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## The Extremal Structure Of Locally Compact Convex Sets

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# Pacific Journal of Mathematics

## **THE EXTREMAL STRUCTURE OF LOCALLY COMPACT CONVEX SETS**

J. C. HANKINS AND ROY MARTIN RAKESTRAW

## THE EXTREMAL STRUCTURE OF LOCALLY COMPACT CONVEX SETS

J. C. HANKINS AND R. M. RAKESTRAW

**Let  $X$  be a locally compact closed convex subset of a locally convex Hausdorff topological linear space  $E$ . Then every exposed point of  $X$  is strongly exposed. The definitions of denting (strongly extreme) ray and strongly exposed ray are given for convex subsets of  $E$ . If  $X$  does not contain a line, then every extreme ray is strongly extreme and every exposed ray is strongly exposed. An example is given to show that the hypothesis that  $X$  be locally compact is necessary in both cases.**

By a locally convex space we mean a real Hausdorff locally convex topological linear space  $E$ .  $E^*$  will denote the topological dual of  $E$ . The set of extreme points of  $X$  will be denoted by  $\text{ext } X$ . The closed line segment between the points  $x$  and  $y$  in  $E$  will be denoted  $[x, y]$ . The following definition was given by M. Rieffel [6, p. 75] for subsets of a Banach space. I. Namioka also studied these points in [4].

**DEFINITION 1.** If  $X$  is a subset of a locally convex space, then  $x \in X$  is called a denting (strongly extreme) point of  $X$  if for any nbhd  $U$  of  $x$ ,  $x \notin \text{cl-conv}(X \setminus U)$ . The set of all denting points of  $X$  will be denoted by  $\text{dent } X$ .

Clearly, every denting point is an extreme point. It follows from the separation theorem for convex sets that  $x_0$  is a denting point of  $X$  iff for each nbhd  $U$  of  $x_0$  there exist  $f \in E^*$  and  $\alpha \in \mathbb{R}$  such that  $x_0 \in \{x: f(x) < \alpha\} \cap X \subseteq X \cap U$ . An example is given in [6, p. 75] to show that not every extreme point is a denting point. However, this is not the case in a locally compact set. For completeness we state the following theorem due to J. Reif and V. Zizler [5, p. 64].

**THEOREM 1.** *Assume  $X$  is a locally compact closed convex set in a locally convex space  $E$ . Then any extreme point of  $X$  is a strongly extreme point of  $X$  with respect to the relative topology from  $E$ .*

A point  $p$  of a set  $X$  in a locally convex space  $E$  is an exposed point of  $X$  if there exists an  $f \in E^*$  such that  $f(x) > f(p)$  for each  $x \in X \setminus \{p\}$ . The following definition was given by J. Lindenstrauss [3, p. 140] for subsets of a Banach space.

DEFINITION 2. A point  $x \in X$ , where  $X \subseteq E$ , is called a strongly exposed point of  $X$  whenever (i) there exists an  $f \in E^*$  such that  $f(y) > f(x)$  for each  $y \in X \setminus \{x\}$ , and (ii) for any net  $\{x_\alpha\} \subseteq X$ ,  $f(x_\alpha) \rightarrow f(x)$  in  $R$  implies that  $x_\alpha \rightarrow x$  in  $E$ . The set of all strongly exposed points of  $X$  is denoted by  $\text{strex} X$ .

It is easy to see from the definition that every strongly exposed point is an exposed point. J. Lindenstrauss in [3, p. 145] gave an example of a set which has an exposed point that is not strongly exposed. However, this is not the case if the set is locally compact.

THEOREM 2. *Let  $X$  be a locally compact closed convex subset of a locally convex space  $E$ , then every exposed point of  $X$  is a strongly exposed point of  $X$ .*

*Proof.* Let  $U$  be a closed convex nbhd of  $x$  such that  $U \cap X$  is compact and assume  $f \in E^*$  such that  $f(x) < f(y)$ , for all  $y \in X \setminus \{x\}$ . Since  $x$  is an exposed point of  $X$ ,  $x$  is an extreme point of  $X$ . By Theorem 1,  $x$  is a denting point of  $X$ . Thus, there exist  $g \in E^*$  and  $\alpha \in R$  such that  $\{x: g(x) < \alpha\} \cap X \subseteq (\text{int } U) \cap X$ .

If  $\{x: g(x) \geq \alpha\} \cap (X \cap U) = \emptyset$ , then it follows immediately that  $U \cap X \subseteq \{x: g(x) < \alpha\} \cap X \subseteq (\text{int } U) \cap X$ . Therefore  $U \cap X$  is a nonempty open and closed set in the connected set  $X$ . Hence,  $U \cap X = X$  which implies  $X$  is compact. Let  $\{x_\alpha\}$  be a net in  $X$  such that  $f(x_\alpha) \rightarrow f(x)$  in  $R$ . Since  $X$  is compact, there is a subnet  $\{x_\beta\}$  of  $\{x_\alpha\}$  and a vector  $y \in X$  such that  $x_\beta \rightarrow y$ . Thus,  $f(x_\beta) \rightarrow f(y) = f(x)$  in  $R$  and so  $y = x$ . For any subnet  $\{x_\gamma\} \subseteq \{x_\alpha\}$  there is similarly a subnet which converges to  $x$ , which proves that  $x_\alpha \rightarrow x$  in  $E$ .

On the other hand, if  $W = \{x: g(x) \geq \alpha\} \cap (X \cap U) \neq \emptyset$ , then  $W$  is a nonempty compact convex subset of  $X$  which does not contain  $x$ . Hence, there is a  $w \in W$  such that  $f(x) < f(w) = \inf f(W)$ . Let  $y \in X \setminus U$ , then  $[x, y] \subseteq X$ .  $U$  is a closed convex nbhd of  $x$ ; hence, there exists a  $z \in \text{Bdry } U$  such that  $z \in [x, y]$ . Since  $z \in \text{Bdry } U$ , then  $z \notin \text{int } U$  and  $z \notin \{x: g(x) < \alpha\}$ . Therefore,  $z \in \{x: g(x) \geq \alpha\} \cap (X \cap U)$  so  $f(z) \geq f(w)$ . But  $y - x = \lambda(z - x)$  where  $\lambda > 1$ . Hence,  $f(y - x) = \lambda f(z - x) > f(z - x)$  which implies  $f(y) > f(z) \geq f(w)$ . Let  $\{y_\alpha\}$  be a net in  $X$  such that  $f(y_\alpha) \rightarrow f(x)$  in  $R$ . Since  $\{y_\alpha\} \subseteq X$  and  $f(y) \geq f(w) > f(x)$  for each  $y \in X \setminus U$ , we may assume that  $\{y, y_\alpha\} \subseteq U \cap X$ . Since  $U \cap X$  is compact, it follows from the previous argument that  $y_\alpha \rightarrow x$  in  $E$ .

As V. Klee has shown in [1] and [2], it is possible to extend the Krein–Milman theorem to certain noncompact convex sets with the aid of the notion of extreme ray. An extreme ray of a closed convex set  $X$

is a closed half-line  $\rho \subseteq X$  such that whenever  $x, y \in X$  and  $\lambda x + (1 - \lambda)y \in \rho$  for some  $\lambda$  with  $0 < \lambda < 1$ ,  $x, y \in \rho$ .

**DEFINITION 3.** A ray  $\rho = \{x + \lambda z : \lambda \geq 0, z \neq 0\}$  of a convex set  $X$  in a topological linear space  $E$  is a denting (strongly extreme) ray of  $X$  if for any nbhd  $U$  of  $0$ ,  $\rho' \cap \text{cl-conv}[X' \setminus (x + \langle z \rangle + U)] = \emptyset$ , where  $X'$  is any bounded convex subset of  $X$ ,  $\rho' = \rho \cap X'$  and  $\langle z \rangle$  denotes the one-dimensional linear subspace generated by  $z$ . Denote the union of all denting rays of  $X$  by  $\text{rdent } X$ .

It is easy to show that every denting ray of a convex set  $X$  is an extreme ray of  $X$ . The following theorem and example show that extreme rays and denting rays coincide in some instances and are distinct in others.

**THEOREM 3.** *Let  $X$  be a locally compact closed convex subset of a locally convex space  $E$ , then every extreme ray of  $X$  is a denting ray of  $X$ .*

*Proof.* Let  $\rho$  be an extreme ray of  $X$ . We may assume without loss of generality that  $\rho = \{\lambda x_0 : \lambda \geq 0\}$ ,  $x_0 \neq 0$ . Let  $X'$  be a bounded convex subset of  $X$  and let  $f_0$  be in  $E^*$  such that  $f_0$  is positive on  $K \setminus \{0\}$ , where  $K$  is the union of all rays in  $X$  which emanate from  $0$ , and  $X \cap \{x : f_0(x) \leq t\}$  is compact, for each  $t \in R$ . Such a functional exists by Theorem 3.2 in [1]. Since  $X'$  is bounded and convex,  $\text{cl}(X')$  is bounded and convex. According to a result of Klee [1, p. 236],  $\text{cl}(X')$  is compact which implies  $\sup f_0(\text{cl}(X')) < \infty$ . Then we may assume  $X' \subseteq \{x : f_0(x) \leq 1\} \cap X = X''$ . Let  $W = \{x : f_0(x) = 1\} \cap X$  and assume  $f_0(x_0) = 1$ . Then  $x_0 \in \text{ext}(W)$  and  $W$  is compact, since  $X''$  is compact. By Theorem 1,  $x_0$  is a denting point of  $W$ . Let  $U$  be a nbhd of zero and let  $g \in E^*$  and  $\alpha > 0$  such that  $x_0 \in \{x : g(x) < \alpha\} \cap W \subseteq (x_0 + U) \cap W$ . Let  $T = \{x : g(x) = \alpha\} \cap W$ . Then  $T$  is compact, convex and  $T \cap \langle x_0 \rangle = \emptyset$ . Let  $f \in E^*$  and  $\beta > 0$  such that  $f(\langle x_0 \rangle) < \beta < \inf f(T)$ . Since  $0 \in \langle x_0 \rangle$ , we have  $0 = f(\langle x_0 \rangle) < \beta < \inf f(T)$ .

If  $y \in W$  such that  $f(y) < \beta$ , then  $f_0(y) = 1$  and  $[x_0, y] \cap T = \emptyset$ , since  $f(x_0) < \beta$ . It follows that  $g(y) < \alpha$  and hence,  $y \in (x_0 + U) \cap W \subseteq \langle x_0 \rangle + U$ .

On the other hand, if  $y \in X$  such that  $f_0(y) < 1$  and  $f(y) < \beta$ , then there is a unique  $\lambda > 0$  such that  $f_0(y + \lambda x_0) = 1$ . Again from Klee [1, p. 235] we have  $y + \lambda x_0 \in X$ . Hence,  $y + \lambda x_0 \in W$  and  $f(y + \lambda x_0) = f(y) < \beta$ . By the previous argument, it follows that  $y + \lambda x_0 \in x_0 + U$  and so  $y \in (1 - \lambda)x_0 + U \subseteq \langle x_0 \rangle + U$ .

In both cases we have  $y \in \{x : f(x) < \beta\} \cap X''$  implies  $y \in \langle x_0 \rangle + U$ . Hence,  $X'' \setminus (\langle x_0 \rangle + U) \subseteq X'' \setminus \{x : f(x) < \beta\} \subseteq \{x : f(x) \geq \beta\}$ . Thus,  $\text{cl-conv}[X' \setminus (\langle x_0 \rangle + U)] \subseteq \{x : f(x) \geq \beta\}$ . Now  $f(\rho') < \beta$ , since

$f(\langle x_0 \rangle) < \beta$  and  $\rho' = (X' \cap \rho)$ , so  $\rho' \cap \text{cl-conv}[X' \setminus (\langle x_0 \rangle + U)] = \emptyset$ . Therefore,  $\rho$  is a denting ray of  $X$ .

EXAMPLE 1. Let the space be  $\ell_2$  with the canonical basis  $\{e_n\}$ , and  $X = \text{cl-conv}(\{e_n; n = 2, 3, \dots\})$ . Then  $0 \in X$  and  $e_1$  is in  $\ell_2 \setminus X$ . Let  $C$  be the cone generated by  $X$  with vertex  $e_1$ , then  $C$  is a closed convex subset of  $\ell_2$ . Let  $\rho$  be the ray of the cone through  $0$ . Clearly,  $\rho$  is an extreme ray of  $C$ . Let  $S_{\frac{1}{2}}(0)$  be the open ball of radius  $1/2$  centered on  $0$ . Clearly,  $e_n \notin S_{\frac{1}{2}}(0)$  so  $e_n \notin \langle e_1 \rangle + S_{\frac{1}{2}}(0)$  and it follows that  $e_n \in \text{cl-conv}[X \setminus (\langle e_1 \rangle + S_{\frac{1}{2}}(0))]$  for  $n \geq 2$ . However,  $\{e_n\}$  converges weakly to  $0$  and  $\text{cl-conv}[X \setminus (\langle e_1 \rangle + S_{\frac{1}{2}}(0))]$  is weakly closed so  $0 \in \text{cl-conv}[X \setminus (\langle e_1 \rangle + S_{\frac{1}{2}}(0))]$ . Hence  $\rho$  is not a denting ray of  $C$ .

A ray  $\rho$  in  $X$ , where  $X \subseteq E$ , is an exposed ray of  $X$  if there exist  $f \in E^*$  and  $\alpha \in \mathbb{R}$  such that  $\rho = \{x: f(x) = \alpha\} \cap X$  and  $f(X \setminus \rho) > \alpha$ . The next definition was given by V. Zizler in [7, p. 55] for subsets of a Banach space.

DEFINITION 4. Let  $X$  be a convex set in a locally convex space  $E$  and  $\rho$  a closed ray in  $X$ . Then  $\rho$  is a strongly exposed ray of  $X$  if (i) there exist  $f \in E^*$  and  $r \in \mathbb{R}$  such that  $f(x) = r$  for  $x \in \rho$  and  $f(x) > r$  for  $x \in X \setminus \rho$ , and (ii)  $\{x_n\}$  is eventually in  $\rho + U$ , whenever  $U$  is a nbhd of  $0$  and  $\{x_n\}$  is a bounded net in  $X$  such that  $f(x_n) \rightarrow r$ . The set of all strongly exposed rays will be denoted by  $\text{rstrep } X$ .

Clearly every strongly exposed ray is an exposed ray. The following proposition, theorem, and examples show the relationships among denting ray, exposed ray and strongly exposed ray.

PROPOSITION 1. Let  $\rho$  be a strongly exposed ray of a convex set  $X$  in a locally convex space  $E$ . Then  $\rho$  is a denting ray of  $X$ .

*Proof.* We may assume  $\rho = \{\lambda x_0: \lambda \geq 0\}$ ,  $x_0 \neq 0$ . Let  $f \in E^*$  such that  $\rho = \{x: f(x) = 0\} \cap X$  and  $f(x) > 0$  for each  $x \in X \setminus \rho$ . Let  $U$  be a nbhd of zero and  $X'$  a bounded convex subset of  $X$ . Assume for each positive integer  $n$  there is an  $x_n \in \{x: f(x) < (1/n)\} \cap X'$  such that  $x_n \notin \langle x_0 \rangle + U$ . Clearly  $\{x_n\}$  is bounded and  $f(x_n) \rightarrow 0$ . Hence, there exists a positive integer  $N$  such that  $x_n \in \rho + U$  for  $n \geq N$ . This is a contradiction; so there is a positive integer  $N'$  such that  $\{x: f(x) < (1/N')\} \cap X' \subseteq (\langle x_0 \rangle + U) \cap X'$ . Thus,  $\text{cl-conv}[X' \setminus (\langle x_0 \rangle + U)] \subseteq \{x: f(x) \geq (1/N')\}$  which implies  $(\rho \cap X') \cap \text{cl-conv}[X' \setminus (\langle x_0 \rangle + U)] = \emptyset$ ; so  $\rho$  is a denting ray of  $X$ .

THEOREM 4. Let  $X$  be a locally compact closed convex subset of a locally convex space  $E$ , then every exposed ray of  $X$  is a strongly exposed ray of  $X$ .

*Proof.* Let  $\rho$  be an exposed ray of  $X$ . We may assume that  $\rho$  emanates from the origin. Let  $f \in E^*$  such that  $\rho = X \cap \{x: f(x) = 0\}$  and  $f(x) > 0$  for  $x \in X \setminus \rho$ . Let  $\{x_\alpha\}$  be a bounded net in  $X$  such that  $f(x_\alpha) \rightarrow 0$  in  $R$  and let  $U$  be a nbhd of 0. There exists a nbhd  $V$  of 0 such that  $V$  is closed, balanced and convex,  $V \subseteq U$  and  $V \cap X$  is compact. Let  $\{x_\beta\}$  denote the set of all vectors in the net  $\{x_\alpha\}$  which lie in  $X \setminus U$ . If  $\{x_\beta\}$  is not a subnet of  $\{x_\alpha\}$ , then  $\{x_\alpha\}$  is eventually in  $U = 0 + U \subseteq \rho + U$  and the conclusion follows.

If  $\{x_\beta\}$  is a subnet of  $\{x_\alpha\}$ , then it suffices to show that  $\{x_\beta\}$  is eventually in  $\rho + U$ . By Theorem 1, 0 is a denting point of  $X$ , since 0 is an extreme point of  $X$ . Let  $g \in E^*$  and  $a > 0$  such that  $\{x: g(x) < a\} \cap X \subseteq V \cap X$ . Since  $x_\beta \notin V$ , then  $g(x_\beta) \geq a$ , for each  $\beta$ . The net  $\{x_\alpha\}$  is bounded, so there exists a number  $b > 0$  such that  $g(x_\beta) \leq b$ , for each  $\beta$ . Hence,  $0 < a \leq g(x_\beta) \leq b$ , for each  $\beta$ . If  $y_\beta = [a/g(x_\beta)]x_\beta$ , then  $y_\beta \in \{x: g(x) = a\} \cap X$ . Since  $\{x: g(x) < a\} \cap X \subseteq V \cap X$  and  $V \cap X$  is compact, then  $\{x: g(x) = a\} \cap X$  is compact; so there is a subnet  $\{y_\gamma\} \subseteq \{y_\beta\}$  and a point  $y \in \{x: g(x) = a\} \cap X$  such that  $y_\gamma \rightarrow y$  in  $E$ . Since  $g(x_\beta)$  is bounded and  $f(x_\beta) \rightarrow 0$  in  $R$ , we have  $y \in \{x: f(x) = 0\} \cap X$  and thus,  $y \in \rho$ . Hence,  $y \in \{x: g(x) = a\} \cap \rho$ . It follows immediately that  $\{y\} = \{x: g(x) = a\} \cap \rho$ . Let  $W = \{x: g(x) = a\} \cap X$  and  $z \in W \setminus \{y\}$ . Then  $z \in W \setminus \rho$  which implies  $f(z) > 0$ . Thus,  $y$  is exposed by  $f$  on  $W$ . Since  $f(y_\beta) \rightarrow 0 = f(y)$ , by Theorem 2 we have  $y_\beta \rightarrow y$  in  $E$ . Hence, there is a  $\lambda_0$  such that  $y_\beta \in y + (a/b)V$ , for  $\beta \geq \lambda_0$ . If  $z_\beta = [g(x_\beta)/a]y$ , then  $z_\beta \in \rho$ , for each  $\beta$ . But  $y_\beta = [a/g(x_\beta)]x_\beta$ , so  $x_\beta \in [g(x_\beta)/a]y + [g(x_\beta)/a](a/b)V \subseteq \rho + V \subseteq \rho + U$ , for all  $\beta \geq \lambda_0$ . Therefore, the net  $\{x_\beta\}$  is eventually in  $\rho + U$  and it follows that  $\rho$  is a strongly exposed ray of  $X$ .

EXAMPLE 2. The ray  $\rho$  defined in Example 1 is exposed by  $f = (0, \frac{1}{2}, \frac{1}{3}, \dots, 1/n, \dots)$  on  $C$ . Therefore  $\rho$  is an exposed ray of  $C$  that is not a denting ray of  $C$  so by Proposition 1  $\rho$  is not a strongly exposed ray of  $C$ .

EXAMPLE 3. Let the space be  $R^3$  and

$$X = \text{conv}\{(x, y, z): x^2 + y^2 \leq 1, -1 \leq y \leq 0 \text{ and } z = 1\} \cup (1, 1, 1).$$

Let  $C$  be the cone generated by  $X$  with vertex  $(0, 0, 0)$ . Then  $C$  is a closed convex subset of  $R^3$ . Let  $\rho$  be the ray of the cone through the point  $(1, 0, 1)$ . It is easy to see  $\rho$  is not an exposed ray of  $C$ , but  $\rho$  is a denting ray of  $C$ .

From the preceding work we can restate two of Klee's theorems ([2, Th. 2.3, p. 91], [1, Th. 3.4, p. 237]) as follows:

THEOREM 5. *Suppose  $X$  is a locally compact closed convex subset of a normed linear space, and  $X$  contains no line. Then  $\text{ext } X \subseteq \text{cl}(\text{strex } X)$  and  $X = \text{cl-conv}(\text{strex } X \cup \text{rstrex } X)$ .*

THEOREM 6. *If  $X$  is a locally compact closed convex subset of a locally convex space, and  $X$  contains no line, then  $X = \text{cl-conv}(\text{dent } X \cup \text{rdent } X)$ .*

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