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# Scattering Of Waves In Many Dimensions

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toward $F^{(l+1)}$ which is required to satisfy $F^{(l+1)} \cdot W^{(l+1)} = 1$. The $(n+1)$th approximation to $F^{(l+1)}$ would be given by $F^{(l)} + F + \delta F$, where we try to satisfy

$$(F^{(l)} + F + \delta F)W^{(l+1)}$$
$$= (F^{(l)} + F)W^{(l+1)} + \delta F \cdot W^{(l)} + \delta F(W^{(1)} - 1) = 1.$$

But, we now assume $\delta F \cdot W^{(1)}$ to be small, and we use our knowledge of $F^{(l)}$ to write

$$\delta F = F^{(l)}[1 - (F^{(l)} + F)W^{(l+1)}]. \qquad \text{(B3)}$$

In this manner $F^{(l+1)}$ is found by iteration, and ultimately so is $F^{(L)}$.

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[1] M. C. Gutzwiller, J. Math. Phys. **8**, 1979 (1967); **10**, 1004 (1969); **11**, 1791 (1970). These papers will be referred to as I, II, and III.
[2] S. Smale, Bull. Am. Math. Soc. **73**, 747 (1967).
[3] W. Kohn and J. M. Luttinger, Phys. Rev. **96**, 1488 (1954).
[4] R. A. Faulkner, Phys. Rev. **184**, 713 (1969).
[5] S. Smale, in *Differential and Combinatorial Topology*, Symposium in Honor of Marston Morse, Stewart S. Cairns, Ed. (Princeton U.P., Princeton, N.J.), pp. 63–80.
[6] For a discussion of the monodromy matrix, cf. L. A. Pars, *A Treatise on Analytical Dynamics* (Heinemann, London, 1965), p. 461. The area preserving map near a periodic orbit is discussed in C. Siegel, *Vorlesungen über Himmelsmechanik* (Springer-Verlag, Berlin, 1956), p. 131.
[7] D. Brouwer and G. M. Clemence, *Methods of Celestial Mechanics* (Academic, New York, 1961), p. 340.
[8] W. J. Eckert and D. A. Eckert, Astron. J. **72**, 1299 (1967).
[9] For references about recent experimental work, cf. Ref. 4.


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# Scattering of Waves in Many Dimensions

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Scattering by a spherical potential is discussed in all dimensions by one formulation using the partial-wave expansion method. The optical theorem relating the total scattering cross section $\sigma$ to the forward scattering amplitude $f(0)$ is derived.

We wish to derive a formula relating the total scattering cross section to the forward-scattering amplitude which holds in all dimensions. The procedure is to use Gegenbauer's expansion[1] of a plane wave in $N$ dimensions in terms of partial waves appropriate for the space under consideration. This allows us to treat scattering in all dimensions on equal footing, without resort to special formulations, and reveals the similarities of wave motion in all dimensions.

The stationary-state Schrödinger equation is

$$[\Delta + k^2 - U(x)]\psi = 0, \qquad (1)$$

where $\Delta$ is the Laplacian operator in $N$ dimensions, $k^2 = (2m/\hbar^2)E$, $E$ is the energy, $m$ is the mass of the particle, and $U(x) = (2m/\hbar^2)V(x)$, where $V(x)$ is the potential and $x$ is the length of the $N$-dimensional position vector $\mathbf{x}$. It is thus assumed that the potential is spherically symmetric. We are interested in the scattering of a wave incident along the $z$ axis, and $\psi$ will depend on $x$ and $\theta$ where $z = x\cos\theta$. The Laplacian takes the form

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{N-1}{x}\frac{\partial}{\partial x} + \frac{N-2}{x^2}\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}. \qquad (2)$$

The wave equation (1) is separable, and the solutions are partial waves of the form $f_n(x)C_n^\alpha(\cos\theta)$, where $f_n(x)$ is the radial solution and $C_n^\alpha(\cos\theta)$ is the $N$-dimensional spherical harmonic of Gegenbauer, in which $n$ is the degree of the polynomial and $\alpha = \frac{1}{2}N - 1$.

These harmonics are defined by the generating function[2,3]

$$(1 - 2h\cos\theta + h^2)^{-\alpha} = \sum_{n=0}^{\infty} h^n C_n^\alpha(\cos\theta) \qquad (3)$$

and satisfy the differential equation

$$\left(\frac{d^2}{d\theta^2} + (N-2)\cot\theta\frac{d}{d\theta} + n(N+n-2)\right)C_n^\alpha(\cos\theta) = 0. \qquad (4)$$

By substituting (4) and (2) into (1) we obtain the radial equation

$$\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx} + k^2 - n(n+2\alpha)x^{-2} - U(x)\right)f_n(x) = 0. \qquad (5)$$

The substitution $f_n(x) = x^{-\alpha}\phi_n(x)$ gives

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + k^2 - (n+\alpha)^2x^{-2} - U(x)\right)\phi_n(x) = 0. \tag{6}$$

The asymptotic solution of (6) is

$$\phi_n(x) \sim Z_{n+\alpha}(kx), \tag{7}$$

where $Z$ is a general Bessel function.

The solution $\psi$ which is of interest must consist of an incident wave $\exp(ikx\cos\theta)$ and a scattered wave. Now, a plane wave can be resolved into partial waves by the Gegenbauer expansion formula:

$$e^{ikx\cos\theta} = 2^\alpha\Gamma(\alpha)\sum_{n=0}^{\infty}(\alpha+n)i^n(kx)^{-\alpha}J_{n+\alpha}(kx)C_n^\alpha(\cos\theta), \tag{8}$$

and, since $\psi$ must have the asymptotic behavior of a plane wave plus outgoing partial waves, we can write the asymptotic formula

$$\psi \sim 2^\alpha\Gamma(\alpha)\sum_{n=0}^{\infty}(\alpha+n)i^n(kx)^{-\alpha}C_n^\alpha(\cos\theta) \times \tfrac{1}{2}\{e^{i\delta_n}h_{n+\alpha}^{(1)}(kx) + e^{-i\delta_n}h_{n+\alpha}^{(2)}(kx)\}e^{i\delta_n}, \tag{9}$$

where $\delta_n$ is the phase shift for the $n$th partial wave and $h_\nu^{(1)}$ and $h_\nu^{(2)}$ are Hankel's functions of the first and second kind. In writing (9),the well-known asymptotic behavior of Bessel functions was used, namely,[4]

$$\begin{aligned} J_\nu(x) &\sim (2/\pi x)^{\frac{1}{2}}\cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi),\\ h_\nu^{(1)}(x) &\sim (2/\pi x)^{\frac{1}{2}}e^{i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)},\\ h_\nu^{(2)}(x) &\sim (2/\pi x)^{\frac{1}{2}}e^{-i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}. \end{aligned} \tag{10}$$

By subtracting (8) from (9) we obtain the asymptotic behavior of the scattered wave $\psi_s$, namely,

$$\begin{aligned} \psi_s &\sim 2^\alpha\Gamma(\alpha)\sum_{n=0}^{\infty}(\alpha+n)i^n(kx)^{-\alpha}(2/\pi kx)^{\frac{1}{2}}\\ &\quad\times C_n^\alpha(\cos\theta)e^{i(kx-\frac{1}{2}(n+\alpha)\pi-\frac{1}{4}\pi)}\tfrac{1}{2}(e^{2i\delta_n}-1)\\ &\equiv i(ix)^{-(\alpha+\frac{1}{2})}e^{ikx}f(\theta), \end{aligned} \tag{11}$$

which also defines the scattering amplitude $f(\theta)$.

The optical theorem relates the total cross section $\sigma = \int |f(\theta)|^2\,d\Omega$ to the forward scattering amplitude $f(0)$. The cross section $\sigma$ can be obtained from (11) by remembering that[3]

$$\int_0^\pi \sin^{2\alpha}\theta C_m^\alpha(\cos\theta)C_n^\alpha(\cos\theta)\,d\theta = \frac{\pi\Gamma(2\alpha+n)}{2^{2\alpha-1}(\alpha+n)n!\,[\Gamma(\alpha)]^2}\delta_{m,n} \tag{12}$$

and that in the integration over solid angle the contribution of the azimuthal angles $(\phi_1, \phi_2, \cdots, \phi_{N-2})$ gives the factor $\Omega_\phi$, where

$$\begin{aligned} \Omega_\phi &= 2\pi^{\alpha+\frac{1}{2}}/\Gamma(\alpha+\tfrac{1}{2}), & \alpha \geq \tfrac{1}{2},\\ &= 1, & \alpha < \tfrac{1}{2}, \end{aligned} \tag{13}$$

and thus we obtain

$$\sigma = \sum_{n=0}^{\infty}\frac{8\pi^{\alpha+\frac{1}{2}}(\alpha+n)\Gamma(2\alpha+n)}{n!\,k^{2\alpha+1}\Gamma(\alpha+\frac{1}{2})}\sin^2\delta_n. \tag{14}$$

For $\theta = 0$, Eq. (3) gives

$$C_n^\alpha(1) = (n!)^{-1}\Gamma(2\alpha+n)/\Gamma(2\alpha), \tag{15}$$

which enables us to deduce from Eq. (11) that

$$\operatorname{Im} f(0) = 2^{\alpha+\frac{1}{2}}\pi^{-\frac{1}{2}}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}k^{-(\alpha+\frac{1}{2})} \times \sum_{n=0}^{\infty}\frac{(\alpha+n)\Gamma(2\alpha+n)(\sin^2\delta_n)}{n!}. \tag{16}$$

By comparing (14) and (16) we obtain the optical theorem:

$$\sigma = 8\pi^{\alpha+1}(2k)^{-(\alpha+\frac{1}{2})}[\Gamma(2\alpha)/\Gamma(\alpha)][\Gamma(\alpha+\tfrac{1}{2})]^{-1}[\operatorname{Im} f(0)]. \tag{17}$$

In three dimensions, $\alpha = \frac{1}{2}$, and we have the well-known result $\sigma = (4\pi/k)\operatorname{Im} f(0)$. In two dimensions $\alpha = 0$ and

$$\sigma(N=2) = 2(2\pi/k)^{\frac{1}{2}}[\operatorname{Im} f(0)]. \tag{18}$$

In one dimension $\alpha = -\frac{1}{2}$ and

$$\begin{aligned} \lim_{\alpha\to-\frac{1}{2}}\frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})} &= \lim_{\alpha\to-\frac{1}{2}}\frac{2\alpha\Gamma(2\alpha)(\alpha+\frac{1}{2})}{2\alpha\Gamma(\alpha)(\alpha+\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}\\ &= \tfrac{1}{4}[\Gamma(2\alpha+2)/\Gamma(\alpha+1)][\Gamma(\alpha+\tfrac{3}{2})]^{-1}\\ &= \tfrac{1}{4}\pi^{-\frac{1}{2}} \end{aligned}$$

(which also could have been foreseen from the duplication formula for the gamma function), and thus we obtain

$$\sigma = 2\operatorname{Im} f(0), \quad N = 1, \tag{19}$$

a result which is equivalent to that obtained by Eberly,[5] since our $f(0)$ as defined by Eq. (11) is $-i$ times the $f(0)$ used by Eberly.

Other applications of the Gegenbauer expansion formula have been given elsewhere.[6]

[1] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge U.P., Cambridge, 1966).
[2] Reference 1, p. 129.
[3] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic, New York, 1949), Appendix IV.
[4] Reference 1, p. 198.
[5] J. H. Eberly, Am. J. Phys. **33**, 771 (1965).
[6] I. Adawi, Phys. Rev. **146**, 379 (1966); Phys. Letters **26A**, 317 (1968).