Time-domain Analysis And Measurement Techniques For Distributed RC Structures. II. Impulse Measurement Techniques

Robert C. Peirson
Missouri University of Science and Technology

Edward C. Bertnolli
Missouri University of Science and Technology

RESEARCH ARTICLE | NOVEMBER 17 2003

Time-Domain Analysis and Measurement Techniques for Distributed *RC* Structures. II. Impulse Measurement Techniques

Robert C. Peirson; Edward C. Bertnolli

Journal of Applied Physics 40, 118–122 (1969)
https://doi.org/10.1063/1.1657014

Time-Domain Analysis and Measurement Techniques for Distributed *RC* Structures. II. Impulse Measurement Techniques*

ROBERT C. PEIRSON AND EDWARD C. BERTNOLLI
University of Missouri at Rolla, Rolla, Missouri 65401

(Received 10 April 1968; in final form 28 August 1968)

Transient analysis for uniform *RC* structures is considered in this paper. A method is presented for determining the parameters of such structures. The measurements are obtained using impulse excitations in open-circuit and short-circuit configurations. The theoretical results obtained predict fairly the experimental results.

INTRODUCTION

The development of thin-film, *RC* microelectronic circuits offers some important distributed-parameter network analysis and measurement problems. Sinusoidal steady-state analyses of these networks have been performed by Happ and Castro[1] and Bertnolli and Halijak.[2] Delay and rise time calculations have been presented by Protonotarios and Wing.[3] Transient analysis in terms of Poisson-derived functions in the reciprocal time domain has been developed by Happ and Gupta.[4]

This paper reviews transient analysis for uniform *RC* networks and presents a method for determining the parameters of an *RC* network. These parameters, *R* and *C*, are measured using impulse excitation with either open or short-circuit terminations.

THE OPEN-CIRCUITED *RC* MICROCIRCUIT

The impulsive response of a distributed-parameter *RC* network is found using the network's transfer function, which is derived from the telegrapher's equations as presented by Happ and Gupta.[4] For an open-circuited uniform *RC* network (see Fig. 1), driven by an ideal voltage generator, the voltage transfer function is

$$V(d,s)/I(0,s)=\operatorname{sech}(sRC)^{1/2}. \tag{1}$$

From this, the impulsive response is found to be

$$V(d,s)=\operatorname{sech}(sRC)^{1/2}, \tag{2}$$

where R is the total distributed resistance and C is the total distributed capacitance, i.e., $R=rd$ and $C=cd$, where r and c are the per unit length parameters. The inverse Laplace transform of Eq. (2) in the time domain is[5]

$$L^{-1}[\operatorname{sech}(s^{1/2}a)]=-(1/a^2)[(\partial/\partial\nu)\theta_1(\nu/2,t/a^2)]_{\nu=0}, \tag{3}$$

where $\theta_1(\nu/2,t/a^2)$ is the theta function defined by,[5] or

$$\theta_1(\nu/2,t/a^2)=[a/(\pi t)^{1/2}]\sum_{-\infty}^{\infty}(-1)^n \times\exp\{(-a^2/t)[(\nu/2)-(1/2)+n]^2\}. \tag{4}$$

Performing the differentiation indicated in (3) yields

$$v(d,t)=L^{-1}[\operatorname{sech}(sRC)^{1/2}] =[-(RC)^{1/2}/\pi^{1/2}t^{3/2}]\sum_{-\infty}^{\infty}(-1)^n(n-\tfrac{1}{2}) \times\exp[(-RC/t)(n-\tfrac{1}{2})^2] \tag{5}$$

for the inverse Laplace transform of (2). Replacing t with $1/r$ casts Eq. (5) into the reciprocal time domain, giving

$$v(d,r)=[r^{3/2}(RC)^{1/2}/\pi^{1/2}]\sum_{-\infty}^{\infty}(-1)^{n+1}(n-\tfrac{1}{2}) \times\exp[-RC(n-\tfrac{1}{2})^2r]. \tag{6}$$

This series (6) can also be expressed as

$$v(d,r)=[r^{3/2}(RC)^{1/2}/\pi^{1/2}]\sum_{0}^{\infty}2(-1)^n(n+\tfrac{1}{2}) \times\exp[-RC(n+\tfrac{1}{2})^2r]. \tag{7}$$

Plotting the first three terms of series (7) with $(RC)^{1/2}=1$ for $n=0$, 1, and 2 vs r generates Fig. 2. It is obvious from this figure that rapid convergence is obtained for large r or small t. An increasing number of terms is needed for r small or t large. This has been shown by Happ and Gupta.[4]

At this point, series (6) is transformed into a form which yields rapid convergence for r small or t large. This new form along with Eq. (7) describes the response of an *RC* microcircuit for both t small and t large.

This series transformation can be derived by the use of Poisson's sum formula[6]

$$\sum_{-\infty}^{\infty}f(x+n)=\sum_{-\infty}^{\infty}\exp(2\pi ikx)\int_{-\infty}^{\infty}f(\xi)\exp(-2\pi ik\xi)d\xi. \tag{8}$$

* Portion of a thesis written in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering at the University of Missouri at Rolla.
[1] P. S. Castro and W. W. Happ, *Proc. Nat. Electron. Conf.*, **16**, 448 (1960).
[2] E. C. Bertnolli and C. A. Halijak, 1966 IEEE Int. Conv. Rec. Pt. 7, 243 (1966).
[3] E. N. Protonotarios and O. Wing, 1965 IEEE Int. Conv. Bd. Pt. 7, 1–6 (1965).
[4] W. W. Happ and S. C. Gupta, J. Appl. Phys. **40**, 109 (1969), preceding paper.
[5] G. E. Robert and H. Kaufman, *Tables of Laplace Transforms* (W. S. Saunders Co., Philadelphia, Pa., 1966).
[6] R. Bellman, *A Brief Introduction to Theta Functions* (Holt Reinhart and Winston, Inc., New York, 1961).

Choosing

$$f(x)=[(RC)^{1/2}/\pi^{1/2}](-1)^{x+1}(x-\tfrac{1}{2})r^{3/2}\exp[-RC(x-\tfrac{1}{2})^2r], \tag{9}$$

and substituting this into Eq. (8) yields

$$\sum_{-\infty}^{\infty}f(x+n)=\sum_{n=-\infty}^{\infty}[(RC)^{1/2}/\pi^{1/2}](-1)^{x+n+1}(x+n-\tfrac{1}{2})r^{3/2}\exp[-RC(x+n-\tfrac{1}{2})^2r]$$

$$=\sum_{n=-\infty}^{\infty}\exp(2\pi ikx)\int_{-\infty}^{\infty}[(RC)^{1/2}/\pi^{1/2}](-1)^{\xi+1}(\xi-\tfrac{1}{2})r^{3/2}\exp[-RC(\xi-\tfrac{1}{2})^2r-2\pi ik\xi]d\xi. \tag{10}$$

Setting $x=0$ in Eq. (10) gives

$$\sum_{-\infty}^{\infty}[(RC)^{1/2}/\pi^{1/2}](-1)^{n+1}(n-\tfrac{1}{2})r^{3/2}\exp[-(n-\tfrac{1}{2})^2rRC]=\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(RC)^{1/2}/\pi^{1/2}](-1)^{\xi+1}(\xi-\tfrac{1}{2})r^{3/2}$$
$$\times\exp[-RC(\xi-\tfrac{1}{2})^2r-2\pi ik\xi]d\xi. \tag{11}$$

The left-hand side of Eq. (11) may be recognized as series (6), $v(d, r)$. Performing the integration indicated in the right-hand side of Eq. (11) gives

$$\sum_{-\infty}^{\infty}[(RC)^{1/2}/\pi^{1/2}](-1)^{n+1}(n-\tfrac{1}{2})r^{3/2}\exp[-RC(n-\tfrac{1}{2})^2r]=\sum_{-\infty}^{\infty}(-1)^{n+1}[(n-\tfrac{1}{2})\pi/(RC)^{1/2}]$$
$$\times\exp[-(n-\tfrac{1}{2})^2\pi^2/rRC], \tag{12}$$

for $v(d, r)$.

Another form of this series is

$$v(d,r)=\sum_{0}^{\infty}[2\pi/(RC)^{1/2}](-1)^n(n+\tfrac{1}{2})\exp[-(n+\tfrac{1}{2})^2\pi^2/rRC]. \tag{13}$$

Plotting the first three terms of the series of Eq. (13), with $(RC)^{1/2}=1$ for $n=0$, 1, and 2, vs r generates Fig. 3. As can be seen from Fig. 3, this series converges rapidly for r small, t large.

Equations (7) and (13) are expressions for the impulsive response of an RC microcircuit that converge rapidly for t small and t large respectively. Using first term approximations gives

$$v(d,t)\doteq[(RC)^{1/2}/\pi^{1/2}t^{3/2}]\exp(-RC/4t)\qquad\text{for small } t, \tag{14a}$$

and

$$v(d,t)\doteq[\pi/(RC)^{1/2}]\exp(-\pi^2t/4RC)\qquad\text{for large } t. \tag{14b}$$

Equations (14) is referred to later in the section on experimental parameter determination and is summarized in Table I.

THE SHORT-CIRCUITED RC MICROCIRCUIT

Consider an RC microcircuit driven by an ideal voltage generator and terminated with a short circuit. The input voltage, output current transfer function is

$$I(d,s)/V(0,s)=(sC/R)^{1/2}\operatorname{csch}(RCs)^{1/2}. \tag{15}$$

Again, since the impulse response is being considered, let $V(0, s)=1$. Equation (15) is now written

$$I(d,s)=(sC/R)^{1/2}\operatorname{csch}(RCs)^{1/2}. \tag{16}$$

The inverse Laplace transform of Eq. (16) is

$$L^{-1}[s^{1/2}\operatorname{csch}(RCs)^{1/2}]=(d/dt)L^{-1}[\operatorname{csch}(RCs)^{1/2}/s^{1/2}]+L^{-1}[\operatorname{csch}(sRC)^{1/2}/s^{1/2}]\,|_{t=0} \tag{17}$$

$$=(d/dt)\{[1/(RC)^{1/2}]\theta_4(0,t/RC)\}+[1/(RC)^{1/2}]\theta_4(0,t/RC)\,|_{t=0}, \tag{18}$$

where

$$\theta_4(\nu,x)=[1/(\pi x)^{1/2}]\sum_{-\infty}^{\infty}\exp[(-1/x)(\nu+\tfrac{1}{2}+n)^2] \tag{19}$$

is a theta function.

Substitution of $\theta_4(\nu, x)$ from Eq. (19) into Eq. (18) yields

$$i(d,t)=(C/R\pi)^{1/2}\{(2/t^{5/2})\sum_{0}^{\infty}RC(n+\tfrac{1}{2})^2\exp[-RC(n+\tfrac{1}{2})^2/t]-(1/t^{3/2})\sum_{0}^{\infty}[RC(n+\tfrac{1}{2})^2/t]\}. \tag{20}$$

Series (19) is an expression that is useful (has good convergence) for r large or t small.

A rapidly converging expression for r small, t large can be found using Equation (20) and a different representation of the theta function. This form of the theta function, derived using Poisson's sum formula as in the first part of this paper, is[7]

$$\theta_4(\nu,\xi)=\sum_{-\infty}^{\infty}\exp[-\pi n^2\xi+2\pi in(\nu+\tfrac{1}{2})]. \tag{21}$$

Let

$$\xi=\pi t/a^2$$

and

$$\nu=0,$$

then

$$\theta_4(0,t/a^2)=\sum_{-\infty}^{\infty}(-1)^n\exp(-\pi^2n^2t/a^2). \tag{22}$$

Therefore, from

$$i(d,t)=(d/dt)\{[1/(RC)^{1/2}]\theta_4(0,t/RC)\}+[1/(RC)^{1/2}]\theta_4(0,t/RC)\,|_{t=0}, \tag{23}$$

$$i(d,t)=(C/R)[\pi^2/RC(RC)^{1/2}]\sum_{-\infty}^{\infty}n^2(-1)^{n+1}\exp(-n^2\pi^2t/RC). \tag{24}$$

Casting this into the reciprocal time domain with $t=1/r$ yields

$$i(d,r)=(\pi^2/R^2C)\sum_{-\infty}^{\infty}n^2(-1)^{n+1}\exp(-n^2\pi^2/RCr). \tag{25}$$

This expression can also be written

$$i(d,r)=(2\pi^2/R^2C)\sum_{-\infty}^{\infty}n^2(-1)^{n+1}\exp(-n^2\pi^2/RCr). \tag{26}$$

Convergence is rapid for r small, t large. Using one term (for $n=1$),

$$i(d,r)\doteq(2\pi^2/R^2C)\exp(-\pi^2/RCr). \tag{27}$$

The preceeding analysis provides rapidly converging expressions for both t large and t small for the short-circuited distributed-parameter RC network.

These expressions are:

$$i(d,t)\doteq(C/R\pi)^{1/2}[-(1/2t^{3/2})\exp(-RC/4t)+(RC/4t^{5/2})\exp(-RC/4t)] \quad \text{for } t \text{ small}, \tag{28a}$$

and

$$i(d,t)\doteq(2\pi^2/R^2C)\exp(-\pi^2t/RC) \quad \text{for } t \text{ large}. \tag{28b}$$

Equation (22) is used in the following section on experimental determination of parameters, and is summarized in Table I.

THE CHARACTERISTIC IMPEDANCE TERMINATION

The preceeding sections have been concerned with terminal characteristics and terminations that are easy to obtain: i.e., short circuit and open circuit. In this section the characteristic impedance termination is considered. This termination serves to illustrate the behavior of the line, since there are no reflected "waves." In the previous section the dependence on the distance variable, x, is suppressed. In this section the dependence on x is explicit: Therefore, r and c are "per unit" parameters instead of the "total" parameters R and C used before.

The impulsive response, at any point x, is

$$V(x,s)=\exp[-(src)^{1/2}x].$$

The inverse Laplace transformation yields

$$v(x,t)=[(RC)^{1/2}x/4\pi^{1/2}t^{3/2}]\exp(-RCx^2/4t).$$

[7] B. Van Der Pol and H. Bremmer, *Operational Calculus Based on Two Sided Laplace Transforms* (Cambridge University Press, Cambridge, England, 1955), 2nd Ed.

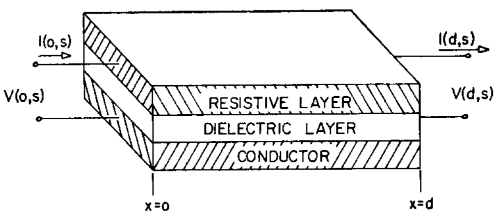


FIG. 1. The uniform RC microcircuit.

This "wave" is plotted in Fig. 4 on a three-dimensional axis, with amplitude vs both distance and time.

The three-dimensional representation of this "wave" clearly illustrates the propagation (note that the peak moves in the positive x direction) and the attenuation of the line. The operation of this RC line is described by the heat or diffusion equation. Therefore, heat flow and diffusion may be described through an analogy to voltage response on an RC line.

EXPERIMENTAL DETERMINATION OF *RC* MICROCIRCUIT PARAMETERS

The transient analysis reviewed in the preceeding sections is now used to develop a procedure for experimentally determining the distributed-network parameters R and C. This knowledge of R and C provides a complete description of the RC microcircuit.

Table I lists responses for both open-circuited and short-circuited RC networks with both impulse voltage and impulse current excitation. The impulse current excitation response is found by using procedures similar to those shown in the preceeding sections of this paper. This response is included here because in actual practice, it may be easier to obtain an ideal current generator than an "ideal" voltage generator.

The procedure is to record open-circuit RC network voltage responses and short-circuit RC network current responses to either impulsive current or voltage excitations. These recorded impulsive responses are then compared with the predicted response (see Table I) in a least-squares sense to obtain the parameters R and C.

The general form of the responses listed in Table I are

$$f(t) = \alpha e^{-\beta t}. \tag{29}$$

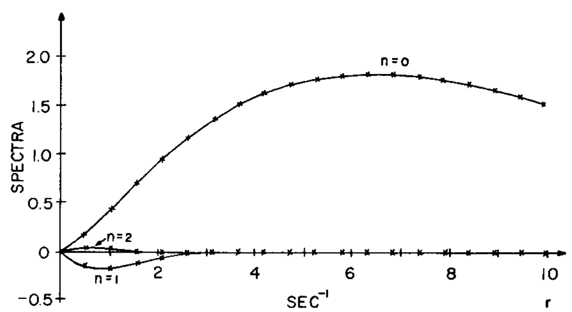


FIG. 2. Impulse-response spectra in the reciprocal time domain.

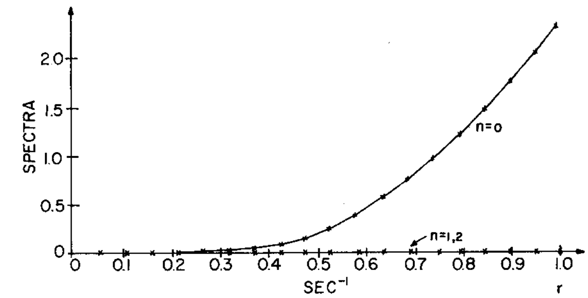


FIG. 3. Impulse-response spectra in the reciprocal time domain.

A curve may be fit into Eq. (23) by first linearizing it,

$$\ln f(t) = \ln\alpha - \beta t.$$

Now, form the difference between corresponding values of observed values (y_i, t_i), and the theoretical values:

$$\ln f(t_i) = (\ln\alpha - \beta t_i). \tag{30}$$

The sum of the squared deviations is

$$s(\alpha, \beta) = \sum_{i=1}^{n} (y_i - \ln\alpha - \beta t_i)^2. \tag{31}$$

Then

$$\partial s/\partial\alpha = 2 \sum_{i=1}^{n} (y_i - \ln\alpha + \beta t_i)(-1/\alpha), \tag{32}$$

$$\partial s/\partial\beta = 2 \sum_{i=1}^{n} (y_i - \ln\alpha + \beta t_i) t_i. \tag{33}$$

Equating these partial derivatives to zero determines the solutions a and b (the least-squares estimates of α and β) of the equations

$$\sum_{i=1}^{n} y_i = n \ln a - b \sum_{i=1}^{n} t \tag{34}$$

and

$$\sum_{i=1}^{n} y_i = n \ln a \sum_{i=1}^{n} t_i - b \sum_{i=1}^{n} t_i^2. \tag{35}$$

The observed data (y_i, t_i) in Eq. (24) is taken from

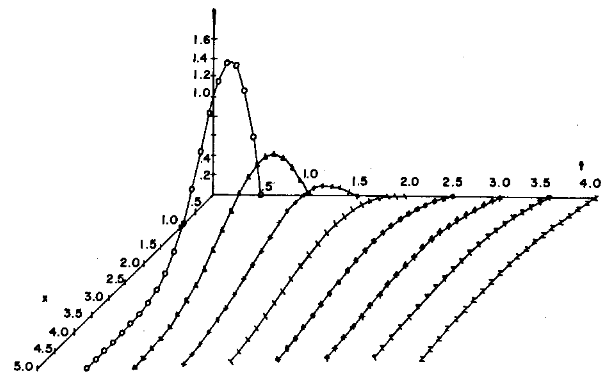
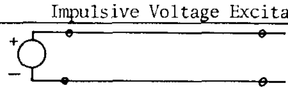
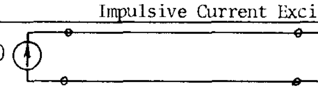
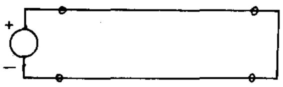


FIG. 4. The impulsive response of a matched RC microcircuit (three-dimensional view).

TABLE I. Impulse response for open-circuit and short-circuit configurations.

Impulsive Voltage Excitation	Impulsive Current Excitation
$\delta(t)$  Open circuit For Small t: $v(d,t) \doteq \frac{\sqrt{RC}}{\sqrt{\pi}\, t^{3/2}} e^{-\frac{RC}{4t}}$ For Large t: $v(d,t) \doteq \frac{\pi}{\sqrt{RC}} e^{-\frac{\pi^2 t}{4RC}}$	$\delta(t)$  Open Circuit For Small t: $v(d,t) \doteq \sqrt{\frac{R}{C}} \frac{2}{\sqrt{\pi t}} e^{-\frac{RC}{4t}}$ For Large t: $v(d,t) \doteq \sqrt{\frac{R}{C}} \{1-2 e^{-\frac{\pi^2 t}{RC}}\}$
$\delta(t)$  Short Circuit For Small t: $i(d,t) \doteq \sqrt{\frac{C}{R\pi}} \{\frac{1}{2t^{3/2}} + \frac{RC}{4t^{5/2}}\} e^{-\frac{RC}{4t}}$ For Large t: $i(d,t) = \frac{2\pi^2}{R^2 C} e^{-\frac{\pi^2 t}{RC}}$	$\delta(t)$  Short Circuit For Small t: $i(d,t) \doteq \frac{\sqrt{RC}}{\sqrt{\pi}\, t^{3/2}} e^{-\frac{RC}{4t}}$ For Large t: $i(d,t) \doteq \frac{\pi}{\sqrt{RC}} e^{-\frac{\pi^2 t}{4RC}}$

an oscillograph of the impulsive response of the RC microcircuit. These data are then compared with the appropriate curve from Table I (depending on the conditions of the test, i.e., open circuit or short circuit and voltage or current excitation). A detailed explanation of this least-square approximation is found in Ref. 8.

It is to be noted that to completely determine R and C, two sets of measurements must be made. For example, if the line is excited with a voltage source, the approximate open-circuit response, for t small, is

$$v(d,t) \doteq [(RC)^{1/2}/\pi t^{3/2}] \exp(-RC/4t). \qquad (36)$$

From the least-squares approximation, the term RC can be determined.

Now a short-circuit test with the same source should be performed. The approximate response here, from Table I, is

$$i(d,t) \doteq (2\pi^2/R^2C) \exp(-\pi^2 t/RC), \qquad (37)$$

for t large.

[8] R. G. Stanton, *Numerical Methods for Science and Engineering* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1961).

The least-squares approximation will give an estimate of R^2C. From these two measurements, R and C may now be determined.

CONCLUSION

The impulsive response of RC distributed networks has been reviewed. It is shown that because of rapid convergence for t large or t small, one term of the resulting response expression is adequate to describe the impulsive response at these extremes.

The parameters of the RC network may be determined by measuring the impulsive response for either open- or short-circuit termination and comparing it (in a least-squares sense) with the expressions presented herein.

ACKNOWLEDGMENTS

The authors wish to express their gratitude to W. W. Happ for posing the measurement problem undertaken in this paper and to NASA-ERC for research support under Grant NGR-26-003-037.