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Kalman Filter Equalization For QPSK Communications

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Kalman Filter Equalization for QPSK Communications

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Abstract—**The discrete complex Kalman filter is considered as an equalizer for quadrature phase-shift keyed (QPSK) systems in the presence of additive noise and intersymbol interference (ISI). When the channel is unknown, an adaptive Kalman equalizer is used in which the channel complex tap gains are estimated by decision feedback.**

I. INTRODUCTION

For additive white Gaussian noise (WGN) and no other signal corruption, the system error-rate performance and optimum receiver structure of various types of digital communication systems are well known [1]. In high-speed data transmission systems, two important causes of detection error are intersymbol interference (ISI) and additive noise. ISI may result from realizable filters, from distortion due to a dispersive transmission medium, or from a nonoptimum choice of sampling instants [2], [3]. As a result, if the rate of transmission is high enough, successive received symbols overlap and significantly degrade error-rate performance. For quadrature phase-shift keyed (QPSK) systems, ISI also causes crosstalk between in-phase and quadrature channels [4].

Lawrence and Kaufman [5] have used the discrete Kalman filter as an equalizer for binary transmission in the presence of noise and ISI. It was shown how the Kalman filter can be used to estimate both the tap weights and the binary signal. More recently, Fitch and Kurz [6] have applied recursive equalization to pulse-amplitude modulated systems. For an unknown or slowly varying channel, an adaptive decision feedback equalizer has been derived [7], [8].

Evaluating the error probabilities of various digital communication systems in the presence of both Gaussian and intersymbol interference has been investigated by many authors [9], [10]. Finding an accurate upper bound or lower bound for the error probability is often more practical from an engineering standpoint than finding the exact closed form for the error probability.

In this concise paper, a discrete complex Kalman filter is developed as an equalizer for a QPSK system in the presence of WGN and ISI. When the channel parameters are unknown, an adaptive discrete complex Kalman filter equalizer is derived which uses decision feedback for channel estimation. A two-component multipath channel QPSK system is used as an example. Simulations using the Chernoff upper bound for the error probability, both for the known channel and for the unknown channel equalizers, indicate better performance than the integrate-and-dump correlator with no equalizer.

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II. SYSTEM MODEL AND STATE VARIABLE REPRESENTATION

Fig. 1 illustrates a typical integrate-and-dump QPSK system used when ISI is negligible. The transmission channel is assumed linear, causal, and time invariant with finite duration impulse response $h(t)$, i.e., $h(t) = 0$ for $t < 0$ and $t > nT_s$. The received signals are perturbed by additive WGN with double-sided power spectral density $N_0/2$. The transmitted signal which is assumed to be generated by an infinite bandwidth modulator is

$$S(t) = \sqrt{\frac{2E}{T_s}} \cos\left[\omega_0 t + p(t)\frac{\pi}{2} - \frac{\pi}{4}\right],$$

$$= \sum_{i=-\infty}^{\infty} S_i(t), \tag{1}$$

where

$$S_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T_s}} \cos(\omega_0 t + \phi_i) & (i-1)T_s \leqslant t < iT_s \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$\phi_i = p_i \frac{\pi}{2} - \frac{\pi}{4}, \tag{3}$$

E is the energy per symbol of $S(t)$, $\omega_0 = 2\pi n_0/T_s$, n_0 is a fixed integer, T_s is the symbol period, and $p(t) = p_i = k$ for $(i-1)T_s < t < iT_s$, $k = 1,2,3,4$ and $i = \cdots, -2, -1, 0, 1, 2, \cdots$. It is assumed that the values of p_i are equally probable, and that p_i is independent of $p_{j \neq i}$. It is also assumed that the receiver is perfectly synchronized with respect to the carrier bit period. Synchronization errors and the effect of finite bandwidth modulation can be included in $h(t)$.

If $\omega_0 \gg 2\pi/T_s$, that is narrow-band modulation relative to the carrier frequency, the complex discrete signal representation of the QPSK system of Fig. 1 is well known and is shown in Fig. 2. The complex output is

$$z_i = \sum_{k=0}^{n} c_k e^{-j\phi_{i-k}} + n_i, \tag{4}$$

where $c_k = a_k + jb_k$ and $n_i = n_c(i) + jn_s(i)$ are defined in the Appendix, and $j = \sqrt{-1}$, $i = 1,2,3, \cdots$. The Gaussian random variables $n_c(i)$ and $n_s(i)$ will have zero mean and variance $N_0/2$.

Thus the model chosen to represent the channel and detector is a complex weighted n-tap delay line with constant delay T_s between taps. From (4), define

$$w_i = e^{-j\phi_i}, \tag{5}$$

$$W_i = (w_i, w_{i-1}, \cdots, w_{i-n})^T, \tag{6}$$

and

$$C = (c_0, c_1, \cdots, c_n). \tag{7}$$

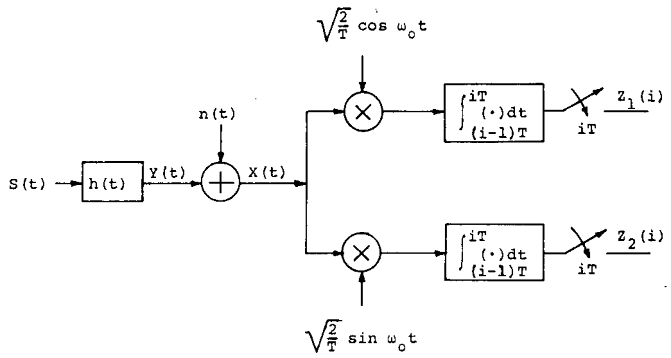

Fig. 1. QPSK system.

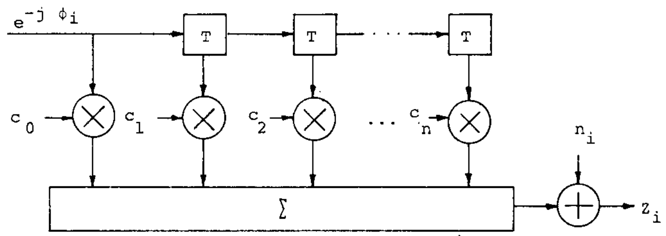

Fig. 2. Complex representation of a QPSK system.

Using (5)–(7) in (4), the complex scalar observation equation is

$$z_i = CW_i + n_i, \tag{8}$$

where C is the channel-coefficient vector, W_i is the signal vector, and n_i is a scalar complex Gaussian random variable. The "dynamics" of the message source can thus be represented by the trivial complex vector state equation [5]

$$W_i = FW_{i-1} + Gu_i, \tag{9}$$

where

$$F = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{10}$$

and

$$u_i = w_i = \exp(-j\phi_i).$$

III. COMPLEX DISCRETE KALMAN FILTER EQUALIZER—KNOWN CHANNEL

The assumption has been made that the values of ϕ_i are *a priori* equally likely, n_i is a zero mean complex Gaussian random variable, and W_i and n_i are uncorrelated. Define n^* to be the complex conjugate of the complex scalar n, F^T to be the transpose of the real matrix, F, C^* to be the complex conjugate transpose of the complex matrix C, $E\{u_i\}$ to be the statistical expectation of u_i, $\text{cov}\{u_i,n_j\} = E\{(u_i - E\{u_i\})(n_j - E\{n_j\})^*\}$ to be the covariance of u_i and n_j, $\text{var}\{W_0\} = E\{(W_0 - E\{W_0\})(W_0 - E\{W_0\})^*\}$, and I to be the identity matrix. The complex discrete Kalman filter derivation is similar to that for the more common real discrete Kalman filter, and the algorithms are changed only in that the transpose of a complex matrix becomes the complex conjugate transpose.

TABLE I
COMPLEX DISCRETE KALMAN FILTER EQUALIZER ALGORITHMS

Message Model	$W_i = F W_{i-1} + G u_i$	(I-1)
Observation Model	$z_i = C W_i + n_i$	(I-2)
Prior Statistics	$E\{u_i\} = 0,\ E\{n_i\} = 0,\ E\{W_o\} = 0$	(I-3)
	$\text{Cov}\{u_i,u_j\} = \delta_k(i-j)$	(I-4)
	$\text{Cov}\{n_i,n_j\} = N_o\ \delta_k(i-j)$	(I-5)
	$\text{Cov}\{u_i,n_j\} = \text{Cov}\{W_o,u_j\} = \text{Cov}\{W_o,n_j\} = 0$	(I-6)
	$\text{Var}\{W_o\} = P_o$	(I-7)
Filter Algorithm	$\hat{W}_i = F\hat{W}_{i-1} + K_i(z_i - CF\hat{W}_{i-1})$	(I-8)
Gain Equation	$K_i = P_{i\|i-1}C^*[CP_{i\|i-1}C^* + N_o]^{-1}$	(I-9)
A Priori Variance Algorithm	$P_{i\|i-1} = FP_{i-1}F^T + GG^T$	(I-10)
A Posteriori Variance Algorithm	$P_i = [I - K_iC]\ P_{i\|i-1}$	(I-11)
Initial Conditions	$W_o = E\{W_o\} = 0$	(I-12)
	$P_{o\|o-1} = P_o$	(I-13)

The complex discrete Kalman filter equalizer algorithms are summarized in Table I.

In the estimate

$$\hat{W}_i = (\hat{w}_i(i), \hat{w}_{i-1}(i), \cdots, \hat{w}_{i-n}(i))^T \tag{11}$$

of W_i, the term $\hat{w}_{i-j}(i)$ is the estimate of w_{i-j} at time i. For the best estimate, w_{i-n} would be estimated by $\hat{w}_{i-n}(i)$ rather than using $\hat{w}_i(i)$ to estimate w_i. This causes a delay of n symbol periods. If the delay could not be tolerated, $\hat{w}_i(i)$ could be used with increased error probability. After the estimate of w_i is obtained, a decision $\overline{\phi}_{i-n}$ on the value of ϕ_{i-n} is made depending on which quadrant $\hat{w}_{i-n}(i)$ lies in.

The algorithms are comparatively easy to implement as the inverse that appears in the gain equation is a complex scalar inverse. The resulting general form for the filter is shown in Fig. 3.

IV. ADAPTIVE KALMAN EQUALIZER—UNKNOWN CHANNEL

If the channel coefficient matrix C is unknown and time invariant (or very slowly time varying), C at time $i - j$ is approximately equal to C at time i, that is

$$\begin{aligned} C &\cong C(i) \\ &\cong C(i-j), \qquad i = 1,2,3,\cdots \\ &\qquad\qquad\qquad j = n, n+1, \cdots, 2n. \end{aligned} \tag{12}$$

Using decision feedback [6], [7], all decisions are assumed correct and (4) may be approximated by

$$z_{i-n} \cong \sum_{k=0}^{n} c_k e^{-j\overline{\phi}_{i-n-k}} + n_i, \tag{13}$$

where $\overline{\phi}_i$ is the decision for ϕ_i.

Define

$$\overline{w}_i = e^{-j\overline{\phi}_i}, \tag{14}$$

$$\overline{W}_i = (\overline{w}_{i-n}, \overline{w}_{i-n-1}, \overline{w}_{i-n-2}, \cdots, \overline{w}_{i-2n}). \tag{15}$$

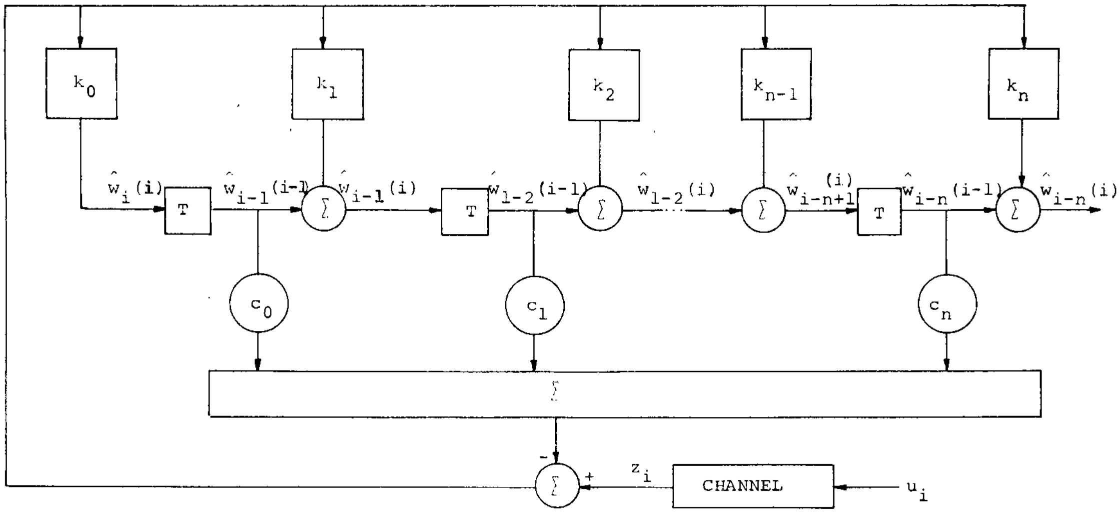

Fig. 3. Complex discrete Kalman filter equalizer.

and

$$C^T = (c_0, c_1, \cdots, c_n)^T. \tag{16}$$

Equation (13) can be written as

$$z_{i-n} = \overline{W}_i C^T + n_i, \tag{17}$$

where $\overline{W}_i$ is the known decision vector, and the approximation has been assumed an equality.

Equation (17) is of the same linear form as (8) if the decisions are known. Thus the channel coefficients can be estimated by a Kalman filter. The algorithms are easily derived and are presented in Table II. If the channel is not stationary but changes very slowly, the algorithms are still applicable if a lower bound is placed on the covariance matrix P_{ci}.

Using these identification algorithms, (8) is approximately

$$z_i \cong \hat{C}_i W_i + n_i \tag{18}$$

where

$$\hat{C}_i = (\hat{c}_0(i), \hat{c}_1(i), \cdots, \hat{c}_n(i)). \tag{19}$$

As the estimate $\hat{C}_i$ is obtained from data prior to i, $\hat{C}_i$ is available at time i and the adaptive equalizer equations are formed by substituting $\hat{C}_i$ for C in the equations in Table I. The block diagram for the adaptive equalizer is shown in Fig. 4. In the steady state, the estimate $\hat{C}_i$ of the coefficient matrix converges to C and the error probability (or upper bound) for the adaptive equalizer may be found by the same procedure developed for the Kalman equalizer. The possibility exists of course that, if the *a priori* values of C are sufficiently in error, the adaptive equalizer may not converge. A known training sequence a few times as long as the number of elements in C will usually assure convergence given reasonable signal-to-noise ratios.

TABLE II
KALMAN FILTER CHANNEL COEFFICIENT IDENTIFICATION ALGORITHMS

Message Model	$C_i^T = C_{i-1}^T = C^T$	(II-1)
Observation Model	$z_{i-n} = \overline{W}_i C_i^T + n_i$	(II-2)
Prior Statistics	$E\{n_i\} = 0, \quad E\{C_o^T\} = \hat{C}_o^T$	(II-3)
	$\mathrm{Cov}\{n_i, n_j\} = N_o \, \delta_k(i-j)$	(II-4)
	$\mathrm{Cov}\{C_o^T\} n_j\} = 0$	(II-5)
	$\mathrm{Var}\{C_o^T\} = P_{co}$	(II-6)
Filter Algorithm	$\hat{C}_i^T = \hat{C}_{i-1}^T + K_{ci}(z_{i-n} - \overline{W}_i \hat{C}_{i-1}^T)$	(II-7)
Gain Equation	$K_{ci} = P_{ci\|i-1} \overline{W}_i^* (\overline{W}_i P_{ci\|i-1} \overline{W}_i^* + N_o)^{-1}$	(II-8)
A Priori Variance Algorithm	$P_{ci\|i-1} = P_{ci-1}$	(II-9)
A Posteriori Variance Algorithm	$P_{ci} = [I - K_{ci}\overline{W}_i] P_{ci\|i-1}$	(II-10)
Initial Conditions	$\hat{C}_o^T = E\{C_o^T\}$	(II-11)
	$P_{co\|0-1} = P_{co}$	(II-12)

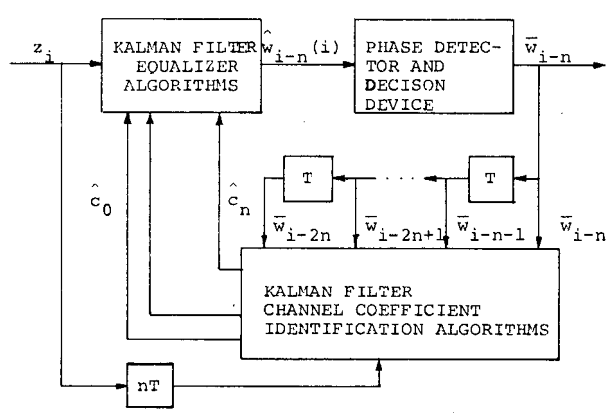

Fig. 4. Adaptive discrete Kalman equalizer.

V. ERROR PROBABILITIES AND COMPUTER SIMULATIONS

Calculating the exact error probability for an arbitrary QPSK system with ISI and WGN is extremely difficult and a direct computer simulation requires an enormous amount of computer time for small error probabilities. Therefore, an upper bound on the error probability at the output of the integrate-and-dump correlator or at the output of the Kalman filter is of considerable value. Prabhu [8] has derived a Chernoff upper bound for QPSK degraded by WGN and ISI. His work can readily be adapted to bounding the steady state probability of error when using the Kalman filter equalizer or adaptive equalizer with decisions based on $\hat{w}_i(i)$, if the required variances are determined by simulation. This also provides a loose upper bound on the probability of error for decision based on $\hat{w}_{i-n}(i)$. An example where this upper bound is found follows.

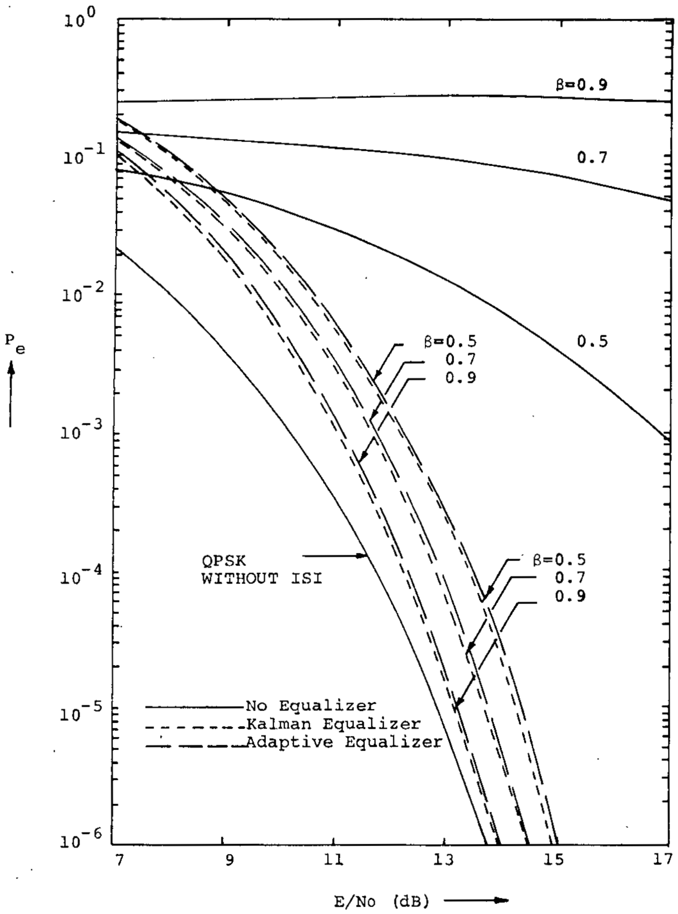

Fig. 5. Probability of symbol error in the presence of two-component multipath and WGN. $m = 0.1$, $\beta = 0.5$, 0.7, and 0.9.

Example—Two Component Multipath Channel: Consider the impulse response, $h(t) = \delta(t) + \beta\delta(t - \tau)$, where $0 \leqslant \beta < 1$ and $0 \leqslant \tau < T_s$. Let $n_0 = 123$ and $m = \tau/T_s$.

For the simulation a 1000-sample equally likely random signal sequence and independent Gaussian random variable sequences $n_c(i)$ and $n_s(i)$ with zero mean and variance $N_0/2$ were generated. For an unknown channel, no ISI was assumed to start the adaptive Kalman equalizer. Fig. 5 compares the performance of the equalized receivers as given by the Chernoff upper bound on probability of symbol error and the performance of the unequalized receiver. It is clear that as β increases the performance of both the Kalman equalizer and adaptive equalizer are much better than the integrate-and-dump without equalizer. Referring to Fig. 5, the curves of the Kalman equalizer and adaptive equalizer, given the same parameters m and β, are very close to each other. This is because after many samples, the estimated channel coefficients are very close to the true channel coefficients. This is illustrated by Table III using parameters defined in the Appendix.

VI. DISCUSSION AND CONCLUSIONS

The calculated error bounds show that the complex Kalman equalizer significantly improves the performance of an integrate-and-dump correlator when ISI is present. The number of delay elements required for the Kalman filter is equal to the number n of interfering symbols, and for optimum performance, the decision is delayed by n symbol periods. The channel coefficients may be estimated by a second Kalman filter to form an adaptive equalizer.

APPENDIX

The noise terms and coefficients in the complex channel representation of (4) are

TABLE III
ESTIMATED CHANNEL COEFFICIENTS AFTER $N = 300$ SAMPLING TIMES FOR $\beta = 0.9$, $m = 0.1$, AND INITIAL VALUES $a_0 = 1, a_1 = b_0 = b_1 = 0$

	True Value of a_0=0.7496	True Value of a_1=-0.0278	True Value of b_0=0.7703	True Value of b_1=0.0856
Estimated Value / E/N_0 (DB)	$\hat{a}_0$	$\hat{a}_1$	$\hat{b}_0$	$\hat{b}_1$
7	0.7526	-0.0218	0.7782	0.1014
9	0.7516	-0.0232	0.7766	0.0973
11	0.7509	-0.0244	0.7751	0.0941
13	0.7505	-0.0253	0.7739	0.0917
15	0.7502	-0.2060	0.7729	0.0900
17	0.7500	-0.0265	0.7721	0.0887

$$c_k = a_k + jb_k = \sqrt{E} \int_{T_s}^{T_s} \left(1 - \frac{|t|}{T_s}\right) e^{j\omega_0 t} h(t + kT) T_s. \tag{A1}$$

and

$$n_i = n_c(i) + jn_s(i) = \int_0^{T_s} \sqrt{\frac{2}{T_s}}\, e^{j\omega_0 t} n[t - (i-1)T_s]\, dt. \tag{A2}$$

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