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A NOTE ON T_1 TOPOLOGIES

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ABSTRACT. Let t be a T_1 topology for a set X. The problem of representing t as the lattice product (intersection) of stronger topologies is considered.

It is well known that the set of all topologies on an infinite set X is a complete lattice with partial order $\geq$ defined as stronger than. Also, the family of T_1 topologies on X is a complete sublattice of X. In [1], R. Lalitha proved that: (1) If t is a T_1 topology for X, then t is the lattice product (intersection) of all stronger T_2 topologies on X. (2) If t satisfies the first axiom of countability, it is the intersection of all stronger metric topologies on X. In this note, we give a stronger theorem with a much shorter proof. Also, several new results follow from the stronger theorem. Nagata's book [2] is a general reference for notation and terminology.

Theorem 1. *If X is an infinite set and t is a T_1 topology for X, then for every subset E of X such that $E \notin t$ there exists a fully normal, completely normal, disconnected topology t_E on X with $t \leq t_E$ and $E \notin t_E$. Also, if t is first countable, t_E is metrizable.*

Proof. Given $E \notin t$, choose $p \in E \cap \mathrm{cl}(X - E)$. Let $\mathfrak{N}(p)$ be an open base at p. For each $N \in \mathfrak{N}(p)$, set

$$V_N = \Delta \cup [\{p\} \cup (N \cap X - E)] \times [\{p\} \cup (N \cap X - E)]$$

where $\Delta = \{(x, x): x \in X\}$. Note that $N \cap (X - E) \neq \emptyset$ and

$$V_N[p] = \{p\} \cup (N \cap X - E) \subset N.$$

It is easy to see that $\Delta \subset V_N, V_N = V_N^{-1}$, $V_N \cap V_M = V_{N \cap M}$, and $V_N \circ V_N = V_N$. Hence $\mathfrak{B} = \{V_N: N \in \mathfrak{N}(p)\}$ is a base for a uniform structure $\mathfrak{U}$. t_E is the induced uniform topology.

(1) If $A \subset X$ and $p \notin A$, then $A \in t_E$. If $x \in A$, $x \neq p$ and t is T_1 implies there exists $N \in \mathfrak{N}(p)$ such that $x \notin N$. Then $\{x\} = V_N[x] \in t_E$.

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(2) The topology t_E is Hausdorff. If $x \neq y$, we may suppose $y \neq p$. Since t is T_1, there exists $N \in \mathfrak{N}(p)$ such that $y \notin N$. Observe that $\{y\} \cap V_N[x] = \emptyset$.

(3) $E \notin t_E$. This follows since $p \in E$ and $V_N[p] = \{p\} \cup (N \cap X - E) \not\subset E$ for $N \in \mathfrak{N}(p)$.

(4) $t \leq t_E$. Suppose $x \in A \in t$. If $x \neq p$, then, by (1), $\{x\} \in t_E$. If $x = p$, there exists $N \in \mathfrak{N}(p)$ such that $N \subset A$. Now $V_N[p] \subset N \subset A$. Thus $A \in t_E$.

(5) t_E is completely normal. Suppose $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. If $p \notin A \cup B$, then, by (1), $A \in t_E$ and $B \in t_E$. Clearly, $A \cap B = \emptyset$. If $p \in A \cup B$, suppose $p \in A$. Then $p \notin B$ implies $B \in t_E$. $X - \overline{B} \in t_E$, $A \subset X - \overline{B}$, and $B \cap X - \overline{B} = \emptyset$.

(6) t_E is fully normal. Let $\mathfrak{U} = \{O_\alpha : \alpha \in I\}$ be an open cover of X. $p \in O_{\alpha'}$ for some $\alpha' \in I$. $X = O_{\alpha'} \cup (\bigcup_{x \neq p} \{x\})$. Denote this open cover by $\mathfrak{B}$. $\mathfrak{B}^{\Delta} < \mathfrak{U}$. For, if $x \in X$ and $x \neq p$, then $S(x, \mathfrak{B}) = O_{\alpha'}$ if $x \in O_{\alpha'}$ and $S(x, \mathfrak{B}) = \{x\}$ if $x \notin O_{\alpha'}$. Also, $S(p, \mathfrak{B}) = O_{\alpha'}$.

(7) t_E is disconnected. Since t is T_1, there exists $N \in \mathfrak{N}(p)$ with $N \neq X$. $V_N[p] \subset N$ implies there exists $O \in t_E$ with $p \in O \neq X$. $p \in O$ implies $\overline{O} = O$.

(8) t is first countable implies t_E is metrizable. In this case, we choose $\mathfrak{N}(p)$ so it is countable and clearly the uniform structure $\mathfrak{U}$ has a countable base. Thus t_E is metrizable.

Corollary 1. *If t is a T_1 topology on an infinite set, then $t = \bigcap\{u : u$ is T_2 and $t \leq u\} = \bigcap\{u : u$ is regular and $t \leq u\} = \bigcap\{u : u$ is normal and $t \leq u\} = \bigcap\{u : u$ is fully normal, completely normal, disconnected, and $t \leq u\}$.*

Corollary 2. *If t is the weakest T_1 topology on an infinite set, then $t = \bigcap\{u : u$ is $T_1\} = \bigcap\{u : u$ is $T_2\} = \bigcap\{u : u$ is regular$\} = \bigcap\{u : u$ is fully normal, completely normal, and disconnected$\}$.*

The reader can observe other consequences of the theorem. For example, if t is a T_2 topology on an infinite set, then $t = \bigcap\{u : u$ is regular and $t \leq u\}$. Actually, the topology t_E in the theorem has other properties. We give one in the following

Lemma. *The uniform structure $\mathfrak{U}$ that gives t_E is complete.*

Proof. Suppose $\mathfrak{F}$ is a $\mathfrak{U}$-Cauchy filter and let $N \in \mathfrak{N}(p)$. Then there exists $F \in \mathfrak{F}$ such that $F \times F \subset V_N$. $F \neq \emptyset$. If $F = \{x_0\}$, $\mathfrak{F}$ converges to x_0. If $F \neq \{x_0\}$, $F \supset \{a, b\}$ where $a \neq b$, and we show that $F \subset N$. Then

$N \in \mathcal{F}$ and $\mathcal{F}$ converges to p. If $c \in F$ and $c \neq p$, choose $d \in F$ such that $c \neq d$. Then $(c, d) \in F \times F \subset V_N$ implies $c \in N \cap X - E \subset N$.

The construction, given in the proof of Theorem 1, involves only elementary concepts. Also, it is useful in constructing examples. We illustrate this fact.

Example. X is the unit interval $[0, 1]$, t is the topology of finite complements, and $E = \{O\} \notin t$. $p = O \in E \cap \mathrm{cl}(X - E)$. If $O \in N \in t$

$$V_N[O] = \{O\} \cup (N \cap X - \{O\}) = N.$$

Thus the t_E open sets containing O are the t open sets containing O and

$$t_E = \{\emptyset, X\} \cup \{A: O \notin A\} \cup \{B: O \in B \text{ and } X - B \text{ is finite}\}.$$

(1) From the proof of Theorem 1, t_E is uniformizable, completely normal, fully normal, and disconnected. It is clearly compact.

(2) We show that t_E is not pseudo-metrizable and also not perfectly normal. If it were pseudo-metrizable it would be metrizable since t_E is T_2. Also, we could write $\{O\} = \bigcap_{n=1}^{\infty} A_n$ where $O \in A_n \in t_E$. Then $X - \{O\} = \bigcup_{n=1}^{\infty}(X - A_n)$ would be countable which is impossible. Since $X - \{O\} \in t_E$, $\{O\}$ is closed and the above argument shows that t_E is not perfectly normal.

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