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Further Comments On “Two Counterexamples To Aizerman's Conjecture”

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and assuming appropriate regularity conditions for the expressions stated, we conclude from (8) and (9) that

$$\hat{x}_{k_{\text{MMSE}}} = z_k - \eta\left(\frac{1-r_k}{1+r_k}\right), \qquad k=1,2,\cdots \tag{10}$$

$$E\left[(x_k - \hat{x}_{k_{\text{MMSE}}})^2 | Z^k\right] = \eta^2 \cdot \left[1 - \left(\frac{1-r_k}{1+r_k}\right)^2\right]. \tag{11}$$

■

Solution B

In the given technical situation the input noise is smaller in magnitude than the measurement noise (but not negligible). In this case we may accept for w_k and v_k the following "unknown-but-bounded" models [6], respectively:

$$0 \leqslant |w_k| < \eta, \qquad \eta > 0 \tag{12}$$

$$|v_k| = \eta. \tag{13}$$

We notice that (13) corresponds to $p_v(v_k) = c\delta(v_k + \eta) + (1-c)\delta(v_k - \eta)$ with not predetermined $c \in (0,1)$.

Considering now S [(4), (5)] together with (12) and (13) we will show that the state of S can exactly be reconstructed from the measurements by

$$\hat{x}_k^* = z_k - \eta\,\text{sgn}(z_k - \varphi(\hat{x}_{k-1}^*)), \qquad \text{if } \hat{x}_0^* = x_0. \tag{14}$$

Indeed for $k=1$,

$$\hat{x}_1^* = x_1 + v_1 - \eta\frac{|x_1 + v_1 - \varphi(\hat{x}_0^*)|}{x_1 + v_1 - \varphi(\hat{x}_0^*)} = x_1 + v_1 - \eta\frac{|w_0 + v_1|}{w_0 + v_1} = x_1.$$

Let $\hat{x}_k^* = x_k$, then we find that

$$\hat{x}_{k+1}^* = x_{k+1} + v_{k+1} - \eta\,\text{sgn}(x_{k+1} + v_{k+1} - \varphi(\hat{x}_k^*)) = x_{k+1}.$$

So (14) is true for $\forall k = 1, 2, \cdots$ ■

Some comments may now be useful on Solutions A and B.

1) The estimator (14), which can also be generalized for invertible multivariable systems, has a structure similar to the Kalman–Bucy filter. It consists namely of the model of the dynamical process and a control element in the feedback loop. One further recognizes this control element as the possible noise generating model, that is $v_k \triangleq \text{sgn}(v_k')$, $v_k' \neq 0$, and $v_k' \sim N(0, \sigma')$.

2) The simplicity of (14) in comparison with the stochastic estimator [(9), (10)] is evident from Solution B, the later requiring an on-line digital implementation.

3) It should be mentioned that the assumption of *a priori* knowledge of x_0, which makes Solution B less attractive, may be relaxed by the use of bounds as in (12).

Conclusion

Some advantages and limitations of two different solutions for a non-Gaussian/nonlinear estimation problem have been discussed. Although the deterministic Solution B seems to correspond to a very restrictive case, it appears that such a modeling of the uncertainties may sometimes better interpret a given physical situation. The stochastic approach of Solution A on the other side indicates that a discrete-type representation of p_v leads to computational schemes without further linearizations and approximations.

Finally let us point out that a combination of both approaches, say for disjoint regions of the observation set Z^k, may lead to "filter-observers" matching even better the nature of the problem.

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Further Comments on "Two Counterexamples to Aizerman's Conjecture"

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Singh and Mukerjee [1] have recently questioned some results of the above paper.[1] Specifically, they considered a system as follows:

$$\begin{aligned} \dot{x} &= -a_3 z + y \\ \dot{y} &= -a_2 x - a_1 y - g(y) y \\ \dot{z} &= x \end{aligned} \tag{1}$$

where a_1, a_2, and a_3 are constants and g is continuous such that

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad g(y) > 0. \tag{2}$$

A Lyapunov function is assumed to be

$$V = a_1 a_2 x^2 + a_1 y^2 + a_1 a_2 z^2 \tag{3}$$

and it is found that

$$\dot{V} = -2a_1^2 y^2 - 2a_1 g(y) y^2. \tag{4}$$

In his reply [2] Fitts correctly points out that $\dot{V}$ is not negative definite, but only negative semidefinite, and that asymptotic stability can not be concluded from their argument. However, it appears that global asymptotic stability can indeed be established for this system by appending the following brief argument.

It is well known [3] that global asymptotic stability can be concluded when V is positive definite and $\dot{V}$ is only negative semidefinite provided it can also be shown that the system can not remain forever at points where $\dot{V}=0$, except at the origin. For the Lyapunov function under consideration, $\dot{V}$ is zero when $y=0$ for arbitrary values of x and z. From the system equations (1), however, it is clear that the system can remain indefinitely at points where $y=0$ only if

$$x = 0, \qquad z = 0 \tag{5}$$

is also satisfied. Thus global asymptotic stability is established. The remainder of the analysis of Singh and Mukerjee remains unchanged.

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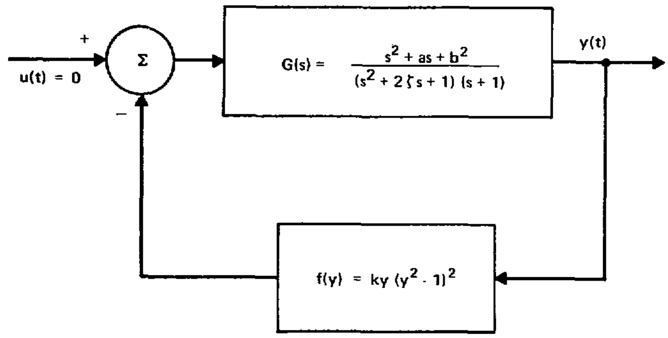

Fig. 1. Third-order nonlinear feedback system.

Author's Reply[2]

R. E. FITTS

I appreciate the interest of D. R. Fannin in my short paper, especially since his correspondence reveals a point which my paper did not make completely clear. In my investigation of the third-order nonlinear system shown in Fig. 1, I wanted the complex poles to lie only in the left-half plane. I established that their relative damping, ζ, was at least 0.0001 and I bounded the entries for ζ in Table III[1] from below to show that fact. Since I did not establish an experimental upper bound on the relative damping for four of the five cases, I did not show an upper bound for those cases in Table III; but I did include Case 4 to show one experimental upper bound ($\zeta = 0.092$). From the nature of Case 4, its upper bound is probably an upper bound for all the other cases, however the short paper did not state that explicitly.

Although the objective of Singh and Mukerjee [1] was to disprove my counterexample, the proof as finally constructed by Fannin is a contribution to the objective of my work, for I undertook my investigation in order to reduce the uncertainty in our knowledge of nonlinear system behavior. Previous investigators, notably Brockett and Willems [2], had established some sufficient theoretical boundaries for the regions of stability of nonlinear systems; at the suggestion of Prof. Brockett I undertook the task of looking for an unstable nonlinear system outside those boundaries so that we could establish the necessary boundary. Ultimately, our goal was a precise statement of the ranges of parameters for which a nonlinear system would be unstable in order to avoid having to use unquantifiable statements to explain why our approximate analytical methods break down, for example, as in a recent paper by Kou and Han [3].

The contribution of Fannin's proof can be seen by examining the behavior of the system of Fig. 1 as the location of the zeros is varied. Initially, let us confine the zeros to the imaginary axis ($a = 0$). The transfer function of the linear part becomes

$$G(s) = \frac{s^2 + b^2}{s^3 + (1 + 2\zeta)s^2 + (1 + 2\zeta)s + 1}.$$

A simple calculation shows that the phase angle of the denominator is 90° when $s = \pm jb_0$, where $b_0 = (1 + 2\zeta)^{-1/2}$. Fannin's proof requires $b = b_0$ (see Singh and Mukerjee [1]). The result of this condition is to confine the Nyquist locus of the linear part to the first and fourth quadrants, as can be seen in Fig. 2, where the locus is drawn for $\zeta = 0.0001$, the value used in Table III.[1]

Fig. 2 displays two other Nyquist locii, both of which characterize unstable systems. One, for $b = 0.797$, represents the upper limit of instability in Table III.[1] The other for $b = b_1$ where $b_1 = \sqrt{1 + 2\zeta(3 + 2\zeta)}$, represents the zero location for which the Nyquist locus passes through $(-1, 0)$ and the linear part of Fig. 1 becomes unstable. Portraying these boundaries in $\zeta - b$ parameter space defines regions of stability and instability of this system. The inner region of instability does not have a precisely defined boundary; the intent of my short paper was only to show that that region was not a single point but covered some area. Several precisely defined inner boundaries could be constructed from Pliss' monograph [4], but the nonlinearity used to obtain the data for Table III ($f(y) = ky(y^2 - 1)^2$) violates Pliss' constraints. Thus one is driven to the conclusion that there is an inner region of instability which must be avoided for stability (the necessary condition) and that a precise determination of the boundary of this region (which depends upon the nonlinearity) has not yet been made for the nonlinearity I used.

Both my short paper and my dissertation [5] discuss the system of Fig. 1 in the conceptual framework of the describing function. In particular, the nonlinearity acts to advance the phase of the fundamental and retard the phase of the harmonics so that the system becomes unstable at two or more frequencies simultaneously. Fig. 2 is annotated to show the location of the fundamental frequency of oscillation. In all cases, the fundamental frequency lies on the portion of the Nyquist locus in the second quadrant where frequency increases as one moves along the locus away from the origin. If all other parameters are held constant, reversing the direction of the Nyquist locus (so that frequency increases as one moves toward the origin) should convert a point of stable oscillation (convergent equilibrium) to a point of unstable oscillation (divergent equilibrium) and the oscillation should cease. This heuristic argument is the basis for labeling the region between the line $b = b_0$ and $b = b_1$ in Fig. 3 as stable.

If the zeros are restricted to the negative real axis excluding the origin then the Nyquist locus can be confined to the first and fourth quadrants by setting $b^2 = \zeta$, and $a = 1 + \zeta$ (in effect reducing the linear part of the system to a second-order system) and the system is globally assymptotically stable. If the leftmost zero moves further away from the origin while the other stays at $(-\zeta, 0)$ the Nyquist locus enters the third quadrant at high frequencies. Although the direction of the Nyquist locus is appropriate for stable nonlinear oscillation, the nonlinearity must produce a phase advance at the fundamental both greater than 90° and greater in magnitude than the phase retards at the harmonics. Such a behavior is contrary to that which I observed so, heuristically, the system of Fig. 1 is stable for this case.

Therefore, based on Fannin's proof and the arguments presented above one concludes that the system of Fig. 1 is potentially unstable if its linear part is lightly damped with zeros lying closer to the origin than the poles, even when the nonlinearity lies completely within the Hurwitz sector.[1] Alternatively, one might say that the system is potentially unstable for nonlinearities lying within the Hurwitz sector if the Nyquist locus of the linear part moves away from the origin in the second quadrant. In addition, there is a region of demonstrated instability when both zeros lie close to or at the origin. The instability need not take the form of sustained oscillations, O'Day and Hyde [6] have shown that subharmonic generation and jump resonance are also possible for these systems.

[2]Manuscript received May 8, 1975.
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