01 Jan 1967

Tolerance Regions For A Joint Exponential Distribution

Lee J. Bain
Missouri University of Science and Technology, ljbain@mst.edu

Recommended Citation

L. J. Bain, "Tolerance Regions For A Joint Exponential Distribution," *IEEE Transactions on Reliability*, vol. R thru 16, no. 3, pp. 111 - 113, Institute of Electrical and Electronics Engineers, Jan 1967.
The definitive version is available at https://doi.org/10.1109/TR.1967.5217478

Tolerance Regions for a Joint Exponential Distribution

LEE J. BAIN

Abstract—**The evaluation of the reliability of a system of components, when the components are assumed to follow a joint exponential distribution, is considered. The approach used is to develop tolerance regions for the joint exponential distribution or to estimate the probability content of the appropriate specification region.**

Introduction

CONSIDER a system of k components, and let x_i denote the variable measured on the ith component. Suppose that the system will function properly if the response of the components $(x_1, \ldots, x_k)$ falls within a certain specification region S. Thus, the reliability of the system is represented by the probability that $(x_1, \ldots, x_k)$ falls in S. Suppose a sample of n observations $x_{(j)} = (x_{1j}, \ldots, x_{kj})$ $j = 1, \ldots, n$ is to be taken, and a confidence statement concerning the reliability of the system is desired. A common procedure in a problem of this type is to determine the number of system successes in a sample of size n, then the probability of success is the parameter of a binomial distribution, and a confidence limit for this parameter can be easily determined. To use the variable's data from components to increase precision, it is necessary to assume some functional form for the distribution function of the variables. As an example, a joint exponential density function given by

$$f(x_1, \ldots, x_k) = \left(1/\prod_{i=1}^{k} \theta_i\right) \exp\left[-\sum_{i=1}^{k} x_i/\theta_i\right], \quad 0 < x_i < \infty$$

is now considered.

Now, determining a confidence limit for the probability of a system success $P[(x_1, \ldots x_k)\epsilon S]$ is similar to a tolerance region problem. A set R is called a (γ, β) tolerance region if $P\{P[(x_1, \ldots, x_k)\epsilon R] \geq \beta\} = \gamma$. That is, at least 100 β percent of the population will fall in the tolerance region, with γ confidence. Thus, one procedure would be to calculate a (γ, β) tolerance region and see if it falls within the specification region. If it does, the confidence would be at least γ that the probability of success is at least β. Of course, the shape of the tolerance region would need to correspond closely to the shape of the specification region in order not to be too conservative. Since the size and shape of a tolerance region usually depends on the sample, a reasonable procedure might be to require the tolerance region to correspond to the specification region, and then calculate the corresponding content β (which would depend on the sample result) as an estimate of the content of the specification region.

Manuscript received October 11, 1966. The work reported here was supported by the U. S. Atomic Energy Commission and was performed while the author was with the Sandia Corporation, Albuquerque, N. Mex.

The author is with the University of Missouri, Rolla, Mo.

Derivation of Tolerance Regions

Case 1. Specification region $S_L = \{(x_1, \ldots, x_k)|x_1 \geq a_1, \ldots, x_k \geq a_k\}$

A lower (γ, β) tolerance region of the form

$$R_L = \{(x_1, \ldots, x_k)|x_1 \geq \ell_1, \ldots, x_k \geq \ell_k\} = \{(\ell_1, \infty), \ldots, (\ell_k, \infty)\}$$

is needed to correspond to the specification region S_L. Let $l_1 = c_i n\bar{x}_i$, $i = 1, \ldots, k$, and let $\chi^2(\nu)$ denote a chi-square variable with ν degrees of freedom. Also, let $P[\chi^2(\nu) \leq \chi^2_\alpha(\nu)] = \alpha$. It can be seen that

$$P = \int_{\ell_1}^{\infty} \cdots \int_{\ell_k}^{\infty} f(x_1, \ldots, x_k)dx_1 \ldots dx_k \geq \beta$$

if $\sum_{i=1}^{k} \ell_i/\theta_i \leq -\ln \beta$, thus

$$\begin{aligned} P\{P[(x_1, \ldots, x_k)\epsilon R_L] \geq \beta\} &= P\left[\sum_{i=1}^{k} \ell_i/\theta_i \leq -\ln \beta\right] \\ &= P\left[\sum_{i=1}^{k} c_i n\bar{x}_i/\theta_i \leq -\ln \beta\right] \\ &= P\left[\sum_{i=1}^{k} c_i \chi_i^2(2n) \leq -\ln \beta\right]. \end{aligned}$$

The exact distribution of a weighted sum of independent chi-square variables, with even integer degrees of freedom, is given by Box.[1] In our notation, let $Y = \sum_{i=1}^{k} c_i\chi_i^2(2n)$, then

$$P[Y > Y_0] = \sum_{i=1}^{k}\sum_{s=1}^{n} \alpha_{is} P[\chi^2(2s) > Y_0/c_i] \quad (1)$$

where $\alpha_{is} = f_i^{(n-s)}(0)/(n-s)!$, $f_i^{(h)}(0)$ is obtained by differentiating $f_i(z)$ h times with respect to z and then putting $z = 0$, and

$$f_i(z) = \prod_{j \neq i}^{k} \left[\frac{c_i - c_j}{c_i} + z\, c_j/c_i\right]^{-n}.$$

Box also considered the accuracy of a simple approximation to the distribution, which should be quite adequate for our problem, especially since the degrees of freedom are constant. In our notation, Y is approximately distributed as $g\chi^2(h)$, where

$$g = \left(\sum_{i=1}^{k} c^2_i\right)\Big/\sum_{i=1}^{k} c_i, \; h = 2n\left(\sum_{i=1}^{k} c_i\right)^2\Big/\sum_{i=1}^{k} c^2_i. \quad (2)$$

Thus, to obtain a prescribed (γ, β) tolerance region, the c_i must be chosen to satisfy $g\chi_\gamma{}^2(h) = -\ln \beta$, where g and h are given by (2). Since choosing the c_i constant would maximize the effective degrees of freedom h, this would probably be the best procedure unless some other shape is particularly desirable. In this case the exact value of c is given by $c = (-\ln \beta)/\chi_\gamma{}^2(2nk)$, and letting $\ell_i = cn\bar{x}_i$ provides a (γ, β) tolerance region R_L.

If a particular specification region S_L is being considered, then the probability content of S_L (relative to a prescribed confidence γ) can be estimated. For a given sample result, let $c_i = a_i/n\bar{x}_i$, then the estimated content of S_L is given by $\beta^* = \exp[-g\chi_\gamma{}^2(h)]$, where g and h are given by (2). That is, on the average, over samples, the confidence is γ that at least 100 β percent of the systems will operate successfully. Also, the calculated β^* could be compared to a previously determined value, to provide a reject or accept type solution.

Case 2. $S_U = \{(0, b_1), \ldots, (0, b_k)\}$

Considerable difficulty is encountered in deriving an upper (γ, β) tolerance region of the form $R_U = \{(0, c_1n\bar{x}_1), \ldots, (0, c_kn\bar{x}_k)\}$ as is needed for this case. It is necessary to determine the c_i such that

$$P\left\{\prod_{i=1}^{k} [1 - \exp(-c_i\bar{x}_i/\theta_i)] \geq \beta\right\} = \gamma.$$

This can be reduced to the problem of solving

$$P\left[\prod_{i=1}^{k} (1 - w_i^{c_i}) \geq \beta\right] = \gamma$$

where the w_i are independent random variables with density

$$f(w_i) = (-\ln w_i)^{n-1}/(n-1)!\ 0 < w_i < 1.$$

This problem could be solved for a particular case with the aid of a computer.

An approximate joint upper tolerance region can also be obtained from the individual tolerance limits on the components. It is well known that $\ell_i = -2n\ln(1-\beta_i)\bar{x}_i/\chi^2_{1-\gamma^{1/k}}(2n)$ provides a $(\gamma^{1/k}, \beta_i)$ upper tolerance limit for the ith component. If the variables are independent, it follows that $R_u = \{(0, \ell_1), \ldots, (0, \ell_k)\}$ is an upper (γ, β) joint tolerance region, where $\prod_{j=1}^{k} \beta_i \leq \beta \leq \min \beta_i$. In particular, choosing $\beta_i = \beta$ provides a conservative (γ, β^k) upper tolerance region, where the true content is between β^k and β.

It may be of interest to compare approximate lower tolerance regions obtained by this method to the exact lower tolerance regions given earlier. Suppose individual $(\gamma^{1/k}, \beta)$ lower tolerance limits are used, then the exact content of the corresponding joint tolerance region is obtained by considering the exact lower tolerance region given earlier for constant c_i. The true content β_t is given by $\beta_t = \beta^d$, where $d = \chi_\gamma{}^2(2nk)/\chi^2_{\gamma^{1/k}}(2n)$. Table I gives the values of d for several values of n, k, and γ.

Case 3. $S_u' = \{(x_1, \ldots, x_k) | (x_1, \ldots, x_k) \in [(a_1, \infty), \ldots (a_k, \infty)]\}$

It may be worth noting that the complement of a lower tolerance region would be one form of an upper tolerance region which could be obtained easily, and would be appropriate for the specification region S_u'. It follows that a (γ, β) tolerance region

$$R_u' = \{(x_1, \ldots, x_k) | (x_1, \ldots, x_k) \in [(\ell_1, \infty), \ldots, (\ell_k, \infty)]\}$$

is given by replacing γ by $1 - \gamma$ and β by $1 - \beta$ in the lower tolerance region R_L given earlier.

Case 4. $S_u'' = \left\{(x_1, \ldots, x_k) \middle| \sum_{i=1}^{k} b_ix_i \leq b\right\}.$

A specification region similar to S_u'' may sometimes be more appropriate than a rectangular region, or a combination of the two types may be applicable. For example, a reasonable specification region for $k = 2$ might be

$$S = \{(x_1, x_2) | x_1 \geq a_1, x_2 \geq a_2, x_1 + x_2 \leq b\}.$$

To obtain a (γ, β) tolerance region of the form

$$R_u'' = \left\{(x_1, \ldots, x_k) \middle| \sum_{i=1}^{k} b_ix_i \leq \tilde{\ell}_u\right\}$$

it is necessary to obtain an upper (γ, β) tolerance limit ℓ_u for the variable $\sum_{i=1}^{k} b_ix_i$. Now setting $P\left[\sum_{i=1}^{k} b_ix_i < L\right] = \beta$ and using the approximate distribution for the weighted sum of chi-square variables gives

$$L = (g/2)\chi_\beta{}^2(h)$$

where

$$g = \sum_{i=1}^{k} (\theta_ib_i)^2 / \sum_{i=1}^{k} \theta_ib_i \qquad h = 2\left(\sum_{i=1}^{k} \theta_ib_i\right)^2 \Big/ \sum_{i=1}^{k} (\theta_ib_i)^2.$$

TABLE I

	$\gamma = 0.90$					$\gamma = 0.95$				
n^k	2	4	8	16	32	2	4	8	16	32
5	1.55	2.53	4.26	7.45	13.53	1.53	2.46	4.13	7.30	12.53
10	1.65	2.82	4.97	8.98	16.24	1.63	2.76	4.84	8.58	15.72
20	1.73	3.09	5.61	10.42	19.33	1.71	3.03	5.50	10.06	18.84
40	1.80	3.33	6.23	11.81	22.57	1.80	3.29	6.14	11.61	22.14

Let $l_u = c\sum_{i=1}^{k} b_i n\bar{x}_i$, then

$$P\left\{P\left[\sum_{i=1}^{k} b_i x_i \le \ell_u\right] \ge \beta\right\} = P[\ell_u \ge L]$$

$$= P\left[c\sum_{i=1}^{k} b_i n\bar{x}_i \ge (g/2)\chi_\beta^2(h)\right]$$

$$= P[(cg/2)\chi^2(nh) \ge (g/2)\chi_\beta^2(h)]$$

$$= P[c\chi^2(nh) \ge \chi_\beta^2(h)] = \gamma.$$

Setting the probability equal to γ gives $c = \chi_\beta^2(h)/\chi_{1-\gamma}^2(nh)$, where $h = 2\left(\sum_{i=1}^{k} \theta_i b_i\right)^2 \Big/ \sum_{i=1}^{k}\left(\theta_i b_i\right)^2$. Thus, c depends on the unknown parameters, unless all the θ_i are equal. An estimate of the probability content of the specification region S_u'' (for a prescribed confidence level γ) can be obtained easily for a given sample result by letting $c = b/\sum_{i=1}^{k} b_i n\bar{x}_i$ and finding the value of β such that

$$\chi_{1-\beta}^2(\hat{h}) = c\chi_\gamma^2(n\hat{h})$$

where

$$\hat{h} = 2\left(\sum_{i=1}^{k} a_i\bar{x}_i\right)^2 \Big/ \sum_{i=1}^{k}(a_i\bar{x}_i)^2.$$

Similarly, suppose $S_L'' = \left\{(x_1, \ldots, x_k) \Big| \sum_{i=1}^{k} a_i x_i \ge a\right\}$, then an estimate of the probability content of S_L'' (for prescribed γ) is the value of β such that

$$\chi_{1-\beta}^2(\hat{h}) = c\chi_\gamma^2(n\hat{h})$$

where

$$\hat{h} = 2\left(\sum_{i=1}^{k} a_i\bar{x}_i\right)^2 \Big/ \sum_{i=1}^{k}(a_i\bar{x}_i)^2$$

and

$$c = a/\sum_{i=1}^{k} a_i n\bar{x}_i.$$

Reference

[1] G. E. P. Box, "Some theorems on quadratic forms applied in the study of analysis of variance problems, I. Effect of inequality of variance in the one-way classification," *Ann. Math. Stat.*, vol. 25, pp. 290–302 June 1954.

Relation of a Physical Process to the Reliability of Electronic Components

CLARENCE F. KOOI, SENIOR MEMBER, IEEE

***Abstract*—An ensemble of electronic components having a random variation of some parameter, such as surface contamination, is considered. A physical process is postulated which leads to a change in one of the operating characteristics of the device. When this operating characteristic attains a value outside an acceptable range, the device is considered to have failed. The failure rate is calculated directly from the time behavior of the physical process and compared, for illustration, to the Weibull failure law. The parameters of the Weibull law are then related to the parameters of the physical process and the distribution of starting parameters.**

THE RELIABILITY of electronic components has increased steadily, so that at the present time the failure rates of common components are extremely low. This improvement is shown in Fig. 1, which also shows that semiconductor diodes and transistors are still the most unreliable. It can be concluded from the values of the failure rates that straightforward life testing can be prohibitively expensive; in time, if a few components are used, in money, if numerous identical components are used. Therefore it is becoming necessary to resort to accelerated life testing. This is done usually by raising either the voltage or the temperature. Thus, a *stressed* failure rate is obtained and it is hoped that this failure rate can be extrapolated to normal operation. Furthermore in many classes of components there occurs a stressed infant mortality, i.e., the stressed failure rate decreases with time. By this *burning-in* process it is hoped that the weak components are weeded out while the remainder are undamaged. These hopes have no theoretical basis at present. It is the object of this paper to provide a theoretical basis for a certain physical process leading to component deterioration.

Manuscript received February 12, 1967. The work reported here was supported by the Lockheed Independent Research Funds and by the National Science Foundation.
The author is with the Lockheed Palo Alto Research Laboratories, Palo Alto, Calif.

Physical Process

Assume that we have a system consisting of N particles in two potential wells separated by a potential hill of height ϵ (see Fig. 2). Initially all N particles are assumed to be in well 2 but due to thermal agitation par-