A Class Of Functional Equations And Mielnik Probability Spaces

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ABSTRACT. Let S be the unit sphere of a normed real linear space N and let (S,p) be a Mielnik space of dimension two. For $p(x,y) = f(\|x + y\|)$, $x, y \in S$, where f is a continuous, strictly increasing function from $[0,2]$ onto $[0, 1]$, it has been shown that (S,p) being two dimensional is equivalent to N being an inner product space.

In some polarization problems modeled on the unit sphere of an inner product space, the transition probability $p(x,y)$ may not be as well behaved as $p(x,y) = f(\|x + y\|)$. In order to provide a more suitable setting, we have constructed wide classes of two-dimensional transitional probability spaces (S,p), all having the same set of bases $\mathfrak{B}$, with $p = \phi \circ f$ where ϕ is a solution of a certain functional equation. In particular, for $p(x,y) = \|x + y\|^2/4$, we answer a question due to B. Mielnik.

1. **Introduction.** In [1], transitional probability spaces, in the sense of Mielnik [2], were utilized to obtain a new characterization of real inner product spaces.

Let $\mathfrak{F} = \{f \mid f\colon [0,2] \text{ onto } [0,1], f, \text{ continuous and strictly increasing}\}$ and let $\mathcal{G} = \{g \mid g\colon [0,2] \text{ onto } [0,2], g, \text{ continuous and strictly decreasing}\}$. Considering those $f \in \mathfrak{F}$ and $g \in \mathcal{G}$ that satisfy the functional equation $f + f \circ g = 1$ where $(f \circ g)(t) = f(g(t))$ and 1 is the identity function on $[0,2]$, the following result is given in [1]: Let N be a normed real linear space, S its unit sphere, and let $p(x,y) = f(\|x + y\|)$, where $f \in \mathfrak{F}$. Then N is an inner product space if and only if, for some $f \in \mathfrak{F}$, (S,p) is a Mielnik probability space of dimension 2.

This result provides an adequate model for some polarization problems. However, there are polarizations of particles other than photons modeled on the unit sphere S of a real inner product space N for which transitional probabilities $p(x,y)$ of the form $p(x,y) = f(\|x + y\|)$ are not adequate.

The above remark motivates an effort to study two-dimensional transitional probability spaces that will provide a more suitable setting for this kind of polarization phenomena.

In order to construct those more suitable transitional probability spaces, we need to consider functions satisfying the following functional equation:

$$(*) \qquad \phi(t) + \phi(1 - t) = 1, \qquad t \in [0, 1].$$

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The size of the class Φ_0 of all functions ϕ from $[0, 1]$ onto $[0, 1]$ and satisfying $(*)$ is too wide for our purposes, as the following proposition indicates.

PROPOSITION 1. *Every $\phi \in \Phi_0$ is of the form*

$$\phi(t) = \psi(t - \tfrac{1}{2}) + \tfrac{1}{2}, \qquad t \in [0, 1],$$

where ψ is an odd function on $[-\frac{1}{2}, \frac{1}{2}]$.

Any of the following subclasses of Φ_0 can be used in the construction of more general transitional probabilities:

$\Phi_1 = \{\phi | \phi \in \Phi_0, \phi(t) = 0 \Leftrightarrow t = 0\}$,

$\Phi_2 = \{\phi | \phi \in \Phi_0, \phi$, strictly increasing$\}$,

$\Phi_3 = \{\phi | \phi \in \Phi_0, \phi$, strictly increasing and continuous$\}$.

It is plain that $\Phi_3 \subset \Phi_2 \subset \Phi_1 \subset \Phi_0$. Even the smallest subclass Φ_3 can exhibit very pathological behavior as is shown by Ganguli [3]; for instance, Φ_3 contains Cantor's singular function.

In general, polarization does not have to be a mapping T: $x \mapsto -x$ where x is a unit vector in N. It can be modeled by a corresponding Mielnik two-dimensional probability space with an automorphism T: $S \to S$ where S is any set of states.

The following proposition makes the above remark even more transparent.

PROPOSITION 2 [1]. *Let (S,p) be a two-dimensional Mielnik space. Then there is an involution T: $S \to S$ such that every basis B of (S,p) is of the form $B = \{x, Tx\}$, $x \in S$.*

In particular, if S is the unit sphere of a normed real linear space N, then $Tx = -x$ as shown in [1].

2. **Construction of two-dimensional transitional probability spaces**. We are now in the position to exhibit the existence of a wide class of two-dimensional probability spaces in a very general setting utilizing the class Φ_1.

THEOREM. *Let (S,p) be a Mielnik probability space of dimension 2. Then for every $\phi \in \Phi_1$, $(S, \phi \circ p)$ is a transitional probability space of dimension 2, where $\phi \circ p = \phi(p)$.*

PROOF. Since ϕ is a self-map of $[0, 1]$ and $0 \leqslant p(x,y) \leqslant 1$, $0 \leqslant \phi(p(x,y)) \leqslant 1$ for all $x, y \in S$.

If $\phi(p(x,y)) = 1$, then from $(*)$ we obtain $\phi(1 - p(x,y)) = 0$. Since $\phi(t) = 0$ if and only if $t = 0$, this implies that $p(x,y) = 1$. Since $p(x,y)$ is a probability function, this forces $x = y$.

If $x = y$, then $p(x,x) = 1$ and, hence, $\phi(p(x,x)) = \phi(1) = 1$ by definition of the class Φ_1. Therefore, $\phi(p(x,y)) = 1$ if and only if $x = y$. The symmetry of $\phi(p(x,y))$ follows from the symmetry of $p(x,y)$. Now, $\phi(p(x, Tx)) = \phi(0)$

$= 0$, since (S,p) is two dimensional with all bases of the form $B = \{x, Tx\}$, $x \in S$, and $\phi(0) = 0$. Moreover, if $\phi(p(x,y)) = 0$, then $p(x,y) = 0$. Since (S,p) is a two-dimensional Mielnik space, this forces $y = Tx$.

Therefore, all bases $\mathfrak{B}$ of $(S, \phi \circ p)$ are of the form $B = \{x, Tx\}$, $x \in S$. Since ϕ is a solution of $(*)$ and $p(x,y)$ satisfies Axiom C in [2], we have, for any $x \in S$ and basis $B = \{y, Ty\}$, $y \in S$,

$$\phi(p(x,y)) + \phi(p(x,Ty)) = \phi(p(x,y)) + \phi(1 - p(x,y)) = 1,$$

i.e., $\phi \circ p$ satisfies Axiom C. Hence, $(S, \phi \circ p)$ is a two-dimensional transitional probability space.

By taking S to be the unit sphere of a normed real linear space and $p(x,y) = f(\|x + y\|)$ to be any function of $\|x + y\|$ satisfying the generalization of the parallelogram law given in [1], we obtain the following corollary.

COROLLARY 1. *Let N be a real inner product space, S its unit sphere, and f be as above. Then $(S, (\phi \circ f)(\|x + y\|))$ is a two-dimensional transitional probability space for every $\phi \in \Phi_3$.*

It is clear that using the subclasses Φ_2 and Φ_3, we can obtain important special cases of the above theorem.

Our next corollary answers a question due to Mielnik[1] concerning necessary and sufficient conditions for the existence of a particular type of two-dimensional probability space in the setting of normed real linear spaces.

COROLLARY 2. *Let N be a normed real linear space, S its unit sphere, and let $p(x,y) = \phi(\|x + y\|^2/4)$ where $\phi \in \Phi_0$. Then (S,p) is a two-dimensional transitional probability space if and only if N is an inner product space and $\phi \in \Phi_3$.*

The proof follows from the observation that if $f \in \mathfrak{F}$ and $\phi \in \Phi_3$, then $\phi \circ f \in \mathfrak{F}$, and the fact that $f(t) = (t/2)^2$ is in $\mathfrak{F}$.

Corollary 2 allows us to construct fairly bizarre two-dimensional transitional probability structures even in a real inner product space. Although $f(\|x + y\|)$ and, in particular, $\|x + y\|^2/4$, are very well-behaved functions of $\|x + y\|$, one can take $\phi \in \Phi_3$ to be a Cantor singular function or any other solution of the functional equation $\phi(t) + \phi(1 - t) = 1$, constructed by the method of Ganguli [3]. This shows that if the model (S,p) is not adequate for the polarization states S, one can construct more suitable models $(S, \phi \circ p)$ due to the variety of functions in Φ_3, or even in Φ_1. In the case where we have so-called "partial" or "incomplete" polarization, the representation space for reasonable transition probabilities must be a uniformly convex normed real linear space as is shown in [4].

[1] In a private communication to the second author.

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