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Univalence of Derivatives of Functions Defined by Gap Power Series. II

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1. INTRODUCTION

In this paper, we continue our work in [9] on the univalence of derivatives of functions defined by gap power series. We consider regular functions, f, defined in some region containing 0, and denote by ρ_n and $\rho_n(c)$ the radius of univalence and the radius of convexity, respectively, of $f^{(n)}$. For ease of notation, we shall sometimes write $\rho_n = \rho(n)$.

Earlier, we considered functions defined by power series with gaps of at least a certain length. By that we meant the following: Suppose $F(z) = \sum_{j=0}^{\infty} A_j z^j$. Then F is defined by a power series with gaps at least of length k provided that k is a nonnegative integer such that, if $A_n \neq 0$ for some n, then $A_{n+j} = 0$ for $1 \leqslant j \leqslant k$. We proved the following [9].

THEOREM A. *There is a strictly-increasing sequence, $\{x_k\}_{k=1}^{\infty}$, of positive numbers with the following properties: Let f be defined in a disc about 0 by a power series with gaps at least of length $k - 1$ and with a radius of convergence, $R > 0$. Assume that f is not a polynomial. Then*

$$Rx_k \leqslant \limsup_{n\to\infty} n\rho_n .$$

If $R = \infty$, then

$$x_k/\delta \leqslant \limsup_{n\to\infty} \rho_n .$$

Further,

$$(k! \log 2)^{1/k} \leqslant x_k < (k!)^{1/k}.$$

Here, $\delta = \liminf_{r\to\infty} \nu(r)/r$, where $\nu(r)$ is the central index of f. This result extends theorems of G. A. Read and V. G. Iyer (see [9]).

In this paper, we investigate functions with gaps no greater than a certain length. By this, we mean the following: Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^{\lambda_j}, \tag{1.1}$$

where $\{\lambda_j\}_{j=0}^{\infty}$ is a strictly-increasing sequence of nonnegative integers and where $a_j \neq 0$ for all j. Let

$$k = \limsup_{j\to\infty}(\lambda_{j+1} - \lambda_j).$$

Then we say that f is defined by a power series with gaps no greater than $k - 1$. If $k = \infty$, we mean that f is defined by a power series with unbounded gaps. (Actually, if $k < \infty$, such an f may have gaps greater than $k - 1$; but only a finite number of these gaps can occur. We note also that in the earlier definition of "gaps at least of length k", it would have been possible to use the requirement that $k + 1 \leqslant \liminf_{j\to\infty}(\lambda_{j+1} - \lambda_j)$, where f would now be defined by (1.1). Theorem A would remain unchanged.)

If f is defined by (1.1), we define

$$\left.\begin{aligned} \bar{R} &= \limsup_{j\to\infty} \\ \underline{R} &= \liminf_{j\to\infty} \end{aligned}\right\} \, | a_j/a_{j+1} |^{1/(\lambda_{j+1}-\lambda_j)}.$$

If R is the radius of convergence of the power series defining f, then $\underline{R} \leqslant R \leqslant \bar{R}$ [2, p. 422].

We prove the following theorems:

THEOREM 1. *Let f be defined by* (1.1) *and have gaps no greater than $k - 1$. Then, for each integer k such that* $1 \leqslant k < \infty$,

$$y_k \underline{R} \leqslant \limsup_{n\to\infty} n\rho_n \leqslant (1 + \epsilon_k)\,(k!)^{1/k}\,\bar{R}, \tag{1.2}$$

where $\{y_k\}_{k=1}^{\infty}$ is a strictly-increasing sequence such that

$$(.58k!)^{1/k} < y_k < (.6568k!)^{1/k}$$

for $k \geqslant 2$, and where $\{\epsilon_k\}_{k=1}^{\infty}$ is a nonincreasing sequence of positive numbers such that $\lim_{k\to\infty} \epsilon_k = 0$. *Further, neither $\{y_k\}_{k=1}^{\infty}$ nor $\{\epsilon_k\}_{k=1}^{\infty}$ depends on f. If $k = \infty$ and $\underline{R} > 0$, then* $\limsup_{n\to\infty} n\rho_n = \infty$.

THEOREM 2. *Let f, k, $\{y_k\}_{k=1}^{\infty}$, and $\{\epsilon_k\}_{k=1}^{\infty}$ be as in Theorem* 1. *Suppose* *that* $\{| a_j/a_{j+1} |^{1/(\lambda_{j+1}-\lambda_j)}\}_{j=1}^{\infty}$ *is ultimately nondecreasing and tends to* ∞. *Then* f *is entire. If* $k < \infty$, *then*

$$y_k/\gamma \leqslant \limsup_{n\to\infty} \rho_n \leqslant (1 + \epsilon_k)(k!)^{1/k}/\delta. \tag{1.3}$$

If $k = \infty$ *and* f *is of exponential type, then* $\limsup_{n\to\infty} \rho_n = \infty$. *Here,* δ *is as in Theorem* A *and* $\gamma = \limsup_{r\to\infty} \nu(r)/r$.

THEOREM 3. *Let* f *be defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{A}$$

and have radius of convergence, R, *where* $0 < R < \infty$. *Suppose that* $\lim_{n\to\infty} | a_n/a_{n+1} |$ *exists. Then*

$$\limsup_{n\to\infty} n\rho_n \leqslant 2\sqrt{3}\, R \tag{1.4}$$

and

$$\limsup_{n\to\infty} n\rho_n(c) \leqslant \sqrt{2}\, R. \tag{1.5}$$

THEOREM 4. *Let* f, *as defined by* (A), *be entire, and suppose that*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}a_{n-1}}{a_n^{\,2}} \right| \leqslant 1. \tag{1.6}$$

Let

$$\left.\begin{matrix}\gamma^* \\ \delta^*\end{matrix}\right\} = \lim_{n\to\infty} \left\{\begin{matrix}\sup \\ \inf\end{matrix}\right\} n \left| \frac{a_n}{a_{n-1}} \right|.$$

Then

$$\lim_{n\to\infty} \left\{\begin{matrix}\sup \\ \inf\end{matrix}\right\} \rho_n \leqslant \left\{\begin{matrix}2\sqrt{3}/\delta^* \\ 2\sqrt{3}/\gamma^*\end{matrix}\right. \tag{1.7}$$

and

$$\lim_{n\to\infty} \left\{\begin{matrix}\sup \\ \inf\end{matrix}\right\} \rho_n(c) \leqslant \left\{\begin{matrix}\sqrt{2}/\delta^* \\ \sqrt{2}/\gamma^*.\end{matrix}\right. \tag{1.8}$$

COROLLARY. *If* $\{| a_n/a_{n+1} |\}_{n=1}^{\infty}$ *is ultimately a nondecreasing sequence, then*

$$\lim_{n\to\infty} \left\{\begin{matrix}\sup \\ \inf\end{matrix}\right\} \rho_n \leqslant \left\{\begin{matrix}2\sqrt{3}/\delta \\ 2\sqrt{3}/\gamma\end{matrix}\right. \tag{1.9}$$

and

$$\lim_{n\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \rho_n(c) \leqslant \begin{Bmatrix}\sqrt{2}/\delta\\ \sqrt{2}/\gamma.\end{Bmatrix} \tag{1.10}$$

Remarks. (i) Theorem 4 improves the corresponding results of Theorem 3 in [8]. The hypothesis (1.6) is less restrictive than the hypothesis there. Further, the numerical constants in the conclusion of Theorem 4 are better. The bounds in (1.7) involve the coefficients of f and not the growth-measuring constants, γ and δ, which are related to the rank.

(ii) Suppose that f is entire. Let $T = \limsup_{r\to\infty} \log M(r)/r$ and $t = \liminf_{r\to\infty} \log M(r)/r$. Then $\delta \leqslant t$ and $T \leqslant \gamma \leqslant eT$ [7]. Further, if $\delta > 0$, $t \leqslant \delta(1 + \log \gamma/\delta)$ [7]. This allows us to put (1.3) in terms of t and T if we desire.

(iii) Theorems 1 and 2 give better results, for power series with "sparse" gaps, than our earlier work. For example, let

$$f(z) = 1 + z + \cdots + \frac{z^{10}}{10!} + \frac{z^{11}}{11!} + \cdots + \frac{z^{10^2}}{10!^2} + \frac{z^{10^2+2}}{(10^2+2)!} + \cdots$$
$$+ \frac{z^{10^j}}{(10^j)!} + \frac{z^{10^j+j}}{(10^j+j)!} + \cdots.$$

Then f is of order 1 and type 1 (see [1, p. 11]). Further, $\gamma = \delta = 1$. The results in [9] do not apply at all to f, and the results in [8] show only that $\log 2 \leqslant \limsup_{n\to\infty} \rho_n$. However, Theorem 2 shows that $\limsup_{n\to\infty} \rho_n = \infty$.

2. Proofs

The proofs of Theorems 1 and 2 require some lemmas.

Lemma 1. *There is a unique, strictly-increasing sequence,* $\{y_k\}_{k=1}^{\infty}$, *of positive numbers with the following properties:*

(1) *For each* k,

$$1 = \sum_{j=k}^{\infty} \frac{y_k^j}{j!}. \tag{2.1}$$

Further,

$$\lim_{k\to\infty} y_k^k/k! = 1 - 1/e. \tag{2.2}$$

(2) *Let* k *be a positive integer. If* $\{\lambda_m\}_{m=1}^{\infty}$ *is a strictly-increasing sequence*

of positive integers, then there is a unique, strictly-increasing sequence, $\{y_{m,k}\}_{m=1}^{\infty}$, *of positive numbers with the following properties:*

(A) $1 = \sum_{j=0}^{\infty} \frac{(\lambda_m + k + j)!}{\lambda_m!(k+j)!} \left(\frac{y_{m,k}}{\lambda_m}\right)^{k+j}.$

(B) $\lim_{m\to\infty} y_{m,k} = y_k$.

Proof. Let

$$\phi_k(x) = \sum_{j=k}^{\infty} \frac{x^j}{j!}.$$

Then ϕ_k is defined for all x, ϕ_k is strictly-increasing for $x > 0$, $\phi_k(0) = 0$, $\lim_{x\to\infty} \phi_k(x) = \infty$, and $\phi_{k+1}(x) < \phi_k(x)$ for $x > 0$. It follows that there is a unique, strictly-increasing sequence, $\{y_k\}_{k=1}^{\infty}$, of positive numbers that satisfy (2.1).

It is immediate that $y_k^k < k!$ Hence,

$$1 = \frac{y_k^k}{k!} \sum_{j=k}^{\infty} \frac{k!\, y_k^{j-k}}{j!} < \frac{y_k^k}{k!} \sum_{j=0}^{\infty} \left(\frac{y_k}{k}\right)^j,$$

on, using this and the first remark,

$$1 - \frac{y_k}{k} < \frac{y_k^k}{k!} < 1. \tag{2.3}$$

Recalling that Stirling's formula implies that $(k!)^{1/k} \sim k/e$, we conclude that

$$\lim_{k\to\infty} y_k/k = 1/e. \tag{2.4}$$

If n is a positive integer, then

$$1 > \frac{y_k^k}{k!}\left[1 + \frac{y_k}{k+1} + \cdots + \frac{y_k^n}{(k+1)(k+2)\cdots(k+n)}\right].$$

From this and (2.4), it follows that

$$1 \geqslant \left(1 + \frac{1}{e} + \cdots + \frac{1}{e^n}\right) \limsup_{k\to\infty} \frac{y_k^k}{k!},$$

or,

$$\left(1 - \frac{1}{e}\right)\Big/\left(1 - \frac{1}{e^{n+1}}\right) \geqslant \limsup_{k\to\infty} \frac{y_k^k}{k!}.$$

This, (2.4), and (2.3) imply (2.2).

Now let k be a positive integer, and suppose that $\{\lambda_m\}_{m=1}^{\infty}$ is a strictly-increasing sequence of positive integers. Let

$$\phi_{m,k}(x) = \sum_{j=0}^{\infty} \frac{(\lambda_m + k + j)!}{\lambda_m!\,(k+j)!} \left(\frac{x}{\lambda_m}\right)^{k+j}.$$

Note that, if $0 \leqslant x < \lambda_m$, then $\phi_{m,k}$ can be rewritten as

$$\phi_{m,k}(x) = \frac{1}{\left(1 - \dfrac{x}{\lambda_m}\right)^{\lambda_m+1}} - \sum_{j=0}^{k-1} \frac{(\lambda_m + j)!}{\lambda_m!\, j!} \left(\frac{x}{\lambda_m}\right)^{j}. \tag{2.5}$$

From these two expressions, it follows that $\phi_{m,k}$ is strictly-increasing on $[0, \lambda_m)$, $\phi_{m,k}(0) = 0$, and $\lim_{x\to\lambda_m^-} \phi_{m,k}(x) = \infty$. Hence, there is a unique positive number $y_{m,k}$, in $(0, \lambda_m)$ such that $\phi_{m,k}(y_{m,k}) = 1$.

Since,

$$\frac{(\lambda_m + k + j)!}{\lambda_m!\,\lambda_m^{k+j}} = \left(1 + \frac{1}{\lambda_m}\right)\left(1 + \frac{2}{\lambda_m}\right)\cdots\left(1 + \frac{k+j}{\lambda_m}\right),$$

it follows that, if $0 < x < \lambda_m$, then $\phi_{m+1,k}(x) < \phi_{m,k}(x)$. This means that $y_{m+1,k} > y_{m,k}$. Clearly,

$$\frac{(\lambda_m + k)!}{\lambda_m!\, k!} \left(\frac{y_{m,k}}{\lambda_m}\right)^{k} < 1,$$

or,

$$y_{m,k} < (k!)^{1/k} \left[\frac{\lambda_m!\,\lambda_m{}^{k}}{(\lambda_m + k)!}\right]^{1/k} < (k!)^{1/k}.$$

Hence, $\lim_{m\to\infty} y_{m,k}$ exists. Call it y_k'. From (2.5), it follows that

$$y_{m,k} = \lambda_m \left[1 - \exp\left\{\frac{-1}{\lambda_m + 1} \log\left[1 + \sum_{j=0}^{k-1} \frac{(\lambda_m + j)!}{\lambda_m!\, j!} \left(\frac{y_{m,k}}{\lambda_m}\right)^{j}\right]\right\}\right]$$

$$= \frac{\lambda_m}{\lambda_m + 1} \log\left[1 + \sum_{j=0}^{k-1} \frac{(\lambda_m + j)!}{\lambda_m!\, j!} \left(\frac{y_{m,k}}{\lambda_m}\right)^{j}\right] + O(1/\lambda_m).$$

Noting that

$$\lim_{m\to\infty} \frac{(\lambda_m + j)!}{\lambda_m!\,\lambda_m{}^{j}} = 1,$$

it follows that

$$y_k' = \log\left[1 + \sum_{j=0}^{k-1} \frac{(y_k')^{j}}{j!}\right].$$

This is equivalent to $\phi_k(y_k') = 1$. Hence, $y_k' = y_k$. The lemma is proved. Numerical estimates on the y_k are given in the third section.

LEMMA 2. *Let f be defined by* (1.1) *and suppose that* $\{| a_j/a_{j+1} |^{1/(\lambda_{j+1}-\lambda_j)}\}_{j=1}^{\infty}$ *is ultimately nondecreasing. Then*

$$\gamma = \limsup_{j\to\infty} \lambda_j/| a_j/a_{j+1} |^{1/(\lambda_{j+1}-\lambda_j)}$$

and

$$\delta = \liminf_{j\to\infty} \lambda_j/| a_j/a_{j+1} |^{1/(\lambda_{j+1}-\lambda_j)}.$$

The proof of this is essentially the same as that given in [6, p. 85, #148] and so is omitted. The proof of the next lemma is in [3, p. 494].

LEMMA 3. *Let* $f(z) = z + a_k z^k + a_{k+1} z^{k+1} + \cdots$. *Suppose that f is univalent in* $| z | < 1$. *Then* $| a_k | \leqslant 2/(k-1)$.

LEMMA 4. *Let* $f(z) = z + a_2 z^2 + \cdots$. *Suppose that* $\sum_{j=2}^{\infty} j \, | a_j | \leqslant 1$. *Then f is univalent in* $| z | < 1$.

A proof of this is in [4].

Proof of Theorem 1. Suppose $k < \infty$. For $m \geqslant 1$, let

$$F_m(z) = \frac{f^{(\lambda_m-1)}(z\rho(\lambda_m-1)) - f^{(\lambda_m-1)}(0)}{\rho(\lambda_m-1) f^{(\lambda_m)}(0)}$$

$$= z + \frac{\lambda_{m+1}!}{\lambda_m!\,(\lambda_{m+1}-\lambda_m+1)!}\left(\frac{a_{m+1}}{a_m}\right)\frac{[z\rho(\lambda_m-1)]^{\lambda_{m+1}-\lambda_m+1}}{\rho(\lambda_m-1)} + \cdots.$$

Then F_m is univalent in D. From Lemma 3, we have that

$$\frac{\lambda_{m+1}!}{\lambda_m!\,(\lambda_{m+1}-\lambda_m+1)!}\left|\frac{a_{m+1}}{a_m}\right|\rho(\lambda_m-1)^{\lambda_{m+1}-\lambda_m} \leqslant \frac{2}{\lambda_{m+1}-\lambda_m}.$$

Since

$$\left(\frac{\lambda_m!}{\lambda_{m+1}!}\right)^{1/(\lambda_{m+1}-\lambda_m)} < \frac{1}{\lambda_m+1},$$

it follows that

$$(\lambda_m+1)\,\rho(\lambda_m-1) \leqslant \left[2\left(1+\frac{1}{\lambda_{m+1}-\lambda_m}\right)(\lambda_{m+1}-\lambda_m)!\left|\frac{a_m}{a_{m+1}}\right|\right]^{1/(\lambda_{m+1}-\lambda_m)}. \tag{2.6}$$

Hence,

$$\limsup_{n\to\infty} n\rho_n < \bar{R} \limsup_{m\to\infty} \left[2\left(1+\frac{1}{\lambda_{m+1}-\lambda_m}\right)(\lambda_{m+1}-\lambda_m)!\right]^{1/(\lambda_{m+1}-\lambda_m)}.$$

Let $b_n = [n!2(1+1/n)]^{1/n}$. Then it can be shown that $\{b_n\}_{n=2}^{\infty}$ is strictly increasing, that $b_1 = 4$, and that $b_7 < 4 < b_8$. Using these facts, we have that

$$\limsup_{n\to\infty} n\rho_n \leqslant \begin{cases} 4\bar{R}, & d < 8 \\ \left[2\left(1+\frac{1}{k}\right)\right]^{1/k} (k!)^{1/k} \bar{R}, & d \geqslant 8 \end{cases}.$$

Let

$$1 + \epsilon_k = \begin{cases} 4/(k!)^{1/k}, & k < 8 \\ [2(1+1/k)]^{1/k}, & k \geqslant 8 \end{cases}. \tag{2.7}$$

Then the right side of (1.2) is established.

Now we establish the left side. If $\underline{R} = 0$, there is nothing to prove. So, suppose $\underline{R} > 0$ and let $0 < r < \underline{R}$. Then

$$r \leqslant | a_m/a_{m+1} |^{1/(\lambda_{m+1}-\lambda_m)},$$

or,

$$| a_{m+1} | r^{\lambda_{m+1}} \leqslant | a_m | r^{\lambda_m}$$

for all large m. An induction argument shows that, for all large m and all $j \geqslant m$,

$$| a_j | r^{\lambda_j} \leqslant | a_m | r^{\lambda_m}. \tag{2.8}$$

Let m be such that (2.8) holds and that $\lambda_{m+1} - \lambda_m = k$. Let $y_{m,k}$ be as in Lemma 1. Then

$$\sum_{j=m+1}^{\infty} (\lambda_j - \lambda_m + 1) \left| \frac{\lambda_j!}{(\lambda_j - \lambda_m + 1)!} a_j \left(\frac{r y_{m,k}}{\lambda_m}\right)^{\lambda_j - \lambda_m + 1} \right|$$

$$\leqslant \frac{r \,| a_m |\, y_{m,k}}{\lambda_m} \sum_{j=m+1}^{\infty} \frac{\lambda_j!}{(\lambda_j - \lambda_m)!} \left(\frac{y_{m,k}}{\lambda_m}\right)^{\lambda_j - \lambda_m}$$

$$\leqslant \frac{r \,| a_m |\, y_{m,k}}{\lambda_m} \sum_{j=0}^{\infty} \frac{(\lambda_m + k + j)!}{(k+j)!} \left(\frac{y_{m,k}}{\lambda_m}\right)^{k+j}$$

$$= \frac{r \,| a_m |\, y_{m,k} \lambda_m!}{\lambda_m}.$$

Hence, if

$$F_m(z) = f^{(\lambda_m - 1)}\left(\frac{r y_{m,k} z}{\lambda_m}\right)$$
$$= f^{(\lambda_m-1)}(0) + \sum_{j=m}^{\infty} \frac{\lambda_j!}{(\lambda_j - \lambda_m + 1)!} a_j \left(\frac{r y_{m,k} z}{\lambda_m}\right)^{\lambda_j - \lambda_m + 1},$$

it follows from Lemma 4 that F is univalent in D. So,

$$r y_{m,k}/\lambda_m \leqslant \rho(\lambda_m - 1). \tag{2.9}$$

Using Lemma 1 again,

$$r y_k \leqslant \limsup_{n\to\infty} n\rho_n . \tag{2.10}$$

Letting $r \to \underline{R}$, we get the left side of (1.2).

Finally, suppose that $k = \infty$ and $\underline{R} > 0$. Note that (2.9) still holds for all large m when k is replaced in it by a positive integer $\leqslant \lambda_{m+1} - \lambda_m$. Since $\lim_{m\to\infty}(\lambda_{m+1} - \lambda_m) = \infty$, (2.10) is true for arbitrarily large k. The last statement in Theorem 1 follows from this and the fact that $\lim_{k\to\infty} y_k = \infty$. The rest of Theorem 1 will be established in Section 3.

Proof of Theorem 2. Since $\underline{R} \leqslant R \leqslant \bar{R}$ and since $\underline{R} = \infty$, f is certainly entire. Suppose $k < \infty$. From (2.6), we have that, for $m \geqslant 1$,

$$\rho(\lambda_m - 1) < \left[2\left(1 + \frac{1}{\lambda_{m+1} - \lambda_m}\right)(\lambda_{m+1} - \lambda_m)!\right]^{1/(\lambda_{m+1}-\lambda_m)}$$
$$\times \frac{1}{(\lambda_m + 1)/| a_m/a_{m+1} |^{1/(\lambda_{m+1}-\lambda_m)}}.$$

Using Lemma 2, the right side of (1.3) now follows in the same way as the right side of (1.2).

Let $\lambda_{m+1} - \lambda_m = k$ and let $r = | a_m/a_{m+1} |^{1/(\lambda_{m+1}-\lambda_m)}$. From (2.8) we have

$$\frac{y_{m,k} | a_m/a_{m+1} |^{1/(\lambda_{m+1}-\lambda_m)}}{\lambda_m} \leqslant \rho(\lambda_m - 1).$$

The left side of (1.3) follows from this, Lemma 2, and the technique used to establish the left side of (1.2).

The case $k = \infty$ is treated the same way as it was in the proof of Theorem 1.

LEMMA 5. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. *If* f *is univalent in* $| z | < 1$, *then* $| a_2^2 - a_3 | \leqslant 1$. *If* f *also maps* $| z | < 1$ *onto a convex set, then* $| a_2^2 - a_3 | \leqslant (1 - | a_2 |^2)/3$.

The first part of the conclusion is well-known. A proof of the second part is in [10].

Proof of Theorem 3. From the hypotheses, it follows that $\rho_n > 0$ for all large n. Let

$$F(z) = \frac{f^{(n)}(\rho_n z) - f^{(n)}(0)}{\rho_n f^{(n+1)}(0)}$$

$$= z + \frac{n+2}{2}\frac{a_{n+2}}{a_{n+1}}\rho_n z^2 + \frac{(n+2)(n+3)}{6}\frac{a_{n+3}}{a_{n+1}}\rho_n{}^2 z^3 + \cdots .$$

Since F is univalent in $|z| < 1$, Lemma 5 shows that

$$\left| \left(\frac{n+2}{2}\frac{a_{n+2}}{a_{n+1}}\rho_n\right)^2 - \frac{(n+2)(n+3)}{6}\frac{a_{n+3}}{a_{n+1}}\rho_n{}^2 \right| \leqslant 1. \tag{2.11}$$

Using the triangle inequality, this becomes

$$n^2\rho_n{}^2 \left[\left(\frac{n+2}{2n}\left|\frac{a_{n+2}}{a_{n+1}}\right|\right)^2 - \frac{(n+2)(n+3)}{6n^2}\left|\frac{a_{n+3}a_{n+2}}{a_{n+2}a_{n+1}}\right|\right] \leqslant 1.$$

Using the hypothesis, it follows that

$$(\limsup_{n\to\infty} n\rho_n)^2 \left(\frac{1}{4R^2} - \frac{1}{6R^2}\right) \leqslant 1,$$

or,

$$\limsup_{n\to\infty} n\rho_n \leqslant 2\sqrt{3}\,R.$$

Let G be defined like F except let ρ_n be replaced by $\rho_n(c)$. Then G maps $|z| < 1$ onto a convex set. Using the second part of Lemma 5 and proceeding as above, it follows that $\limsup_{n\to\infty} n\rho_n(c) \leqslant \sqrt{2}R$.

Proof of Theorem 4. The hypothesis implies that $\rho_n > 0$ for all large n. Define F as in the proof of Theorem 3 and note that (2.11) remains true. Using the triangle inequality, we have that

$$\left[(n+2)\left|\frac{a_{n+2}}{a_{n+1}}\right|\rho_n\right]^2 \left[\frac{1}{4} - \frac{n+3}{6(n+2)}\left|\frac{a_{n+1}a_{n+3}}{a_{n+2}^2}\right|\right] \leqslant 1.$$

Using (1.6), it follows that

$$\left[\limsup_{n\to\infty}(n+2)\left|\frac{a_{n+2}}{a_{n+1}}\right|\rho_n\right]^2 \leqslant 12.$$

Using the fact that, if $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are sequences of positive numbers, then $(\limsup_{n\to\infty} p_n)(\liminf_{n\to\infty} q_n) \leqslant \limsup_{n\to\infty} p_n q_n$, both parts of (1.7) now follow.

The proof of (1.8) follows from this in the same way that the proof of (1.5) followed from the proof of (1.4). The proof of the corollary is immediate because, under the assumptions of the corollary, $\delta = \delta^*$ and $\gamma = \gamma^*$.

Remark. The constant, $2\sqrt{3}$, on the right-hand side of (1.4) cannot be replaced by any number smaller than π, and the constant, $\sqrt{2}$ on the right-hand side of (1.5) cannot be replaced by any number smaller than 1. This is shown by $z/(1-z)$, for which $\rho_n = \sin \pi/(n+1)$ and $\rho_n(c) = 1/(n+1)$.

3. Numerical Estimates on $\{y_k\}_{k=1}^{\infty}$

The sequence, $\{y_k\}_{k=1}^{\infty}$, is defined by either

$$y_k = \log\left(1 + \sum_{j=0}^{k-1} y_k^j/j\right) \tag{3.1}$$

or

$$1 = \frac{y_k^k}{k!}\left(1 + \frac{y_k}{k+1} + \frac{y_k^2}{(k+1)(k+2)} + \cdots\right). \tag{3.2}$$

Using (3.1), Table I was compiled.

TABLE I

k	y_k	$y_k^k/k!$
1	log 2	.6931 . . .
2	1.1461 . . .	.6567 . . .
3	1.5681 . . .	.6426 . . .
4	1.9761 . . .	.6353 . . .
5	2.3761 . . .	.6311 . . .
6	2.7709 . . .	.6286 . . .
7	3.1619 . . .	.6269 . . .

We also note that [5, p. 184], for $k \geqslant 2$,

$$[2\pi k]^{1/2}\left(\frac{k}{e}\right)^k < k! < [2\pi k]^{1/2}\left(\frac{k}{e}\right)^k e^{1/12k}. \tag{3.3}$$

From (3.2), we have that

$$1 < \frac{y_k^k}{k!}\left[1 + \frac{y_k}{k+1} + \left(\frac{y_k}{k+1}\right)^2 + \cdots\right]$$
$$= \frac{y_k^k}{k!}\left[\frac{1}{1 - y_k/(k+1)}\right].$$

Hence,

$$1 - y_k/(k+1) < y_k^k/k! \tag{3.4}$$

Using (3.3) and the fact that $y_k < (k!)^{1/k}$, we have that

$$\frac{y_k}{k+1} < \frac{k(2\pi k)^{1/2k}\, e^{1/24k}}{e(k+1)}.$$

For $k \geqslant 2$, the right-hand side of this inequality is a decreasing function of k. If $k \geqslant 8$,

$$\frac{y_k}{k+1} < \frac{8(16\pi)^{1/16}\, e^{1/192}}{9e} < .42.$$

Using this in (3.4), $y_k^k > .58k!$ Using (3.3) again,

$$\frac{y_k}{k+1} > \frac{k(.58)^{1/k}\,(2\pi k)^{1/2k}}{e(k+1)}.$$

The right-hand side of this inequality is decreasing for $k \geqslant 8$. Using Table I, it follows that, for $k \geqslant 2$,

$$\frac{1}{e} < \frac{y_k}{k+1} < \frac{1.1415}{e}.$$

Finally, using (3.2) again, we have for $k \geqslant 8$,

$$1 > \frac{y_k^k}{k!}\left[1 + \frac{y_k}{k+1} + \frac{y_k^2}{(k+1)(k+2)} + \frac{y_k^3}{(k+1)(k+2)(k+2)}\right]$$
$$> \frac{y_k^k}{k!}\left[1 + \frac{1}{e} + \frac{1}{e^2}\frac{k+1}{k+2} + \frac{1}{e^3}\frac{(k+1)^2}{(k+2)(k+3)}\right]$$
$$\geqslant \frac{y_k^k}{k!}\left(1 + \frac{1}{e} + \frac{.9}{e^2} + \frac{8.1}{11e^3}\right) > \frac{y_k^k}{k!}(1.5263).$$

Hence, for $k \geqslant 8$, $y_k^k < .6552k!$ Using Table I, it follows that for $k \geqslant 2$,

$$.58 < y_k^k/k! < .6568.$$

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