01 Jan 1975

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Recommended Citation

K. P. Berkbigler and J. K. Byers, "System Reliability: Exact Bayesian Intervals Compared With Fiducial Intervals," *IEEE Transactions on Reliability*, vol. R thru 24, no. 3, pp. 199 - 200, Institute of Electrical and Electronics Engineers, Jan 1975.
The definitive version is available at https://doi.org/10.1109/TR.1975.5215151

System Reliability: Exact Bayesian Intervals Compared with Fiducial Intervals

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Abstract—This paper compares numerically two different, widely used lower limit estimates for the reliability of a series system: Bayesian limits and fiducial limits. The fiducial limits are obtained by Monte Carlo simulation because of its simplicity and ease of computer programming. Subsystem failures are *s*-independent and exponentially distributed; life test data are available for estimating the failure rate of each system.

Reader Aids:
Purpose: Report of calculations
Special math needed for explanations: Bayesian probability
Special math needed for results: Same
Results useful to: Reliability theoreticians, statisticians

INTRODUCTION

Two of the most widely used techniques for computing lower limits for the reliability of series subsystems are the exact Bayesian limits [2, Springer and Thompson] and the fiducial limits using Monte Carlo simulation [1, Levy and Moore]. This paper presents some numerical results obtained by these two methods for identical systems. The Bayesian limits are for two different prior distributions. The uniform prior distribution was chosen because it is often used by Bayesian statisticians when they have no prior knowledge of a system. The fiducial prior distribution [3, Mann] was tried because the Monte Carlo simulation technique seems to be essentially a fiducial approach to interval estimation. These intervals are not *s*-confidence intervals in the traditional sense because we treat the life test data as fixed and compute random values of subsystem reliability based on the data.

NOTATION

t_m	given mission time for a subsystem
θ	mean time between failures
n	Monte Carlo sample size
T	total time of the life test of a subsystem
r	observed number of failures in the life test
T_0	value of T observed in previous experience with a subsystem or a similar one
r_0	value for r observed in previous experience with a subsystem or a similar one
χ^2_ν	chi-square variate with ν degrees of freedom

Fiducial Intervals Using Simulation

For the exponential failure-time distribution, the reliability is

$$R(t_m) = \text{expfc}(t_m/\theta) .$$

Now $2rT/\theta r$ is distributed as χ^2_{2r}. By treating $1/\theta$ as a random variable, fiducial intervals for the reliability of a single subsystem can be obtained.

$$R(t_m) = \text{expfc}((r\chi^2_{2r}/2r)/(T/t_m)) \qquad (1)$$

(The expression is written in this form because r and T/t_m appear in the Bayesian formulation.)

Monte Carlo lower fiducial limits for systems composed of a series of exponential subsystems are formed by the following procedure. Random values of reliability for each subsystem are generated according to (1) by choosing random values for χ^2_{2r}. A system reliability is formed by taking the product of subsystem reliabilities. This process is repeated n times. These system reliabilities are then arranged in ascending order from smallest to largest. From the theory of order statistics it is known that n order statistics partition the range of the reliability into $n + 1$ intervals, and that the probability of an additional value of the reliability being less than order statistic k is $k/(n + 1)$. We chose $n + 1$ to be 1000, and the 50-th order statistic gives an exact 95% Monte Carlo lower fiducial limit.

Exact Bayesian Intervals

Springer and Thompson [2] derived exact Bayesian confidence (sic) intervals using the Mellin integral transform. They used the general prior pdf

$$p(R) = [(T_0/t_m) + 1]^{r_0+1}[\Gamma(r_0 + 1)]^{-1}R^{T_0/t_m}[\ln(1/R)]^{r_0} .$$

When no prior experience with a subsystem exists, the uniform distribution, viz., $T_0/t_m = 0$ and $r_0 = 0$, is often used. The prior pdf corresponding to the fiducial approach is obtained by letting $T_0/t_m = r_0 = -1$, which gives a u-shaped distribution. Mann [3] has also done work with the Bayesian technique using these prior distributions.

Comparison of Results

The Table presents numerical comparisons for the 95% lower limit obtained by

1. Monte Carlo simulation; fiducial limit: R_{L1}

TABLE

95% Lower Limit R_L

no. of subsystems	data: (r, T/t_m)	R_{L1}	R_{L2}	R_{L3}
2	(4,20), (7,15)	.357	.330	.356
2	(3,100), (5,15)	.853	.830	.852
2	(1,100), (3,150)	.942	.921	.943
2	(5,10), (6,6)	.100	.097	.096
2	(3,15), (7,25)	.464	.425	.466
2	(2,20), (4,50)	.706	.661	.717
5	(1,12), (3,20), (6,50), (8,100), (5,200)	.505	.412	.495
5	(2,200), (3,225), (2,480), (5,400), (4,500)	.931	.914	.931
5	(2,9), (6,30), (3,8), (4,25), (5,20)	.167	.131	.171
5	(2,60), (7,300), (4,200), (3,120), (3,70)	.803	.762	.806
10	(3,50), (4,60), (3,100), (5,100), (2,40), (1,30), (4,75), (5,200), (2,20), (4,150)	.504	.410	.503
10	(2,210), (2,250), (1,100), (1,250), (7,1000), (1,150), (2,225), (5,1200), (2,100), (1,175)	.893	.843	.882

r = observed number of failures
T = total life test time
t_m = subsystem mission time
R_{L1} = fiducial limit, Monte Carlo simulation
R_{L2} = Bayesian limit, uniform prior distribution
R_{L3} = Bayesian limit, u-shaped prior distribution

2. Bayesian limit
   a) uniform prior distribution: R_{L2}
   b) u-shaped prior distribution: R_{L3}

These data represent a variety of situations for a range of low to high system reliabilities, with various 'numbers of observed failures' and 'test time to mission time ratios', and 2, 5, 10 subsystems. R_{L1} and R_{L3} always agree well. Although we know of no mathematical proof that the Monte Carlo simulation method used here and the Bayesian method with a u-shaped prior are equivalent, the results for our systems are equivalent. R_{L2} is consistently lower than R_{L3}; we offer no explanation.

We hope the readers will be interested in these results and correspond with us about them and perhaps investigate this subject more.

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Manuscript received September 20, 1974; revised April 24, 1975, and May 23, 1975

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