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R. C. Batra
Missouri University of Science and Technology

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Deformation produced by a simple tensile load in an isotropic elastic body

R. C. BATRA

Engineering Mechanics Department, University of Missouri, Rolla, Missouri 65401, USA

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ABSTRACT
It is shown that a simple tensile load produces a simple extension provided the empirical inequalities (Truesdell and Noll [1], eqn. 51.27) hold.

One form of the general constitutive equation for an unconstrained isotropic homogeneous elastic material (*Cf.* § 47 of [1]) is

$$\boldsymbol{T}=f_0\mathbf{1}+f_1\boldsymbol{B}+f_{-1}\boldsymbol{B}^{-1}, \tag{1}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{B}$ is the left Cauchy–Green tensor with respect to an undistorted configuration, and the response coefficients f_α ($\alpha=-1, 0, 1$) are functions of the principal invariants of $\boldsymbol{B}$. In order that equation (1) describe a response that is physically reasonable, Truesdell and Noll ([1], eqn. 51.27) proposed that f_α satisfy the following inequalities.

$$f_0\leqslant 0\,, \quad f_1>0\,, \quad f_1\leqslant 0\,. \tag{2}$$

The Cauchy–Stress tensor for a simple tensile load along the x_3-axis of a rectangular Cartesian set of axes is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{bmatrix} \tag{3}$$

where T is a positive constant. The constitutive relation (1) requires that $\boldsymbol{TB}=\boldsymbol{BT}$. This together with (3) gives that

$$B_{13}=B_{23}=0\,.$$

Thus

$$\boldsymbol{B}=\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix}. \tag{4}$$

Since $\boldsymbol{B}$ is positive definite,

$$B_{33}>0\,, \quad C \equiv B_{11}B_{22}-B_{12}^2>0\,. \tag{5}$$

Substituting from (3) and (4) into (1), we conclude that

$$\begin{aligned} 0 &= \left(f_1 - \frac{1}{C}f_{-1}\right)B_{12}\,, \\ 0 &= f_0+f_1B_{11} + \frac{1}{C}f_{-1}B_{22}\,, \\ 0 &= f_0+f_1B_{22} + \frac{1}{C}f_{-1}B_{11}\,, \\ T &= f_0+f_1B_{33} + \frac{1}{B_{33}}f_1\,. \end{aligned} \tag{6}$$

Substraction of $(6)_3$ from $(6)_2$ yields

$$0 = \left(f_1 - \frac{1}{C}\,f_{-1}\right)(B_{11}-B_{22})\,. \tag{7}$$

When inequalities (2) hold, it follows from $(6)_1$ and (7) that

$$B_{12}=0\,, \quad B_{11}=B_{22}\,. \tag{8}$$

Thus B has the form

$$\boldsymbol{B} = \mathrm{diag}(\alpha^2 v^2, \alpha^2 v^2, v^2)\,, \tag{9}$$

where

$$\begin{aligned} v^2 &= B_{33}\,, \\ \alpha^2 &= B_{11}/B_{33}\,. \end{aligned} \tag{10}$$

In $(10)_2$ the positive definiteness of $\boldsymbol{B}$ has been used. $\boldsymbol{B}$ given by (9) corresponds to simple tension (*Cf.* [2], eqn. 44.1). That $\alpha < 1$ follows from (9), (2) and the equation obtained by subtracting $(6)_2$ from $(6)_4$.

The constitutive relation for an incompressible isotropic elastic material is

$$T = -p\mathbf{1} + f_1\boldsymbol{B} + f_{-1}\boldsymbol{B}^{-1}, \tag{11}$$

where p is an arbitrary hydrostatic pressure and f_1 and f_{-1} are functions of the first and second invariant of $\boldsymbol{B}$; the third invariant of $\boldsymbol{B}$ equals 1. The E inequalities, suggested as plausibie by Truesdell (Eqn. (41.14) of [3]) in 1952, are

$$f_1>0\,, \quad f_{-1}\leqslant 0\,. \tag{12}$$

Proceeding as we did before, we obtain (4), $(6)_1$ and (7) and, in view of (12), we conclude that for incompressible isotropic elastic materials $\boldsymbol{B}$ should have the form given by (9) except that α now equals $v^{-\frac{3}{2}}$ since the determinant of $\boldsymbol{B}$ must be 1.

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