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The problem of transient thermal stresses in a solid, elastic, homogeneous, and isotropic sphere is solved for uniform and nonuniform, local surface heating. The temperature solutions are obtained by using separation of variables and integral transformation. The corresponding thermal stresses are derived by superposing a particular displacement potential function on Boussinesq solutions. Numerical solutions for two particular cases of localized heating of a typical brittle spherical solid have been obtained and presented. The results indicate a tensile stress concentration in the interior of the solid below the heated zone.

Introduction

A knowledge of thermal stresses caused by heating of brittle solids is important to understand thermal fracture and fragmentation processes. This study was carried out to understand and evaluate a method of breaking rocks and related brittle solids by surface heating. Theoretical analyses were made to obtain stress distributions during localized heating of simple spherical solids to help select optimum heating conditions for fragmentation. Sternberg, et al. [1],[1] Sharma [2], and Holden [3] have considered thermal stress problems in solid spheres with steady-state heating conditions. Warren [4] has studied the transient thermal stresses on the surface of a sphere for an assumed surface temperature distribution. This study presents the transient temperature and stress distributions in a sphere when locally heated on its surface with uniform and nonuniform heating conditions.

Analysis

Temperature Solution. Consider a homogeneous, isotropic, and elastic sphere which is initially at zero temperature. At time $t \geq 0$, part of its surface $0 \leq \theta \leq \theta_0$ is exposed to heat flux of various intensities $F(\theta)$ where θ is the angle measured from the axis of symmetry and the rest of the surface is assumed to be insulated. The radial coordinate r is measured from the center of the sphere. The temperature field $T(r, \mu, t)$ is governed by the mathematical system

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

$$\begin{aligned} K\frac{\partial T}{\partial r} &= F(\mu) \text{ at } r = r_0,\ \mu_0 \leq \mu \leq 1,\ t \geq 0 \\ &= 0 \quad \text{ at } r = r_0,\ -1 \leq \mu \leq \mu_0,\ t \geq 0 \end{aligned} \tag{1}$$

$$T(r,\mu,0) = 0 \text{ for all } r \text{ and } \mu \text{ at } t \leq 0$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial}{\partial \mu}\right] \tag{2}$$

and

$$\mu = \cos\theta, \qquad \mu_0 = \cos\theta_0 \tag{3}$$

In the equations, K is the thermal conductivity of the material; $\kappa = K/\rho c_p$ is the thermal diffusivity, ρ and c_p being density and specific heat at constant pressure, respectively; and r_0 is the radius of the sphere. $F(\mu)$ is a specified function for a certain range of μ or θ.

The solution to the system (1) can be written as

$$T(r,\mu,t) = \Omega(t) + T_s(r,\mu) + T_1(r,\mu,t) \tag{4}$$

where the steady temperature solution $T_s(r, \mu)$ satisfies the system

$$\nabla^2 T_s = \frac{1}{\kappa}\frac{d\Omega}{dt} \tag{5}$$

[1] Numbers in brackets designate References at end of paper.

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$$K\frac{\partial T_s}{\partial r} = F(\mu) \text{ at } r = r_0,\ \mu_0 \le \mu \le 1$$
$$= 0 \quad \text{at } r = r_0,\ -1 \le \mu \le \mu_0 \qquad (5)\ (Cont.)$$
$$\int_{-1}^{1}\int_0^{r_0} T_s r^2 dr d\mu = 0$$

and $T_1(r, \mu, t)$, the difference temperature, satisfies

$$\nabla^2 T_1 = \frac{1}{\kappa}\frac{\partial T_1}{\partial t}$$
$$K\frac{\partial T_1}{\partial r} = 0 \text{ at } r = r_0, -1 \le \mu \le 1,\ t \ge 0 \qquad (6)$$
$$T_1(r,\mu,0) = -T_s(r,\mu) \text{ at } t \le 0$$

The solution $\Omega(t)$ represents the difference between the average temperature at time t and the initial average temperature and is given by [5]

$$\Omega(t) = \frac{\kappa}{Kv}2\pi r_0^2\int_0^t\left[\int_{\mu_0}^1 F(\mu)d\mu\right]dt \qquad (7)$$

where $v = (4/3)\pi r_0^3$ is the volume of the sphere. It is evident that T_1 approaches zero as t approaches infinity.

When the surface heat flux $F(\mu)$ is specified, $\Omega(t)$ can be readily evaluated. With $\Omega(t)$ known, the steady temperature T_s can then be determined. This gives

$$T_s = -\frac{c}{10}r_0^2 + \frac{c}{6}r^2 + r_0\sum_{n=1}^{\infty}\frac{2n+1}{2n}\left(\frac{r}{r_0}\right)^n \times\left\{\int_{-1}^{1}\left[\frac{F(\mu)}{K} - \frac{c}{3}r_0\right]P_n d\mu\right\}P_n(\mu) \qquad (8)$$

where

$$c = \frac{1}{\kappa}\frac{d\Omega}{dt} \qquad (9)$$

and $P_n(\mu)$ is Legendre function of degree n of the first kind. The solution for T_1 can be derived by integral transform and has the form

$$T_1(r,\mu,t) = -\frac{r_0}{K}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{2n+1}{\beta_{nm}^2 - n(n+1)}\left(\frac{r}{r_0}\right)^{1/2}\frac{J_{n+1/2}\left(\beta_{nm}\frac{r}{r_0}\right)}{J_{n+1/2}(\beta_{nm})} \times\left[\int_{\mu_0}^{1}P_n(\mu)F(\mu)d\mu\right]P_n(\mu)e^{-\kappa\beta_{nm}^2 t/r_0^2} \qquad (10)$$

where the eigenvalues β_{nm} are the positive roots of

$$J_{n-1/2}(\beta_{nm}) = \frac{n+1}{\beta_{nm}}J_{n+1/2}(\beta_{nm}) \qquad (11)$$

and J_ν is the Bessel function of the first kind of fractional order ν.

The sum of $\Omega(t)$, $T_s(r, \mu)$, and $T_1(r, \mu, t)$ then gives the complete temperature solution $T(r, \mu, t)$ for a specified function $F(\mu)$.

The Stress Solution. In this section, the thermoelastic stress problem will be formulated. Using tensor notations, the linear thermoelastic equilibrium equations expressed in terms of the displacement are

$$u_{k,ki} + (1 - 2\nu)u_{i,kk} = 2\alpha(1 + \nu)T_{,i} \qquad (12)$$

$$\sigma_{ij} = G\left[u_{i,j} + u_{j,i} + \frac{\nu}{1-2\nu}u_{k,k}\delta_{ij} - \frac{2(1+\nu)}{1-2\nu}\alpha T\delta_{ij}\right] \qquad (13)$$

where u_i denotes the components of the displacement vector, G is the shear modulus, ν is Poisson's ratio, α is the coefficient of thermal expansion, and δ_{ij} is the Kronecker delta. Equations (12) and (13) are to be solved subject to the stress-free boundary conditions

$$\sigma_{ij}n_j = 0 \qquad \text{at } r = r_0 \qquad (14)$$

where n_j are the scalar components of the unit normal vector to the surface at $r = r_0$.

The solution of the system consisting of (12)–(14) can be represented as the sum of a particular solution of the nonhomogeneous system of equations and the complementary solution of the homogeneous system. The particular solution of (12) can be derived by introducing the displacement potential Φ in the form

$$u_i = \Phi_{,i} \qquad (15)$$

Substituting u_i from (15) into (11) yields

$$\nabla^2\Phi = \alpha l T \qquad (16)$$

where

$$l = \frac{1+\nu}{1-\nu} \qquad (17)$$

A particular solution of (16) has the form

$$\Phi = \Phi_1 + \Phi_2 \qquad (18)$$

where

$$\Phi_1 = l\alpha\kappa\int_0^t T_1(r,\mu,t)dt \qquad (19)$$

and Φ_2 satisfies

$$\nabla^2\Phi_2 = l\alpha T - \nabla^2\Phi_1 \qquad (20)$$

The particular solution Φ_1 can be readily obtained by substituting T_1 from (10 into (19) and carrying out the integration. Φ_2 can be then determined from (20) with the known Φ_1 and T solutions from (19) and (4).

Once the solution for Φ is found, the stress components corresponding to this function are obtained from the expressions [6]

$$\begin{aligned}
\tilde{\tilde{\sigma}}_{RR} &= 2\left(\frac{\partial^2\tilde{\Phi}}{\partial R^2} - \tilde{T}\right)\\
\tilde{\tilde{\sigma}}_{R\theta} &= 2\bar{\mu}\left(\frac{1}{R^2}\frac{\partial\tilde{\Phi}}{\partial\mu} - \frac{1}{R}\frac{\partial^2\tilde{\Phi}}{\partial R\partial\mu}\right)\\
\tilde{\tilde{\sigma}}_{\theta\theta} &= 2\left(\frac{1}{R}\frac{\partial\tilde{\Phi}}{\partial R} + \frac{1}{R^2}(1-\mu^2)\frac{\partial^2\tilde{\Phi}}{\partial\mu^2} - \frac{1}{R^2}\mu\frac{\partial\tilde{\Phi}}{\partial\mu} - \tilde{T}\right)\\
\tilde{\tilde{\sigma}}_{\varphi\varphi} &= 2\left(\frac{1}{R}\frac{\partial\tilde{\Phi}}{\partial R} - \frac{1}{R^2}\mu\frac{\partial\tilde{\Phi}}{\partial\mu} - \tilde{T}\right)\\
\tilde{\tilde{\sigma}}_{R\varphi} &= \tilde{\tilde{\sigma}}_{\theta\varphi} = 0
\end{aligned} \qquad (21)$$

with $\bar{\mu} = (1 - \mu^2)^{1/2}$. The dimensionless variables are

$$R = \frac{r}{r_0},\quad \tilde{T} = \frac{T}{\frac{q_0 r_0}{K}},\quad \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{Gl\frac{q_0 r_0\alpha}{K}},\quad \tilde{\Phi} = \frac{\Phi}{l\frac{q_0 r_0^3\alpha}{K}},\quad \tau = \frac{\kappa t}{r_0^2} \qquad (22)$$

The complete solution $[\tilde{\sigma}_{ij}]$ to the thermoelastic equilibrium problem governed by (1), (12), and (13) subject to the traction-free boundary condition (14) may be represented in the form

$$[\tilde{\sigma}_{ij}] = [\tilde{\tilde{\sigma}}_{ij}] + [\tilde{\bar{\sigma}}_{ij}] \qquad (23)$$

where $[\tilde{\tilde{\sigma}}_{ij}]$ is a particular solution of the field equations generated by $\tilde{\Phi}$ and $[\tilde{\bar{\sigma}}_{ij}]$ is the solution of a residual problem. The latter solution satisfies the homogeneous system, equations (12) and (13) without the temperature terms and counteracts the surface traction induced by $[\tilde{\tilde{\sigma}}_{ij}]$. The solution for $\tilde{\bar{\sigma}}_{ij}$ is obtained from the spherical harmonic stress functions χ and Ψ in the form

$$\chi(r,\mu) = r^n P_n(\mu), \qquad \Psi(r,\mu) = r^n P_n(\mu) \qquad (24)$$

as introduced by Sternberg, et al. [1]. The stress solutions $[C_n]$ and $[F_n]$ corresponding, respectively, to χ and Ψ are combined in the form [2]

$$[E_n] = (2n+1)[F_n] - (n - 3 + 4\nu)[C_{n+1}] \tag{25}$$

The stresses associated with $[C_n]$ are

$$\begin{aligned} \tilde{\sigma}_{RR}^{*} &= n(n-1)R^{n-2}P_n(\mu) \\ \tilde{\sigma}_{R\theta}^{*} &= -(n-1)\bar{\mu}R^{n-2}P_n'(\mu) \\ \tilde{\sigma}_{\theta\theta}^{*} &= R^{n-2}[\mu P_n'(\mu) - n^2 P_n(\mu)] \\ \tilde{\sigma}_{\varphi\varphi}^{*} &= R^{n-2}[nP_n(\mu) - \mu P_n'(\mu)] \\ \tilde{\sigma}_{\theta\varphi}^{*} &= \tilde{\sigma}_{R\varphi}^{*} = 0 \end{aligned} \tag{26}$$

and those associated with $[E_n]$ are

$$\begin{aligned} \tilde{\sigma}_{RR}^{**} &= [n^2(n-3) - 2\nu n]R^{n-1}P_{n-1}(\mu) \\ \tilde{\sigma}_{R\theta}^{**} &= (2 - n^2 - 2\nu)\bar{\mu}R^{n-1}P_{n-1}'(\mu) \\ \tilde{\sigma}_{\theta\theta}^{**} &= [(n + 4 - 4\nu)\mu P_{n-1}'(\mu) - n(n^2 + 2n - 1 + 2\nu)P_{n-1}(\mu)]R^{n-1} \\ \tilde{\sigma}_{\varphi\varphi}^{**} &= [-(n + 4 - 4\nu)\mu P_{n-1}'(\mu) \\ &\qquad + n(n - 3 - 4\nu n + 2\nu)P_{n-1}(\mu)]R^{n-1} \\ \tilde{\sigma}_{\theta\varphi}^{**} &= \tilde{\sigma}_{R\varphi}^{**} = 0 \end{aligned} \tag{27}$$

In the foregoing equations, $P_n'(\mu) = dP_n(\mu)/d\mu$. The solution $[\tilde{\tilde{\sigma}}_{ij}]$ can be put into the form

$$[\tilde{\tilde{\sigma}}_{ij}] = \sum_{n=0}^{\infty} c_n[C_n] + \sum_{n=0}^{\infty} d_n[E_{n+1}] \tag{28}$$

wherein the coefficients c_n and d_n are evaluated by imposing the stress-free conditions at $r = r_0$, equation (14). For the case under consideration, $\tilde{\sigma}_{RR} = \tilde{\sigma}_{R\theta} = 0$ at $R = 1$. This gives

$$c_n = \frac{\eta_n[3n + 2 + 2\nu - n^3 + 2\nu n] + \xi_n[2 - (n+1)^2 - 2\nu]}{2(n-1)[n^2 + n + 1 + \nu(2n+1)]} \qquad n = 2,3,4\ldots \tag{29}$$

$$d_n = \frac{\xi_n + n\eta_n}{2[(n^2 + n + 1 + \nu(2n+1)]} \qquad n = 0,1,2,3\ldots \tag{30}$$

where ξ_n and η_n are evaluated from $\tilde{\tilde{\sigma}}_{RR}$ and $\tilde{\tilde{\sigma}}_{R\theta}$ at $r = r_0$ as

$$\tilde{\tilde{\sigma}}_{RR}(1,\mu,\tau) = \sum_{n=0}^{\infty}\xi_n(\tau)P_n(\mu), \quad \tilde{\tilde{\sigma}}_{R\theta}(1,\mu,\tau) = \bar{\mu}\sum_{n=0}^{\infty}\eta_n(\tau)P_n'(\mu) \tag{31}$$

With these values of c_n and d_n, equations (23), (21), (26)–(28) constitute the complete solution of the thermoelastic problem. It should be pointed out, however, that the expressions for ξ_n and η_n are dependent on Φ and T solutions which in turn depend on the nature of surface heating.

Application for Uniform and Nonuniform Surface Heating. In the following sections, the special cases of uniform and nonuniform surface heating are presented because these two cases are very representative of most surface heating. Uniform heating is represented by $F(\mu) = q_0$ where q_0 is a constant. For nonuniform heating, the case of cosine flux variation is considered.

Case 1—Uniform Surface Heat Flux. When the surface heat flux is a constant q_0 for $\mu_0 \le \mu \le 1$, the temperature solution is

$$\tilde{T} = 3A\tau - \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} B_{nm}D_n \frac{J_{n+1/2}(\beta_{nm}R)}{R^{1/2}}P_n(\mu)[e^{-\beta_{nm}^2\tau} - 1] \tag{32}$$

where

$$A = \frac{1}{2}(1 - \mu_0) \tag{33}$$

$$D_n = \int_{\mu_0}^{1} P_n(\mu)d\mu \tag{34}$$

$$B_{nm} = \frac{2n+1}{[\beta_{nm}^2 - n(n+1)]J_{n+1/2}(\beta_{nm})} \tag{35}$$

and β_{nm} are the positive roots of (11). By employing (16), the displacement potential Φ can be obtained as

$$\tilde{\Phi} = \frac{1}{2}A\tau R^2 + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{B_{nm}D_n}{\beta_{nm}^2}\frac{J_{n+1/2}(\beta_{nm}R)}{R^{1/2}}P_n(\mu)[e^{-\beta_{nm}^2\tau} - 1] \tag{36}$$

The stresses corresponding to Φ and T are then evaluated from (21).

The residual stresses are given by (28). The corresponding $[C_n]$, $[E_n]$, c_n, and d_n are expressed by (26), (27), (29), and (30). The expressions for ξ_n and η_n appearing in (29) and (30) are found to be

$$\xi_n(\tau) = 2\sum_{m=1}^{\infty} B_{nm}D_n\frac{n(n+1)}{\beta_{nm}^2}J_{n+1/2}(\beta_{nm})[e^{-\beta_{nm}^2\tau} - 1] \tag{37}$$

and

$$\eta_n(\tau) = 2\sum_{m=1}^{\infty} B_{nm}D_n\frac{J_{n+1/2}(\beta_{nm})}{\beta_{nm}^2}[e^{-\beta_{nm}^2\tau} - 1] \tag{38}$$

The total stresses in dimensionless form are, from (23)

$$\begin{aligned} \tilde{\sigma}_{RR} &= 2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}D_n \\ &\times\left[-\frac{2}{\beta_{nm}R}\frac{J_{n-1/2}(\beta_{nm}R)}{R^{1/2}} + \frac{(n+1)(n+2)}{\beta_{nm}^2R^2}\frac{J_{n+1/2}(\beta_{nm}R)}{R^{1/2}}\right]P_n(\mu) \\ &\times[e^{-\beta_{nm}^2\tau} - 1] + \sum_{n=2}^{\infty}n(n-1)R^{n-2}c_nP_n(\mu) \\ &+ \sum_{n=1}^{\infty}(n+1)[(n+1)(n-2) - 2\nu]R^nd_nP_n(\mu) \end{aligned} \tag{39}$$

$$\begin{aligned} \tilde{\sigma}_{R\theta} &= 2\bar{\mu}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}D_n \\ &\times\left[-\frac{1}{\beta_{nm}R}\frac{J_{n-1/2}(\beta_{nm}R)}{R^{1/2}} + \frac{n+2}{\beta_{nm}^2R^2}\frac{J_{n+1/2}(\beta_{nm}R)}{R^{1/2}}\right]P_n'(\mu) \\ &\times[e^{-\beta_{nm}^2\tau} - 1] - \bar{\mu}\sum_{n=2}^{\infty}(n-1)R^{n-2}c_nP_n'(\mu) \\ &+ \bar{\mu}\sum_{n=1}^{\infty}[2 - (n+1)^2 - 2\nu]R^nd_nP_n'(\mu) \end{aligned} \tag{40}$$

$$\begin{aligned} \tilde{\sigma}_{\theta\theta} &= 2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}D_n\left\{\frac{1}{\beta_{nm}R}\frac{J_{n-1/2}(\beta_{nm}R)}{R^{1/2}}P_n(\mu)\right. \\ &+\left[\left(1 - \frac{n+1}{\beta_{nm}^2R^2}\right)P_n(\mu) - \frac{\mu}{\beta_{nm}^2R^2}P_n'(\mu) + \frac{1-\mu^2}{\beta_{nm}^2R^2}P_n''(\mu)\right] \\ &\times\left.\frac{J_{n+1/2}(\beta_{nm}R)}{R^{1/2}}\right\}[e^{-\beta_{nm}^2\tau} - 1] + \sum_{n=2}^{\infty}R^{n-2}c_n[\mu P_n'(\mu) - n^2P_n(\mu)] \\ &+ \sum_{n=1}^{\infty}R^nd_n\{(n + 5 - 4\nu)\mu P_n'(\mu) - (n+1)[(n+1)^2 + \\ &\qquad 2(n+1) - 1 + 2\nu]P_n(\mu)\} \end{aligned} \tag{41}$$

$$\begin{aligned} \tilde{\sigma}_{\varphi\varphi} &= 2\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\beta_{nm}D_n\left\{\frac{1}{\beta_{nm}R}\frac{J_{n-1/2}(\beta_{nm}R)}{R^{1/2}}P_n(\mu)\right. \\ &+\left.\left[\left(1 - \frac{n+1}{\beta_{nm}^2R^2}\right)P_n(\mu) - \frac{\mu}{\beta_{nm}^2R^2}P_n'(\mu)\right]\times\frac{J_{n+1/2}(\beta_{nm}R)}{R^{1/2}}\right. \\ &\times[e^{-\beta_{nm}^2\tau} - 1] + \sum_{n=2}^{\infty}c_nR^{n-2}[nP_n(\mu) - \mu P_n'(\mu)] \\ &+ \sum_{n=1}^{\infty}d_nR^{n-2}\{-(n + 5 - 4\nu)\mu P_n'(\mu) \\ &\qquad + (n+1)[n - 2 - 2(2n+1)\nu]P_n(\mu)\} \end{aligned} \tag{42}$$

$$\tilde{\sigma}_{R\varphi} = \tilde{\sigma}_{\theta\varphi} = 0 \tag{43}$$

For the stresses given by (39)–(43) to be independent of μ as $R \to 0$, the $n = 2$ terms appearing in all the series have to be excluded. The temperature and stresses at the center of the sphere can be obtained by letting the dimensionless radius R approach zero. Thus

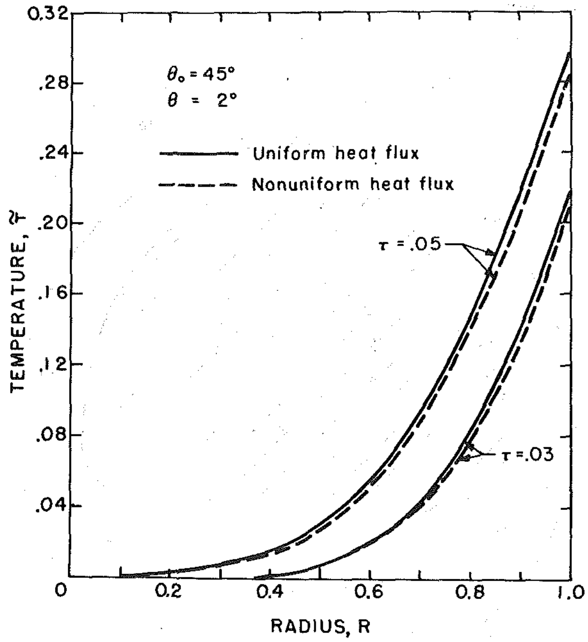

Fig. 1 Temperature distributions in a sphere

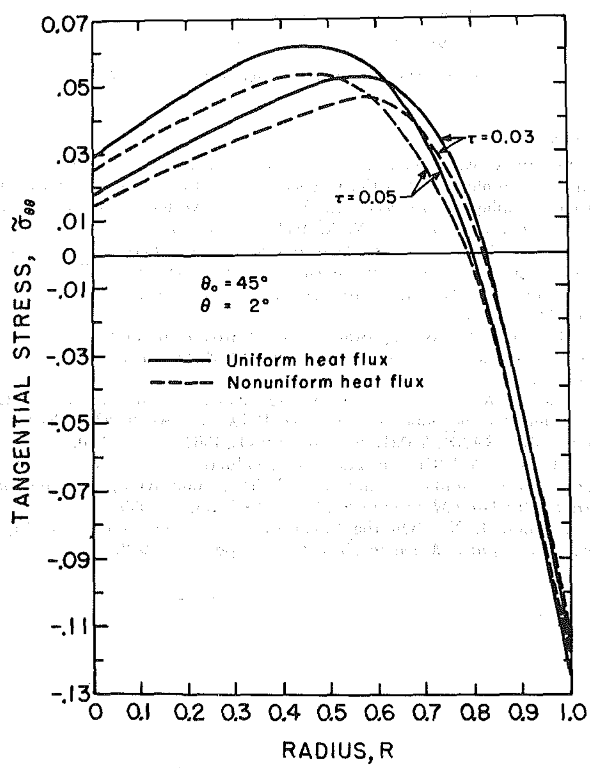

Fig. 2 Tangential stress distributions in a sphere

$$\tilde{T}(0,\mu,\tau) = 4A\left[\frac{3}{4}\tau - \frac{1}{\sqrt{2\pi}}\sum_{m=1}^{\infty}\frac{e^{-\beta_{0m}^2\tau}-1}{\beta_{0m}^{3/2}J_{1/2}(\beta_{0m})}\right] \quad (44)$$

$$\tilde{\sigma}_{RR}(0,\mu,\tau) = \tilde{\sigma}_{\theta\theta}(0,\mu,\tau) = \tilde{\sigma}_{\varphi\varphi}(0,\mu,\tau) = \frac{8}{3}\sqrt{\frac{2}{11}}\,A\sum_{m=1}^{\infty}\frac{e^{-\beta_{0m}^2\tau}-1}{\beta_{0m}^{3/2}J_{1/2}(\beta_{0m})} \quad (45)$$

$$\tilde{\sigma}_{R\theta}(0,\mu,\tau) = 0 \quad (46)$$

When the entire surface of the sphere is heated under constant

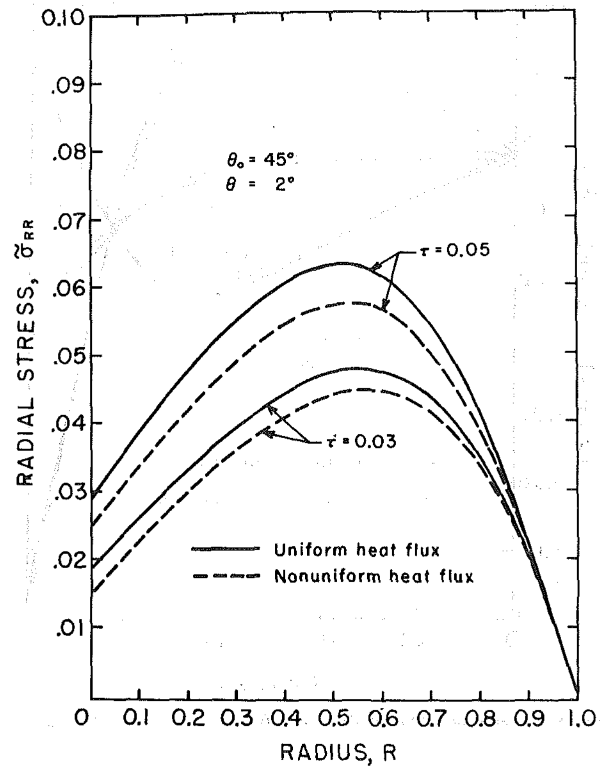

Fig. 3 Radial stress distributions in a sphere

flux, the expressions for σ_{ij} simplify considerably. The resulting expressions are

$$\tilde{\sigma}_{RR}\Big|_{\mu_0=-1} = \frac{8}{R^2}\sum_{m=1}^{\infty}\frac{1}{\beta_{0m}^3\sin\beta}\left[-\cos\beta_{0m}R + \frac{\sin\beta_{0m}R}{\beta_{0m}R}\right] \times \left[e^{-\beta_{0m}^2\tau}-1\right] \quad (47)$$

$$\tilde{\sigma}_{\theta\theta}\Big|_{\mu_0=-1} = \tilde{\sigma}_{\varphi\varphi}\Big|_{\mu_0=-1} = \frac{4}{R^2}\sum_{m=1}^{\infty}\frac{1}{\beta_{0m}^3\sin\beta_{0m}}\left[\cos\beta_{0m}R + \frac{(\beta_{0m}^2R^2-1)\sin\beta_{0m}R}{\beta_{0m}R}\right]\left[e^{-\beta_{0m}^2\tau}-1\right] \quad (48)$$

$$\tilde{\sigma}_{R\theta}\Big|_{\mu_0=-1} = \tilde{\sigma}_{R\varphi}\Big|_{\mu_0=-1} = 0 \quad (49)$$

Equations (47)–(49) are the familiar results for stresses in a sphere due to uniform heat flux over its entire surface.

Case 2—Nonuniform Surface Heat Flux. When the surface heat flux is given by $F(\mu) = q_0\mu$ for $\mu_0 \leq \mu \leq 1$, equations (32) and (36)–(46) still apply except that for this heating condition, A and D_n are replaced, respectively, by

$$A = \frac{1}{4}(1 - \mu_0^2) \quad (50)$$

and

$$D_n = \int_{\mu_0}^{1}\mu P_n(\mu)d\mu \quad (51)$$

Results and Discussion

The analytical solutions are programmed and an example of the dimensionless temperature and stress distributions for heating angle $\theta_0 = 45$ deg ($\mu_0 = 0.707$) are presented in Figs. 1–5 for both cases of heating for a typical brittle solid having Poisson's ratio of 0.25. Fig. 1 shows the dimensionless temperature T, Fig. 2 shows the dimensionless tangential stress $\tilde{\sigma}_{\theta\theta}$, and Fig. 3 shows the radial stress $\tilde{\sigma}_{RR}$ as a function of the radial distance R for dimensionless heating times of $\tau = 0.03$ and $\tau = 0.05$ for both the

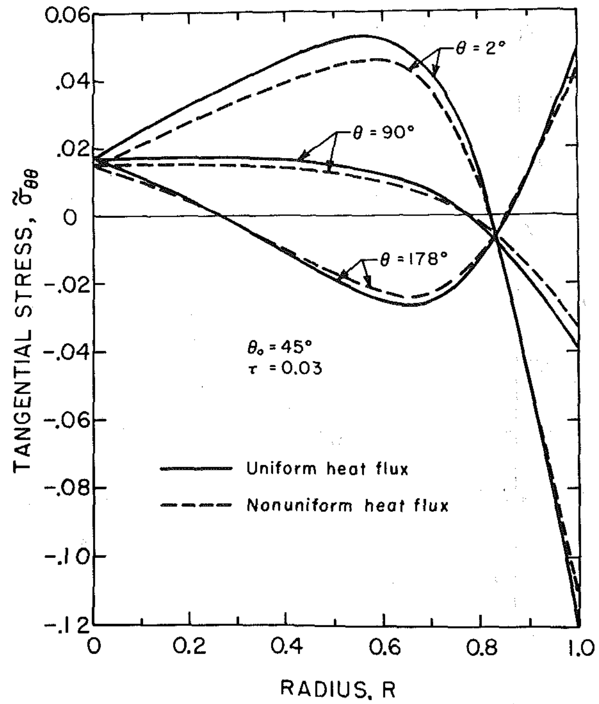

Fig. 4 Tangential stress distributions in a sphere for various angles θ

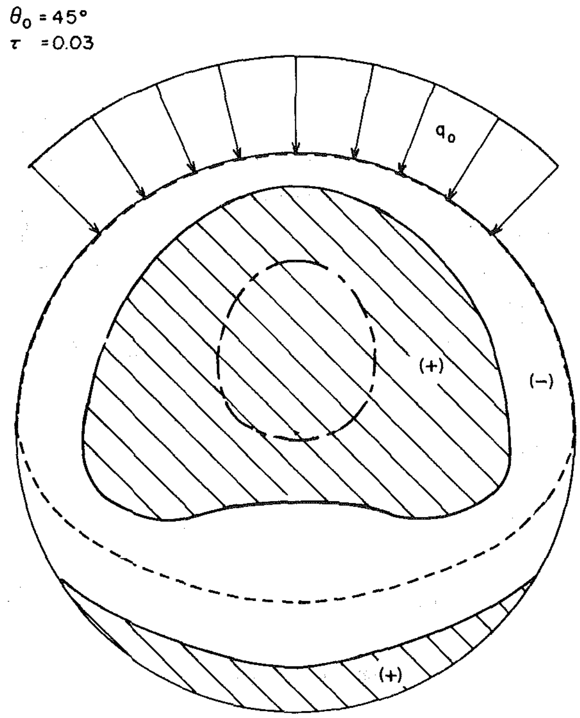

Fig. 5 Tangential stress reversal in a sphere

uniform and nonuniform heating conditions at $\theta = 2$ deg. The results indicate a tensile stress concentration within the heated zone in the interior of the solid. For similar localized heating geometry, the magnitude of stresses for uniform heating exceeds the stress magnitudes induced by nonuniform heating. Fig. 4 shows dimensionless tangential stress $\tilde{\sigma}_{\theta\theta}$ against radial distance R for various angles θ at $\tau = 0.03$. Fig. 5 shows the reversal of the tangential stress due to uniform heating. The results indicate that the tensile stress concentrations induced in the interior of the spherical solid decrease with increasing angle θ from the heated zone. The stresses are reversed in zones directly beneath the heated portion of the sphere.

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