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Vibration Characteristics Of Free Thin Cylindrical Shells

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This paper considers the flexural vibrations of free thin circular cylinders. A frequency equation is derived using free-free characteristic beam functions to represent the variation of mid-surface shell displacement components, u, v and w, with respect to the axial direction. Timoshenko strain-displacement relations for thin cylinders are used to determine elastic vibratory strain energy. Energy methods are applied to obtain the frequency equation and associated amplitude ratios for each of its roots. This energy solution is checked experimentally using a vibration exciter and numerically using the SABOR IV finite element program. With minor modification, the frequency equation conforms to the one obtained in a similar way by Arnold and Warburton for cylinders with clamped ends and simply supported ends. Thus the proposed form of frequency equation, by accommodating a greater variety of boundary conditions, simplifies the task of determining cylinder vibration characteristics.

Introduction

The modal vibration characteristics of thin circular cylindrical shells are needed in the design and evaluation of many structures and machines. An exact method for determining characteristics was outlined by Flügge [1][1] in which the solution of his differential equations of motion requires the evaluation of a third order determinant. Furthermore, analysis of homogeneous end boundary conditions leads to an eighth order eigenvalue problem that is coupled to the third order determinant. Papers by Forsberg [2] using numerical iteration and Warburton [3] using other techniques give practical results for this lengthy method.

Shorter approximate methods for determining modal characteristics are also available. Arnold and Warburton [4] derive frequency equations for cylinders with simply supported ends and clamped ends using the corresponding characteristic beam functions to represent the variation of mid-surface shell displacement components with respect to the axial direction. They use Timoshenko strain-displacement relations for cylinders and the Rayleigh-Ritz energy method to obtain a cubic frequency equation. Parameters for the equation are given to account for simply supported ends and clamped ends.

Another paper by Warburton [3] presents a frequency determinant based on Flügge strain-displacement relations, and the characteristic beam functions and energy methods used in his earlier work. He applies this formulation to cylinders with clamped ends and free ends and notes frequency root errors of less than 10 percent. For free ends, the exact solution in the error calculation was based on the shear stress resultant $N_{x\phi} = 0$ at the ends. In a recent communication, he provides small corrections to the published exact solution values to account for effective shear stress resultant $T_x = 0$ (instead of $N_{x\phi}$) at free ends. Warburton and Higgs [5] discuss free end conditions further and give results for cantilevered cylinders.

The objective of the work herein is to provide an accurate and easy-to-use frequency equation for free thin cylinders by applying the approximate energy method of Arnold and Warburton [4] to the free-free case. Formulation of the free cylinder problem with Timoshenko strain-displacement relations should result in a frequency equation that is similar to previously published equations representing clamped ends and simply supported ends. Identification of the similarities should lead to a general form of frequency equation that accommodates all of these end conditions.

The theoretical expressions to be developed pertain to $m \geq 1$ and $n \geq 2$. When there are no wave variations in shell motion with respect to the axial direction ($m = 0$), Love's [6] frequency equation applies. When the cylindrical shell has axisymmetric ($n = 0$) motion and beam-type ($n = 1$) motion, Forsberg [7] shows that behavior can be adequately predicted by considering the cylinder as a ring for $n = 0$ modes and as a compact beam for $n = 1$ modes. Since minimum natural frequency is usually associated with a mode having two or more circumferential waves ($n \geq 2$), the following energy solution is relevant to typical cylindrical shell vibration problems.

[1] Numbers in brackets designate References at end of paper.

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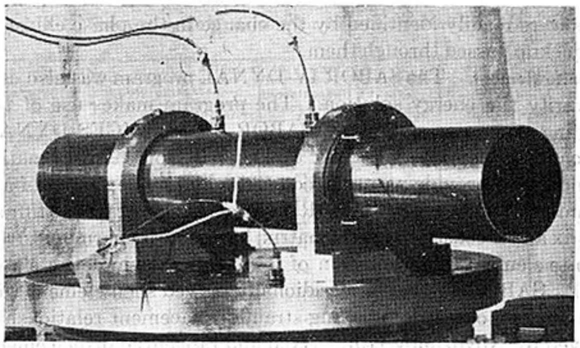

Fig. 1 Free-free cylinder mounted on vibration exciter

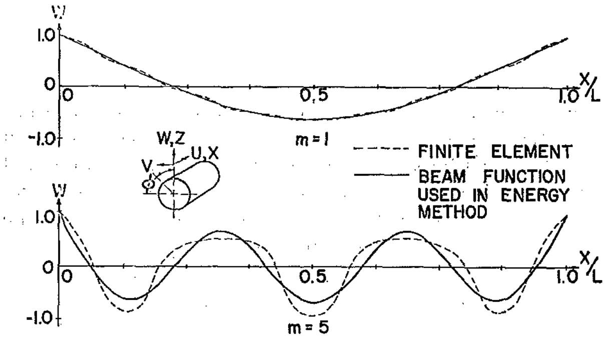

Fig. 2 Mode shape comparison, radial motion component, $L/a = 20$, $n = 2$

Energy Solution

Free Cylinders. In determining the strain energy needed for the free cylinder solution, the following mid-surface displacement functions are used in Timoshenko [8] strain displacement relations

$$u = U\left(\frac{a}{\mu_m}\right)\frac{dF_m(x)}{dx}\cos n\phi \qquad (1a)$$

$$v = VF_m(x)\sin n\phi \qquad (1b)$$

$$w = WF_m(x)\cos n\phi \qquad (1c)$$

where U, V, and W are functions of time only and $F_m(x)$ is the mth ($m = 1, 2, 3 \ldots$) characteristic function for a vibrating uniform free-free Bernoulli-Euler beam whose length is that of the cylinder. In the assumed displacement functions $u \sim \partial w/\partial x$ which gives zero normal strains, $\partial u/\partial x$, at the ends of the cylinder and maxima normal strains, $\partial u/\partial x$, near all other radial motion antinodes. The sinusoidal variations of shell displacement components with respect to the circumferential direction, ϕ, are exact in the sense that they satisfy Flügge [9] differential equations of motion.

Using the foregoing formulation and integrating the strain energy over the volume of elastic shell material, the total strain energy is

$$\begin{aligned} S = \frac{\pi EhL}{4a(1-\nu^2)}\{&\mu^2\theta_1U^2 + \beta\mu^4\theta_1W^2 \\ &+ \theta_1[(nV - W)^2 + \beta(nV - n^2W)^2] \\ &+ 2\nu\theta_2[-\mu nUV + \mu UW + \beta(\mu^2n^2W^2 - \mu^2nVW)] \\ &+ {}^1/_2(1-\nu)\theta_3[n^2U^2 + \mu^2V^2 - 2\mu nUV + 4\beta(\mu^2V^2 \\ &+ \mu^2n^2W^2 - 2\mu^2nVW)]\} \end{aligned} \qquad (2)$$

where values of μ are given by the frequency equation for free-free beams, $\cosh\frac{\mu}{a}L\cos\frac{\mu}{a}L = 1$, and

$$\theta_1 = 1 + (-1)^{m+1}k^2 \qquad (3a)$$

$$\theta_2 = 1 + (-1)^{m+1}\left(\frac{2a}{\mu L}\sin\frac{\mu}{a}L - k^2\right) \qquad (3b)$$

$$\theta_3 = 1 + (-1)^m\left(\frac{6a}{\mu L}\sin\frac{\mu}{a}L + k^2\right) \qquad (3c)$$

$$k = \frac{\cosh\frac{\mu}{a}L - \cos\frac{\mu}{a}L}{\sinh\frac{\mu}{a}L - \sin\frac{\mu}{a}L} \qquad (3d)$$

$$\beta = \frac{h^2}{12a^2} \qquad (3e)$$

Neglecting rotatory inertia, the total kinetic energy is

$$T = \frac{\pi\rho haL}{4g}[\theta_3U^2_{,t} + \theta_1(V^2_{,t} + W^2_{,t})] \qquad (4)$$

Assuming harmonic motion and performing a standard Rayleigh-Ritz analysis, the frequency equation and associated amplitude ratio expressions are found:

$$\Delta^3 - R_2\Delta^2 + R_1\Delta - R_0 = 0 \qquad (5a)$$

where $R = 1/(\theta_1{}^2\theta_3)$

$$\begin{aligned} R_2 &= R(J\theta_1\theta_3 + H\theta_1\theta_3 + G\theta_1{}^2) \\ R_1 &= R(GJ\theta_1 + GH\theta_1 - HJ\theta_3 - y_3z_2\theta_3 - y_1x_2\theta_1 - x_3z_1\theta_1) \\ R_0 &= R(x_2y_3z_1 - x_3y_1z_2 - Gy_3z_2 - y_1x_2J - x_3z_1H + GHJ) \\ G &= \mu^2\theta_1 + {}^1/_2(1-\nu)n^2\theta_3 \\ H &= \{n^2\theta_1 + {}^1/_2(1-\nu)\mu^2\theta_3 + \beta[n^2\theta_1 + 2(1-\nu)\mu^2\theta_3]\} \\ J &= \{\theta_1 + \beta[\mu^4\theta_1 + n^4\theta_1 + 2\nu\mu^2n^2\theta_2 + 2(1-\nu)\mu^2n^2\theta_3]\} \\ y_1 &= -[\nu\mu n\theta_2 + {}^1/_2(1-\nu)\mu n\theta_3] \\ z_1 &= \nu\mu\theta_2 \\ z_2 &= -\{n\theta_1 + \beta[n^3\theta_1 + \nu\mu^2n\theta_2 + 2(1-\nu)\mu^2n\theta_3]\} \\ x_2 &= y_1, \quad x_3 = z_1, \quad y_3 = z_2 \\ \Delta &= \rho a^2\omega^2(1-\nu^2)/Eg \end{aligned}$$

$$(V/U) = \frac{-[x_3 - (G - \Delta\theta_3)(J - \Delta\theta_1)/z_1]}{y_3 - y_1(J - \Delta\theta_1)/z_1} \qquad (5b)$$

Nomenclature

a = mean radius of cylinder
h = thickness of cylinder wall
L = length of cylinder
t = time
x = axial coordinate
ϕ = circumferential coordinate
z = radial coordinate
u, v, w = components of displacement at the shell mid-surface in the axial, tangential, and radial directions
m = axial half-wave number
n = number of circumferential waves
E = Young's modulus of elasticity
μ/a = roots of $\cosh\frac{\mu}{a}L\cos\frac{\mu}{a}L = 1$
ν = Poisson's ratio
ρ/g = mass density of shell material
ω = circular frequency
ω_0 = lowest extensional natural frequency of a ring in plane strain = $[Eg/\rho a^2(1-\nu^2)]^{1/2}$
ω/ω_0 = frequency factor
S = total strain energy of shell
T = total kinetic energy of shell

$$(W/U) = -\frac{1}{z_1}[x_1 + y_1(V/U)] \qquad (5c)$$

Application to Cylinders With Clamped Ends and Simply Supported Ends. It may be noted that the frequency equations developed by Arnold and Warburton [4] for fixed-fixed and simply supported thin cylinders are special cases of the frequency equation developed for free-free cylinders. If equations (5) remain the same, and equation (3c) is changed such that

$$\theta_3 = 1 + (-1)^{m+1}\left(\frac{2a}{\mu L}\sin\frac{\mu}{a}L - k^2\right)$$

the results apply to the case of a fixed-fixed cylinder. If equations (5) remain the same and equations (3) are changed such that $\theta_1 = \theta_2 = \theta_3 = 1$ and $\frac{\mu}{a}L = m\pi$, the results now apply to the case of a simply supported cylinder. Frequencies calculated in this way for the fixed-fixed and simply supported cases agree exactly with those obtained from the expressions of Arnold and Warburton [4].

Methods of Verification

Experimental. For experimental verification of the energy solution, thin cylinders were mounted in the free-free condition to a 3500 pound-force MB C25HHA vibration exciter by continuous ring contacts at axial nodal locations as shown in Fig. 1. A 0.063-diameter-wire snap ring provided continuous contact around the circumference at the axial nodal locations for the tangential and radial components of motion. The nodal locations were determined from finite element program output for each nodal mode to be investigated. The desired distance between normal supports was obtained by two adjustments. First the aluminum fixtures could be attached at one-inch intervals to the base plate through a predrilled hole pattern. Second, the fixture inserts containing the wire snap ring supports could be located within the fixtures anywhere within the one-inch intervals by means of the four adjusting screws. Thus, any location along the cylinder, within the limits (16-in. maximum) of the base plate, could be obtained in order to support the cylinder at nodal locations. The cylinders were lengths of cold-drawn seamless steel tubing having a 4.0-in. OD and a 0.065-in. wall thickness.

A combination of experimental techniques were used to determine particular modal characteristics. Natural frequencies were determined during scanning of the frequency spectrum by watching the 180-deg phase reversals of two accelerometers mounted on the wall of the cylinder, by noting the maximum levels on the g-meters, and by observing the maximum sand activity inside the cylinder. Mode shapes were determined by observing the sand distribution inside the cylinder and by a tracing method using a phonograph cartridge as a vibration pickup. In the tracing method, the signal from an accelerometer mounted on the base plate was fed to one plate of a cathode-ray tube and the signal from the vibration pickup was fed to the other plate. Nodal lines were readily identified by the change in the phase ellipse as the pickup passed through them.

Finite Element. The SABOR IV-DYNAL program was also used to verify the energy solution. The program makes use of two existing computer programs, SABOR IV and ICES-DYNAL, which have been integrated by McDonnell Douglas Automation Company to provide natural frequencies and mode shapes of axisymmetric shells. The SABOR IV program generates a stiffness matrix and a consistent mass matrix, harmonic by harmonic, from a finite element representation of a general axisymmetric elastic shell. SABOR IV uses a meridionally curved shell element with eight degrees of freedom having strain displacement relationships derived by Novozhilov [10]. Axial and tangential displacements are allowed to vary linearly over the element length, while radial displacement is allowed to take on a cubic variation over the element length. Both slope, $\partial W/\partial x$, and displacement compatibility exist at the boundaries between elements. The DYNAL program subsequently uses the mass and stiffness matrices to generate the corresponding frequencies and mode shapes for each harmonic provided by SABOR IV ($n = 2$ and $n = 3$ are the harmonics which were investigated.) The program is extensively documented in reference [11].

The number of axial half-waves handled accurately depended upon the number of elements used in our model. For each L/a ratio, two finite element models were used in order to determine the effect of element length on accuracy of results. First eight and then 32 cylindrical elements were used. Comparing the eight and 32 element models, differences in the predicted natural frequencies indicated an increase in accuracy as element length was decreased. While the differences for long shells, $L/a \geq 10$, were large, the differences for short shells, $L/a \leq 5$, were small and, therefore, the 32 element model was chosen as being satisfactory for this investigation.

The program's output consisted of frequency in Hertz and normalized mode shapes. The mode shapes were used to identify the different axial modes and to locate the nodes for the experimental work.

Results

Frequency Comparison. Table 1 presents natural frequencies obtained by each of the three methods used: energy, experimental, and finite element with 32 elements. These frequencies correspond to the predominately radial modes of vibration. In calculating energy and finite element frequencies, the material properties assumed for the steel cylinders are: Poisson's ratio, $\nu = 0.3$; elastic modulus, $E = 30{,}000$ ksi; and specific weight, $\rho = 0.283$ lb/in³. To facilitate application of results to cylinders with other material properties, the nondimensional frequency factor, ω/ω_0, which does not depend on the values of E and ρ, is also presented for the energy solution. Percent deviations with respect to the energy method results are presented.

The experimental natural frequencies deviate by less than 8 percent from the corresponding energy method frequencies with a

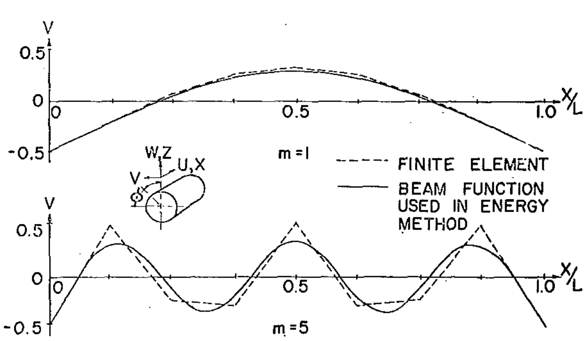

Fig. 3 Mode shape comparison, tangential motion component, $L/a = 20$, $n = 2$

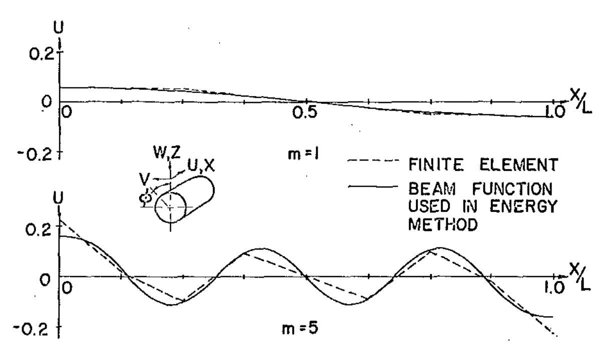

Fig. 4 Mode shape comparison, axial motion component, $L/a = 20$, $n = 2$

Table 1 Frequency comparison for free-free cylinders, $a/h = 30.3$

NODAL NUMBERS n	NODAL NUMBERS m	METHOD OF SOLUTION: Energy Method frequency factor	Energy Method (freq. hz) (1)	Experimental (freq. hz) (2)	Finite Element (freq. hz) (3)	PERCENT DEVIATION $= 100 \cdot \frac{(2)-(1)}{(1)}$	PERCENT DEVIATION $= 100 \cdot \frac{(3)-(1)}{(1)}$
		L/a = 5					
2	1	0.1734	2974.3	**	2866.5	-	-3.6
"	2	0.3773	6471.1	**	5973.0	-	-7.7
"	3	0.5528	9480.3	**	8792.3	-	-7.3
"	4	0.6760	11,593.6	**	10,919.6	-	-5.8
"	5	0.7593	13,022.1	**	12,490.7	-	-4.1
3	1	0.1207	2069.6	2118	2034.4	+2.3	-1.7
"	2	0.2357	4043.0	**	3864.1	-	-4.4
"	3	0.3731	6398.3	**	6065.9	-	-5.2
"	4	0.5001	8577.2	**	8155.6	-	-4.9
"	5	0.6062	10,397.2	**	9978.0	-	-4.0
		L/a = 10					
2	1	0.0547	938.0	940	920.6	+0.0	-1.9
"	2	0.1245	2136.6	2035	2063.7	-4.7	-3.4
"	3	0.2153	3693.2	**	3547.4	-	-3.9
"	4	0.3119	5349.3	**	5130.1	-	-4.1
"	5	0.4045	6937.4	**	6673.1	-	-3.8
3	1	0.0786	1348.2	1395	1342.2	+3.5	-0.4
"	2	0.0996	1709.1	1765	1695.3	+3.3	-0.8
"	3	0.1407	2413.0	**	2386.8	-	-1.1
"	4	0.1963	3367.2	**	3326.6	-	-1.2
"	5	0:2596	4451.7	**	4402.6	-	-1.1
		L/a = 15					
2	1	0.0341	585.5	590	580.5	+0.8	-0.8
"	2	0.0633	1085.8	*	1071.0	-	-1.4
"	3	0.1093	1875.7	1780	1849.4	-5.1	-1.4
"	4	0.1655	2838.2	**	2799.2	-	-1.4
"	5	0.2267	3888.4	**	3842.6	-	-1.2
3	1	0.0744	1275.4	1345	1272.6	+5.5	-0.2
"	2	0.0802	1376.9	*	1375.5	-	-0.1
"	3	0.0938	1609.6	1680	1615.4	+4.3	+0.4
"	4	0.1166	1999.7	1985	2018.4	-0.7	+0.9
"	5	0.1472	2525.4	**	2565.5	-	+1.6
		L/a = 20					
2	1	0.0288	495.1	*	493.2	-	-0.4
"	2	0.0423	725.8	*	725.1	-	-0.1
"	3	0.0678	1162.6	1130	1166.9	-2.8	+0.4
"	4	0.1020	1749.6	*	1763.0	-	+0.8
"	5	0.1421	2437.6	**	2466.0	-	+1.2
3	1	0.0733	1257.9	*	1256.1	-	-0.1
"	2	0.0757	1298.7	*	1298.8	-	0.0
"	3	0.0812	1392.8	1500	1400.5	+7.7	+0.6
"	4	0.0912	1565.0	*	1588.9	-	+1.5
"	5	0.1064	1825.5	1875	1876.1	+2.7	+2.7

* Nodal locations were not within the range of test fixture..
** Frequency was not within the range of vibration exciter.

tendency to be higher when $n = 3$ and lower when $n = 2$. Since energy frequencies are theoretically higher than true values, experimental error, such as constraint by the fixtures, is responsible for the positive deviations noted.

Finite element frequencies deviate from energy frequencies more when L/a is small ($L/a \leq 10$). This is to be expected, since the displacement functions used in the energy methods do not satisfy the end boundary conditions and, therefore, give poorer approximations when end influences become more dominant. Finite element deviations are negative except when $L/a \geq 15$. The small positive deviations for long cylinders would probably be eliminated by using more than 32 elements in the finite element model. However, the finite element results are believed to be very accurate, particularly for $m < 3$, since the 32 element model possesses 132 degrees of freedom (33 nodes $\times$ 4 d.f. per node).

Mode Shape Comparison. Typical mode shapes from the eight finite element model and the energy solution are shown in Figs. 2, 3, and 4. The radial motion component of mode shape is shown in Fig. 2, the tangential motion component in Fig. 3, and the axial motion component in Fig. 4. In presenting the finite element results, the normalized mode shape data obtained from the computer output are plotted as points at the stations between finite elements. These points are then connected with a dashed line giving attention to the particular form of variation over the element length that applies to the motion component being charted. The solid lines show the beam function displacements assumed in the energy solution. These functions are normalized to $W = 1$ which establishes numerical values for U and V given by the amplitude ratio expressions, equations (5*b*) and (5*c*).

Mode shapes from the 32 element model are not shown because they nearly coincide with those of the energy solution. When $m = 1$, the 32 element model gives antinode (maxima) values for u, v, and w that deviate less than 3 percent from energy solution values. The nodal location for w deviate by less than $0.002L$ and similarly for v. When $m = 5$, the 32 element results deviate from the energy solution as follows:

1 Antinode (maxima) values of w are 2 to 5 percent higher with the percentage increasing toward the center of the cylinder. Absolute deviations for u and v are similar, but percentages are 3 to 15 percent.

2 The outer radial motion nodes are $0.004L$ closer to the ends of the cylinder, while the more central nodes are $0.01L$ closer to the center of the cylinder. Outer u and v nodes are $0.015L$ closer to the ends, while the inner nodes are $0.01L$ closer to the center.

Conclusions

The approximate energy method of Arnold and Warburton [4] has been applied to cylinders with free ends. A frequency equation was derived and presented in a form that also accommodates cylinders with clamped ends and simply supported ends. Frequencies from this solution compare closely with experimental and finite element results for free cylinders. Deviations are greater for short cylinders ($L/a < 10$) vibrating in a mode having more than one axial half-wave ($m > 1$), but otherwise appear to be less than 8 percent. Good agreement also exists between the assumed displacement functions of the energy solution and the mode shapes predicted by the finite element model.

The energy solution provides frequencies that are higher than the true frequencies of the modes whose shapes are being approximated with beam functions. This is of little practical importance for the lower m modes of long cylinders, since the error is small. However, for short cylinders, frequencies from the energy solution can be reduced a few percent to more closely estimate true natural frequencies. It is also important to remember that in free cylinder problems two lower frequency modes, Rayleigh and Love modes, also exist. Since they have no wave variations of displacement components with respect to the axial direction, they are not accounted for in this energy solution.

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