union of a countable number of disjoint open intervals $\{(a_i, b_i): i = 1, 2, \ldots\}$. We fix a point c_i in (a_i, b_i) and with f restricted to $[a_i, b_i]$ we associate f_i^t for each real number t, in the special way outlined earlier. Now we define f^t in the obvious way so that f^t agrees with f_i^t on $[a_i, b_i]$ for all i. Finally we define $\pi: R \times R \to R$ by $\pi(t, x) = f^t(x)$.

It is clear that π is a q-flow and hence by Lemma 2.2, it is a continuous flow on R. The proof of the theorem is now complete.

4. Embedding Two Discrete Flows Together

In this section we find necessary and sufficient conditions so that two given homeomorphisms on R may both be generated by a continuous flow. Note that this settles the problem of embedding two discrete flows together in a continuous flow when the underlying motion space is R.

Two homeomorphisms f and g on a topological space X are called *cocyclic* provided there exist nonzero integers m and n such that $f^m = g^n$. If f and g are cocyclic and if we can find a pair (m, n) of nonzero and relatively prime integers such that $f^m = g^n$, then we say that f and g are *primely cocyclic*.

Theorem 4.1. *Let f and g be two cocyclic homeomorphisms on R. Then there exists a continuous flow generating both f and g if and only if the following three conditions are satisfied*:

(a) *f and g are both monotonically increasing.*

(b) *f and g commute; i.e., $f \circ g = g \circ f$.*

(c) *f and g are primely cocyclic.*

Proof. Suppose f and g are embeddable together in a continuous flow π: $R \times R \to R$. So there exist $s, t \in R$ such that $\pi^s = f$ and $\pi^t = g$. Clearly f and g satisfy conditions (a) and (b). Also, if both f and g are identity homeomorphisms on R then condition (c) would be satisfied too. Since f and g are cocyclic and increasing, if one of them is a nonidentity homeomorphism so must the other be a nonidentity homeomorphism. Let m and n be two nonzero integers such that $f^m = g^n$. Then $\pi^{sm} = \pi^{tn}$, and so by Lemma 2.4, we must have $sm = tn$. Now, let k be the greatest common divisor of m and n. As $s(m/k) = t(n/k)$ so $\pi^{s(m/k)} = \pi^{t(n/k)}$ and consequently $f^{m/k} = g^{n/k}$, and so condition (c) is satisfied.

Now suppose f and g satisfy conditions (a), (b), and (c). Let m and n be two relatively prime integers such that $f^m = g^n$. We can find integers p and q such that $m \cdot p + n \cdot q = 1$. Let $h = f^q \circ g^q$. Using the commutativity of f and g, it is easily seen that $h^m = g$ and $h^n = f$. Clearly h is increasing and so by Theorem 3.8, there is a continuous flow π on R generating h. Surely f and g are also generated by π.

Now we turn our attention to a noncocyclic pair of homeomorphisms. We call a pair (f, g) of homeomorphisms on R *well blended* if all of the following conditions are satisfied:

(1) f and g are both monotonically increasing.

(2) f and g commute.

(3) f and g have the same fixed points.

(4) For each x in R, $(x, f(x))$ and $(x, g(x))$ lie on the same side of the diagonal line $y = x$. More precisely, the product of $f(x) - x$ and $g(x) - x$ must be nonnegative.

(5) For any two nonzero integers m and n with the same sign, the inequality $|f^m(x) - x| < |g^n(x) - x|$ must either hold for all x in R except the fixed points of f or it must hold for no x in R. This means that the graphs of f^m and g^n intersect nowhere outside the diagonal line $y = x$ and the graph of one of them is nearer to the diagonal for all x.

(6) The biorbit $A_x = \{f^m g^n(x): m, n \text{ are integers}\}$ must have connected closure for all x in R.

PROBLEM. Prove or disprove that conditions (1)–(5) imply condition (6).

THEOREM 4.2. *Let f and g be two noncocyclic homeomorphisms on R and suppose neither of these is the identity map. Then there is a continuous flow on R generating both f and g if and only if one of the pairs (f, g) or (f, g^{-1}) is well-blended.*

Proof. Suppose f and g are both generated by a continuous flow π on R. Then, there exist s, t in R such that $f = \pi^s$ and $g = \pi^t$. Since f and g are noncocyclic and neither of these is the identity map, the set $\{ms + nt: m, n \text{ are integers}\}$ is a dense subgroup of R. Furthermore, as π has no periodic points except the critical ones, π^s and π^t have the same fixed points. It is clear now that the pair $(\pi^{|s|}, \pi^{|t|})$ is well blended. Now we show that the condition of well-blendedness is sufficient. Throughout the remaining proof we assume that the pair (f, g) is well blended and that there is some fixed z in R such that $z < f(z) < g(z)$. These assumptions merely clarify the notation and do not impair the validity of the argument.

Let $P = \{m/n: m, n \text{ are positive integers and } f^m(z) < g^n(z)\}$. Clearly the set P is bounded above. Moreover if m, n, p, q are positive integers and $p/q < m/n \in P$ then $p/q \in P$. Let δ be the supremum of P. We will show now that δ is irrational. If possible, suppose $\delta = a/b$ is rational, where a and b are some positive integers.

Case (1). $a/b \in P$. Then for any positive integer n, $f^{na}(z) < g^{nb}(z) < f^{na+1}(z)$. Therefore $z < f^{-na} \circ g^{nb}(z) < f(z)$. But this last inequality cannot hold because the map $f^{-a} \circ g^b$ has the same fixed points as f and g.

Case (2). $a/b \notin P$. Then for any positive integer n, $f^{na-1}(z) < g^{nb}(z) < f^{na}(z)$ and consequently $f^{-1}(z) < f^{-na} \circ g^{nb}(z) < z$. For the reason stated in case (1) above, the last inequality is again not possible. We, therefore, conclude that δ must be irrational.

Let $D = \{m + n\delta$: m and n are integers$\}$. Clearly D is a dense subgroup of R. Now define π: $D \times R \to R$ by $\pi(m + n\delta, x) = f^m \circ g^n(x)$. In view of Theorem 2.7, the proof of this theorem will be complete if we show that π is separately continuous.

It is clear that for each $t \in D$, the map π^t is a homeomorphism. It remains to be seen that π_x is continuous for each x in R. Clearly if x is a fixed point of f and g then π_x is constant and hence continuous. So take $x \in R$ such that x is not a fixed point of f. Since f and g satisfy the condition of being well blended (and in particular condition (6) above), the range of π_x has a connected closure. So if we prove that π_x is monotone, the continuity will follow.

Suppose $f(x) > x$. Take $t \in D$ such that $t > 0$. If $t = m + n\delta$, where m and n are both nonnegative integers, then $\pi^t(x) = f^m g^n(x) > x$. If $t = m - \delta n$ with m and n both nonnegative integers then $t > 0$ implies $m > \delta n$ and so $f^m(x) > g^n(x)$. Consequently $f^m g^{-n}(x) > x$, and so $\pi(t, x) > x$. Similarly if $t = \delta \cdot n - m$, where m and n are nonnegative integers then $\pi(t, x) > x$. It is clear now that for any positive t in D and for any x in R satisfying $f(x) > x$, we have $\pi(t, x) > x$; and therefore, since π satisfies (A2), π_x must be monotonically increasing. A similar argument shows that for all x in R satisfying the condition $f(x) < x$, the map π_x is monotonically decreasing. The proof of the theorem is now complete.

It is possible to apply Theorems 4.1 and 4.2 to determine whether or not a given finite collection of homeomorphisms is generated by some continuous flow. However, it would not be worthwhile to attempt to list necessary and sufficient conditions for such a case. If any two of the homeomorphisms of such a collection are well blended, then it may even be possible to explicitly determine the "essentially unique" continuous flow which generates them. Then, of course, it would be easy to see whether or not all of the remaining homeomorphisms of the given set are generated by the flow. The following example shows how it works.

EXAMPLE. Consider the homeomorphisms f and g on R defined by $f(x) = x^3$ and $g(x) = x^5$. Following the proof of Theorem 4.2 we find $\delta = (\ln 5)/(\ln 3)$. One might take $\pi(1, x) = x^3$ and $\pi((\ln 5/\ln 3)), x) = x^5$. A better choice would be to take $\pi(\ln 3, x) = x^3$ and $\pi(\ln 5, x) = x^5$. The two choices are essentially the same, but the second one is more revealing and suggests that we define:

$$\begin{aligned} \pi(t, x) &= x^{\exp t}, & &\text{if } x \geqslant 0 \\ &= -[|x|^{\exp t}], & &\text{if } x < 0. \end{aligned}$$

It is now immediately clear that π is a continuous flow and generates f and g. In fact every homeomorphism of type $x \to x^{2n-1}$, n a positive integer, is generated by π.

Finally we note that the techniques used in this paper do not work in R^2. The corresponding problem of embedding a discrete flow on R^2 in a continuous flow seems to be far more complex.

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