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A Closed-form Solution For The Radiosity At The Edge Of A Rectangular Cavity

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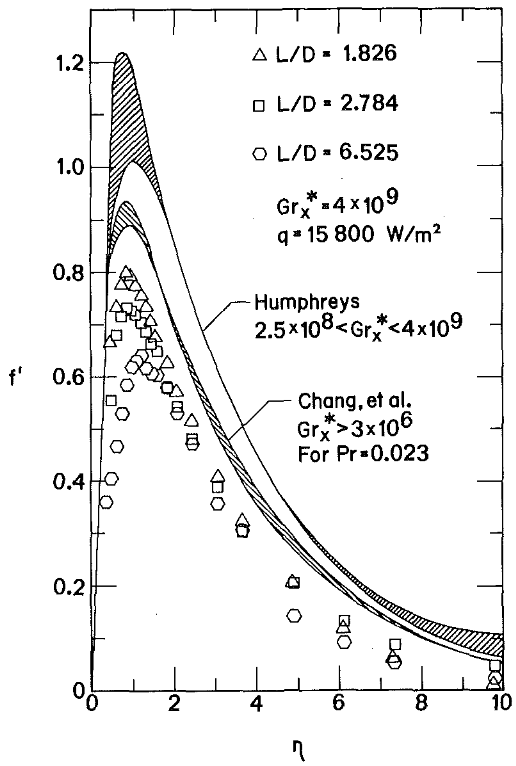

Fig. 4 Dimensionless velocity profiles, highest heat flux

and Chang, et al., observed a "crossing over" of their profiles, i.e., the Grashof number resulting in highest peak velocity yielded the lowest velocity far from the heated wall. Some evidence of this tendency is observable in these figures, particularly at $Gr_x^* = 3 \times 10^6$.

In each figure velocities adjacent to cylindrical surfaces are below those for a plane wall, the difference being less for increased values of Gr_x^*.

Experimental profiles display similar trends but do not coincide as would be true if the similarity parameters, η and f', were sufficient to describe the flow behavior. Definite variation is apparent with the amount of curvature and with Grashof number, variations with Gr_x^* due principally to a varying heat flux level. Since all measurements were accomplished in mercury no Prandtl number effect was investigated; the Prandtl number of mercury (Pr = 0.023) is representative of the liquid metals.

Conclusions that may be reached regarding natural convection in low Prandtl number fluids adjacent to heated vertical cylinders are the following:

1 Velocities adjacent to curved surfaces are below analytical results for flat plates, $L/D = 0$.

2 Velocity data, when reduced to f' versus η form, do not exhibit similarity. Differences are present both with heat flux level and with cylinder diameter.

3 Velocity profiles indicate the hydrodynamic boundary layer thickness to be only slightly affected by heat flux level or cylinder diameter.

4 The principal effect of heat flux and cylinder diameter is on magnitude and location of the velocity peak.

5 With increased heat flux the peak velocity increases and occurs nearer to the heated surface.

6 With decreasing diameter the peak velocity decreases in magnitude and occurs further away from the heated surface.

Indications from this work are that, with a given cylindrical surface oriented vertically, an increase in wall heat flux causes flow rates to increase in the near vicinity of the surface without appreciable effect on the extent of the flow field. This region is also where higher fluid temperatures exist. Thus, an increase in wall heat flux causes a significant increase in flow rate with an increase in energy concentration in the near-surface fluid layers.

Acknowledgment

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A Closed-Form Solution for the Radiosity at the Edge of a Rectangular Cavity

A. L. Crosbie[1]

Recently, Crosbie and Sawheny [1, 2][2] have applied Ambarzumian's method to the following integral equation describing the radiosity in a rectangular cavity:

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[2] Numbers in brackets designate References at end of technical note.

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$$B(x, m) = e^{-mx} + \frac{\rho}{2} \int_0^\infty B(y, m) K(|x - y|) dy \quad (1)$$

with kernel

$$K(x) = \int_0^\infty n e^{-nx} J_1(n) dn = (1 + x^2)^{-3/2} \quad (2)$$

where $J_1(n)$ is the Bessel function of order one. The depth into the cavity is denoted by x. $\rho = 1$ corresponds to a cavity subject to exponentially decaying wall heat flux, i.e., $q_w(x) = q_0 \exp(-mx)$ [1], while $\rho < 1$ corresponds to a cavity subject to an exponentially decaying wall emissive power, i.e., $\sigma T_w{}^4(x) = \sigma T_0{}^4 \exp(-mx)$ [2]. The radiosity at the edge of the cavity satisfies either of the following two nonlinear integral equations:

$$B(0, m) = 1 + \frac{\rho}{2} B(0, m) \int_0^\infty \frac{n J_1(n) B(0,n)}{n + m} dn \quad (3)$$

$$\frac{}{B(0,m)} = \sqrt{1 - \rho} + \frac{\rho}{2} \int_0^\infty \frac{m J_1(n) B(0,n)}{n + m} dn \quad (4)$$

Inspection of equation (4) reveals that for $m = 0$

$$B(0, 0) = (1 - \rho)^{-1/2} \quad (5)$$

In the study of noncoherent scattering [3, p. 205] in a semiinfinite medium, the following integral equation for the source function arises:

$$S(\tau, z) = e^{-\tau/z} + \frac{\lambda}{2} \int_0^\infty S(t, z) K_1(|\tau - t|) dt \quad (6)$$

with kernel

$$K_1(\tau) = \int_0^\infty e^{-\tau/z'} G(z') dz'/z' \quad (7)$$

where τ is optical depth into the medium and λ is the albedo. The source function at the boundary, $S(0, z) = H(z)$ satisfies the following nonlinear integral equation [3, p. 212]:

$$H(z) = 1 + \frac{\lambda}{2} z H(z) \int_0^\infty \frac{H(z') G(z')}{z + z'} dz' \quad (8)$$

This equation can be solved [3, p. 216], i.e.,

$$H(z) = \exp\left\{-\frac{z}{\pi} \int_0^\infty \frac{\ln[1 - \lambda V(u)]}{1 + z^2 u^2} du\right\} \quad (9)$$

where

$$V(u) = \int_0^\infty \frac{G(z)}{1 + u^2 z^2} dz \quad (10)$$

Comparison of equations (1)–(3) with (6)–(8) reveals that $\lambda = \rho$, $z = 1/m$, $G(z) = J_1(1/z)/z^2$, and $B(0, m) = H(1/m)$. Thus,

$$V(u) = \int_0^\infty \frac{J_1(1/z)}{1 + u^2 z^2} \frac{dz}{z^2} = \int_0^\infty \frac{m^2 J_1(m)}{m^2 + u^2} dm = u K_1(u) \quad (11)$$

where $K_1(u)$ is the modified Bessel function of order one. The last integral was obtained from reference [4]; therefore, the radiosity at the edge of the cavity is given by

$$B(0, m) = \exp\left\{-\frac{m}{\pi} \int_0^\infty \frac{\ln[1 - \rho u K_1(u)]}{m^2 + u^2} du\right\} \quad (12)$$

This closed-form expression was evaluated numerically for a wide range of m and ρ values. The results agreed with the previous calculated values [1, 2] which were obtained from equation (4) by iteration.

Equation (12) is more suitable than equation (4) for investigating the nature of the edge singularity in the wall heat flux case ($\rho = 1$, $m \to 0$). When m is small, the main contribution to the integral in equation (12) occurs at small u. For small u

$$u K_1(u) \simeq 1 + \frac{1}{2} u^2 \ln \frac{u}{2} + \frac{1}{4} u^2 (2\gamma - 1) \quad (13)$$

with $\gamma = 0.5772156649\ldots$ Substitution of this approximation into equation (12) yields

$$B(0, m) \simeq \exp\left\{-\frac{m}{\pi} \int_0^\infty \ln\left[(1 - \rho) - \frac{1}{2} \rho u^2 \ln \frac{u}{2} - \frac{1}{4} \rho u^2 (2\gamma - 1)\right] \frac{du}{m^2 + u^2}\right\} \quad (14)$$

When $\rho < 1$ only the first term in equation (13) is required to yield equation (5). However, for the special case of $\rho = 1$, the next term is important. For this case the transformation $x = u/m$ yields

$$B(0, m) \simeq \exp\left\{-\frac{1}{\pi} \int_0^{2/m} \ln\left[-\frac{1}{2} m^2 x^2 \ln \frac{mx}{2} - \frac{1}{4} m^2 x^2 (2\gamma - 1)\right] \frac{dx}{1 + x^2}\right\} \quad (15)$$

or

$$B(0, m) \simeq \exp\left\{-\frac{1}{\pi} \int_0^{2/m} \left[2 \ln x + \ln\left(\frac{1}{2} \Delta m^2\right) + \ln\left(1 - \frac{\ln x}{\Delta}\right)\right] \frac{dx}{1 + x^2}\right\} \quad (16)$$

with

$$\Delta = \ln(2/m) - \frac{1}{2}(2\gamma - 1)$$

Utilization of expansion

$$\ln(1 - \epsilon) = -\epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{3}\epsilon^3 - \ldots \quad (17)$$

in the integrand of equation (16) gives

$$B(0, m) \simeq \exp\left\{-\frac{2}{\pi} \int_0^{2/m} \frac{\ln x}{1 + x^2} dx - \frac{1}{\pi} \ln\left(\frac{1}{2} m^2 \Delta\right) \int_0^{2/m} \frac{dx}{1 + x^2} + \frac{1}{\pi \Delta} \int_0^{2/m} \frac{\ln x}{1 + x^2} dx + \frac{1}{2\pi \Delta^2} \int_0^{2/m} \frac{(\ln x)^2}{1 + x^2} dx + \ldots\right\} \quad (18)$$

Letting m become very small and introducing the following integrals:

$$\int_0^\infty \frac{\ln x}{1 + x^2} dx = 0, \int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}, \text{ and } \int_0^\infty \frac{(\ln x)^2}{1 + x^2} dx = \frac{\pi^3}{8} \quad (19)$$

into equation (18) yields

$$B(0, m) \simeq \exp\left[-\frac{1}{2} \ln\left(\frac{1}{2} m^2 \Delta\right) + \frac{\pi^2}{16\Delta^2}\right] \quad (20)$$

or

$$B(0, m) \simeq \frac{\sqrt{2}}{m\sqrt{\Delta}} \exp(\pi^2/16\Delta^2) \quad (21)$$

Table 1 Comparison of various approximations of $B(0,m)$ with exact results for $\rho = 1$

m	$B(0,m)$	$\frac{\sqrt{2}}{m\sqrt{\Delta}} \exp\left(\frac{\pi^2}{16\Delta^2}\right)$	$\frac{\sqrt{2}}{m\sqrt{\Delta}}$	$\frac{\sqrt{2}}{m\sqrt{\ln(2/m)}}$
1.0	2.03112	9.16015	1.80198	1.69864
0.5	2.94387	3.54314	2.47208	2.40224
0.2	5.38909	5.36883	4.74006	4.65991
0.1	9.07591	8.89990	8.27816	8.17078
0.01	63.7895	63.3084	61.8919	61.4393
0.001	522.220	521.233	515.584	512.959
0.0001	4542.95	4540.41	4511.50	4493.87
0.00001	40786.7	40778.0	40607.4	40478.7

When m is very small, $\Delta = \ln(2/m)$ and equation (21) becomes

$$B(0, m) = \sqrt{2}/[m\sqrt{\ln(2/m)}] \tag{22}$$

Physically, equation (22) means the temperature at the edge of the cavity is inversely proportional to $[m \ln(2/m)]^{1/4}$. This behavior is somewhat unexpected since the overall heat transfer from the cavity is inversely proportional to m.

Approximations (21) and (22) are compared to the exact numerical results in Table 1. These approximations yield imaginary numbers when $m > 2$ and thus are not included. This behavior is due to the truncation of the series for $uK_1(u)$ to three terms. Inspection of Table 1 reveals that equation (21) is a more accurate approximation than equation (22) except when $m > 0.5$. For the range of m values reported in Table 1, approximation (22) underestimates the radiosity.

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Critical Thickness of Insulation Accounting for Variable Convection Coefficient and Radiation Loss

L. D. Simmons[1]

Nomenclature

F = shape factor, fraction of energy leaving the outside surface of the insulation which is incident on the portion of the environment having temperature T_∞

h = mean convection heat transfer coefficient for the outside surface

K = coefficient in the expression for variable convection heat transfer coefficient

k = thermal conductivity of the insulation

m = exponent of r_0 in the expression for variable convection heat transfer coefficient

n = exponent of $(T_0 - T_\infty)$ in the expression for variable convection heat transfer coefficient

q' = heat transfer rate per unit length of tube, wire, or cable

r = radius of cylindrical insulation; r_i and r_0 are inside radius and outside radius, respectively; $r_{0_{crit}}$ is the outside radius which maximizes q'

T = absolute temperature; T_i, T_0, and T_∞ are inside surface temperature, outside surface temperature, and temperature of the environment, respectively

ϵ = total hemispherical emissivity of the outside surface of the insulation

σ = Stefan-Boltzmann constant

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Introduction

If the outside surface of a small diameter tube, wire, or cable has approximately constant temperature, there is a critical thickness of insulation which maximizes energy loss by heat transfer. This critical thickness of coating can be utilized when it is desired to cool the cylindrical tube, wire, or cable as effectively as possible. It is easily shown [1][2] that, if the insulation (with thermal conductivity k) around a long, slender cylinder has its inside radius r_i and its inside surface temperature T_i fixed, has an outside convection coefficient h which can be considered constant, and has negligible energy loss to its environment by radiation, the energy loss rate-per-unit length q' will be maximum when

$$r_0 = \frac{k}{h} \tag{1}$$

This maximizing radius is usually referred to as the critical radius $r_{0_{crit}}$ and $(r_{0_{crit}} - r_i)$ is referred to as the critical thickness of insulation. McAdams has shown [2] that if radiation is included in linearized form such that the radiation from the surface is given by $h_r A(T_0 - T_\infty)$, then the critical radius is given by

$$r_{0_{crit}} = \frac{k}{h + h_r}$$

However, both h (for convection) and h_r (for radiation) were considered constant, not varying with r_0 or T_0. Sparrow has shown [3] that if the variation of h with r_0 and outside surface temperature T_0 is considered in the form

$$h = K(T_0 - T_\infty)^n/r_0^m \tag{2}$$

and radiation neglected, then the critical radius is given by

$$r_{0_{crit}} = \left(\frac{1-m}{1+n}\right)\frac{k}{h} \tag{3}$$

Note, however, that now h on the right-hand side of equation (3) is a function of r_0 and T_0, which itself is a function of r_0, and so the value of $r_{0_{crit}}$ must be found by a trial-and-error or iterative procedure. In all three cases given in the foregoing the resulting expressions for $r_{0_{crit}}$ are simple in form (although not necessarily easily solved). However, none of the three could be expected to correctly predict the true critical radius because of simplifying assumptions. The purpose here is to develop an expression for $r_{0_{crit}}$ for insulation of cylinders with the variation of radiation and h with r_0 and T_0 accounted.

Analysis

The energy loss by conduction through the insulation is given by

$$q' = \frac{2\pi k(T_i - T_0)}{\ln(r_0/r_i)} = f[r_0, T_0(r_0)] \tag{4}$$

and the loss rate by convection and radiation from the surface is given by

$$q' = 2\pi h\, r_0(T_0 - T_\infty) + 2\pi r_0 \epsilon F\sigma(T_0^4 - T_\infty^4) \tag{5}$$

where h is given by equation (2). At steady state the energy loss rates will be equal so that

$$\frac{k(T_i - T_0)}{\ln(r_0/r_i)} = hr_0(T_0 - T_\infty) + r_0\epsilon F\sigma(T_0^4 - T_\infty^4) \tag{6}$$

Now if dq'/dr_0 is found from equations (2), (4), and (6) by using chain rule and the result set to zero, the critical radius is found to be

$$r_{0_{crit}} = \frac{k[h(1-m)(T_0 - T_\infty) + \epsilon F\sigma(T_0^4 - T_\infty^4)]}{[(n+1)h + 4\epsilon F\sigma T_0^3][h(T_0 - T_\infty) + \epsilon F\sigma(T_0^4 - T_\infty^4)]} \tag{7}$$

[2] Numbers in brackets designate References at end of technical note.