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1. INTRODUCTION

If X is any non-empty set, the symbol $F(X)$ will be used to denote the semigroup of all functions which map X into X, where the binary operation is that of composition. The symbol $\bar{x}$ will denote the function which maps every element of X onto x. The function $\bar{x}$ will be referred to as a constant function, and $Z(X)$ will denote the collection of all constant functions of $F(X)$. The symbol $\alpha(X)$ will be used exclusively to denote a subsemigroup of $F(X)$ which contains $Z(X)$. If X is a topological space, $C(X)$ will denote the semigroup of continuous functions on X. $C(X)$ is a subsemigroup of $F(X)$ that contain $Z(X)$. Suppose X and Y are topological spaces and $\alpha(X)$ and $\alpha(Y)$ are isomorphic. This paper deals principally with the following question. What conditions on X and Y are sufficient to imply that X and Y are homeomorphic?

The following two lemma's are straightforward. Magill [1] proved them for the case $\alpha(X) = C(X)$, and his proof carries over.

LEMMA 1. *Let $f \in \alpha(X)$. $f \circ g = f$ for every $g \in \alpha(X)$ if and only if $f = \bar{x}$ for some $x \in X$.*

LEMMA 2. *Let $f \in \alpha(X)$ and put $H(f) = \{x : f(x) = x\}$. $x \in H(f)$ if and only if $f \circ \bar{x} = \bar{x}$.*

Suppose ϕ is an isomorphism from $\alpha(X)$ onto $\alpha(Y)$ and $\phi^* = \phi \mid Z(X)$. Consider the following diagram.

$$\begin{array}{ccc}
\alpha(X) & \xrightarrow{\phi} & \alpha(Y) \\
\cup & & \cup \\
Z(X) & \xrightarrow{\phi^*} & Z(Y) \\
\uparrow x^* & & \uparrow y^* \\
X & \xrightarrow{h} & Y
\end{array}$$

$x^*(a) = \bar{a}$ for every $a \in X$ and $y^*(b) = \bar{b}$ for every $b \in Y$. Clearly x^* and y^* are one-to-one and onto. If ϕ^* maps $Z(X)$ onto $Z(Y)$, it will follow that ϕ^* is an isomorphism onto $Z(Y)$ and $h = y^{*-1} \circ \phi^* \circ x^*$ maps X one-to-one and onto Y.

LEMMA 3. *ϕ^* maps $Z(X)$ onto $Z(Y)$.*

PROOF. If $g \in \alpha(Y)$ there exists $f \in \alpha(X)$ such that $\phi(f) = g$. For $x \in X$, $\phi(\bar{x}) \circ g = \phi(\bar{x} \circ f) = \phi(\bar{x})$. By Lemma 1, $\phi(\bar{x}) \in Z(Y)$. Let $\bar{y} \in Z(Y)$. There exists $f \in \alpha(X)$ such that $\phi(f) = \bar{y}$. Let $g \in \alpha(X)$.

$$\phi(f \circ g) = \phi(f) \circ \phi(g) = \bar{y} \circ \phi(g) = \bar{y} = \phi(f).$$

Since ϕ is one-to-one, $f \circ g = f$. By Lemma 1, $f \in Z(X)$.

REMARK 1. $h(x) = y$ if and only if $\phi(\bar{x}) = \bar{y}$. In [1] Magill proved that $\phi(f) = h \circ f \circ h^{-1}$ for every $f \in \alpha(X)$ for the case $\alpha(X) = C(X)$ and $\alpha(Y) = C(Y)$, and his proof carries over. We note that if T is any one-to-one mapping from X onto Y such that $\phi(f) = T \circ f \circ T^{-1}$ for every $f \in Z(X)$, then $T = h$. For, if $x \in X$ let $y = h(x)$ and then

$$\phi(\bar{x})(a) = (T \circ \bar{x} \circ T^{-1})(a) = T(x) = \overline{T(x)}(a)$$

for every $a \in Y$. Hence

$$\overline{h(x)} = \bar{y} = \phi(\bar{x}) = \overline{T(x)}, \qquad \text{and} \qquad h(x) = T(x).$$

THEOREM 1. *Suppose $\beta(X)$ and $\beta(Y)$ are semigroups such that*

$$\alpha(X) \subset \beta(X) \subset F(X) \qquad \text{and} \qquad \alpha(Y) \subset \beta(Y) \subset F(Y).$$

ϕ can be extended to an isomorphism ψ from $\beta(X)$ onto $\beta(Y)$ if and only if

(1) *$h \circ f \circ h^{-1} \in \beta(Y)$ for every $f \in \beta(X)$, and*

(2) *$h^{-1} \circ g \circ h \in \beta(X)$ for every $g \in \beta(Y)$.*

If ϕ can be extended ψ is unique; $\psi(f) = h \circ f \circ h^{-1}$ for every $f \in \beta(X)$.

PROOF. The last statement follows from the preceding remark. If ϕ can be extended, (1) is clear. Given $g \in \beta(Y)$, there exists $f \in \beta(X)$ such that

$$\psi(f) = h \circ f \circ h^{-1} = g. \qquad h^{-1} \circ g \circ h = f \in \beta(X).$$

If the conditions hold, it is clear that ψ maps $\beta(X)$ into $\beta(Y)$, ψ is one-to-one, and a homomorphism. Given $g \in \beta(Y)$, $h^{-1} \circ g \circ h \in \beta(X)$ by (2) and $\psi(h^{-1} \circ g \circ h) = g$.

The following lemma is essentially contained in the proof of Theorem 3.1 of [1]. We give a proof for the sake of completeness.

LEMMA 4. *$h(H(f)) = H(\phi(f))$ for every $f \in \alpha(X)$ and*

$$h^{-1}(H(g)) = H(\phi^{-1}(g))$$

for every $g \in \alpha(Y)$.

PROOF. Let $x \in X$ and put $y = h(x)$. Using Lemma 2, we see that the following statements are equivalent: $y \in h(H(f))$, $x \in H(f)$, $f \circ \bar{x} = \bar{x}$, $\phi(\bar{x}) = \phi(f) \circ \phi(\bar{x})$, $\bar{y} = \phi(f) \circ y$, $y \in H(\phi(f))$.

THEOREM 2. *Suppose X and Y are topological spaces such that $\{H(f) : f \in \alpha(X)\}$ and $\{H(f) : f \in \alpha(Y)\}$ form bases for the closed sets of X and Y respectively. If ϕ is an isomorphism from $\alpha(X)$ onto $\alpha(Y)$, then h is a homeomorphism from X onto Y.*

PROOF. Since h is one-to-one and onto, this follows immediately from Lemma 4.

2. M-SPACES AND S-SPACES

DEFINITION 2.1. A topological space X is said to be an M-space if $\{H(f) : f \in C(X)\}$ is a base for the closed sets of X.

De Morgan's laws yield X is an M-space if and only if $\{0(f) : f \in C(X)\}$ is a base for the topology of X, where $0(f) = \{x : f(x) \neq x\}$.

An open set G containing a point x in X was defined in [1] to be an S-neighborhood if $G = \{x\}$ or there exists a continuous function f from $c(G)$ into X such that $f(x) \neq x$ and $f(y) = y$ for each y in $c(G) - G$, where $c(G)$ denotes the closure of G. A space was defined to be an S-space if it is Hausdorff and each point has a basis of S-neighborhoods. It was shown in [1] that every S-space is an M-space. In Section 3, an example is given of a space with a basis of S-neighborhoods that is not a T_1-space.

THEOREM 3. *Every M-space is a T_1-space. For Hausdorff spaces, X is an M-space if and only if it is an S-space.*

PROOF. Suppose X is an M-space. If $x \in X$, $\{x\} = H(\bar{x})$ which is a closed set. It suffices to prove that x has a basis of S-neighborhoods. Suppose G is an open set containing x. $X - G = \cap \{H(f) : f \in T \subset C(X)\}$. Since $x \notin X - G$, there exists $f \in T$ such that $f(x) \neq x$ and $f(y) = y$ for every $y \in X - G$. Since $c(G) - G \subset X - G$, $f(y) = y$ for every $y \in c(G) - G$.

REMARK 2. If h is a homeomorphism from X onto Y, ϕ defined by $\phi(f) = h \circ f \circ h^{-1}$ is an isomorphism from $C(X)$ onto $C(Y)$. In view of Theorem 2, two M-spaces X and Y are homeomorphic if and only if the semigroups $C(X)$ and $C(Y)$ are isomorphic. This result for S-spaces is the main theorem in [1], but the proof given there holds for M-spaces.

In [3], an isomorphism ϕ from $\alpha(X)$ onto $\alpha(Y)$ was said to be induced by a homeomorphism if there exists a homeomorphism T from X onto Y such that $\phi(f) = T \circ f \circ T^{-1}$ for every $f \in \alpha(X)$. In view of Remark 1, ϕ is induced by a homeomorphism if and only if $h = y^{*-1} \circ \phi^* \circ x^*$ is a homeomorphism. In [2], Magill proved that a mapping ϕ from $\alpha(X)$ onto $\alpha(Y)$ is an isomorphism if and only if there exists a one-to-one mapping T from X onto Y such that $\phi(f) = T \circ f \circ T^{-1}$ for every $f \in \alpha(X)$. This also follows from Remark 1. Furthermore, if such a mapping exists it must be h.

EXAMPLE 1. Let $X = \{a, b\}$ with topology $t = \{\varnothing, X, \{a\}\}$, and let Y be the same set with topology $u = \{\varnothing, Y, \{b\}\}$. $C(X) = C(Y) = \{\bar{a}, \bar{b}, i\}$. The identity mapping on $C(X)$ is an isomorphism and $h(x) = x$ for every x in X. h is not continuous but f defined by $f(a) = b$ and $f(b) = a$ is a homeomorphism. This shows that an isomorphism may not be induced by a homeomorphism even though X and Y are homeomorphic.

THEOREM 4. *Suppose P is a topological property such that for any two topologies u and v for the set X which possess P, any isomorphism of $C(X, u)$ onto $C(X, v)$ is induced by a homeomorphism. Then any isomorphism of $C(X, u)$ onto $C(Y, t)$ is induced by a homeomorphism if t possesses P.*

PROOF. Let ϕ be an isomorphism from $C(X, u)$ onto $C(Y, t)$. Then $h = y^{*-1} \circ \phi^* \circ x^*$ is a one-to-one map from X onto Y. Let the topology induced from Y by h on X be denoted by v. Clearly h is a homeomorphism from (X, v) to (Y, t) and $\psi(f) = h \circ f \circ h^{-1}$ is an isomorphism from $C(X, v)$ onto $C(Y, t)$. We now have $\psi^{-1} \circ \phi$ an isomorphism from $C(X, u)$ onto $C(X, v)$. We observe that $\psi^{-1} \circ \phi$ is the identity isomorphism and hence is induced by the identity function i which must be a homeomorphism from (X, u) to (X, v). It follows that $h \circ i = h$ is a homeomorphism from (X, u) to (Y, t) and since h induces ϕ the proof is complete.

3. FURTHER EXAMPLES

EXAMPLE 2. Let $X = (0, 1) \cup \{2\}$, and define the topology u for X to be the relative topology for $(0, 1)$ together with the set X. Certainly the space is not Hausdorff. It is a space with S-neighborhoods. Let G be any neighborhood of 2, then $c(G) - G = X - X = \varnothing$, hence the constant function $f(x) = \frac{1}{2}$ suffices for G to be an S-neighborhood. Therefore, X has a basis of S-neighborhoods at 2. Let P be any other point in X, then X is locally

Euclidean at P. Magill [1] has shown that every locally Euclidean space has a basis of S-neighborhoods. To show that X has a basis of S-neighborhoods at P, it will suffice to show that f is in $C(X)$ whenever f is in $C(X - \{2\})$ and $f(2) = 2$ or f is constant. Certainly if f is in $C(X - \{2\})$ and $f(2) = 2$ or if f is constant, f is in $C(X)$. Suppose f is in $C(X)$ and $f(2) = a$, then for any $\epsilon > 0, f^{-1}[(a - \epsilon, a + \epsilon)] = X$. Hence $f(x) = a$ for all x in X and f is constant.

The following is an example of a space (X, t) which is not an M-space. However, if $C(X, t)$ is isomorphic to $C(X, s)$ then $C(X, t) = C(X, s)$. Furthermore, h is a homeomorphism from (X, t) onto (X, s) and $t = s$.

EXAMPLE 3. Let X be an infinite set, and let t denote the topology consisting of the empty set and all subsets of X whose complements are finite.

To show (X, t) is not an M-space, let $\{x_n\}$ be a sequence of distinct points in X. Observe that $C(X, t)$ consists of those functions which are constant or finite to one, where finite to one means, for each $a \in X$, $\{x : f(x) = a\}$ is empty or finite. Hence $f(x) = x$ for x in $X - \{x_n : n \text{ is even}\}$ and $f(x_{2k}) = x_{2k+2}$ is a continuous function. However,

$$H(f) = X - \{x_n : n \text{ is even}\}$$

is not closed.

Let s be any other topology on X such that $C(X, t)$ and $C(X, s)$ are isomorphic. We note that h is a homeomorphism in the t topology. Therefore, if f is in $C(X, t)$ then $h \circ f \circ h^{-1}$ is in $C(X, t)$. Since each function in $C(X, s)$ is of this form, $C(X, s) \subset C(X, t)$. Furthermore, $h^{-1} \circ f \circ h$ is in $C(X, t)$. It follows that $h \circ (h^{-1} \circ f \circ h) \circ h^{-1} = f$ is in $C(X, s)$ and $C(X, t) \subset C(X, s)$. Therefore, $C(X, t) = C(X, s)$.

We now show that $t = s$. It suffices to show that the only topology u on an infinite set X which allows $C(X)$ to be precisely the finite to one and constant functions is t. Let U be a non-empty element of u, and $X - U \neq \varnothing$. Choose q in U and p in $X - U$. We define $f(x) = x$ if $x \neq p$ and $f(p) = q$. f is in $C(X)$, hence $f^{-1}(U) = U \cup \{p\}$ is in u. Therefore, if U is in u and $A \subset X - U$, $(U \cup A) \in u$. Suppose now that U is in u and $X - U$ is not finite. Let q be in $X - U$ and define $f(x) = x$ for x in U, $f(x) = q$ for x in $X - U$. f is not constant or finite to one. To show f is continuous, let $V \in u$. If $q \notin V$ then $f^{-1}(V) = (V \cap U) \in u$. If $q \in V$ then $f^{-1}(V) = (V \cap U) \cup (X - U)$. Now $(V \cap U) \in u$ and from what we have just shown, $(V \cap U) \cup (X - U)$ is in u. Therefore, if U is in u, $X - U$ is finite. To verify the converse of this statement, let $U \subset X$ such that $X - U$ is finite. There exists a V in u such that V is a proper subset of X, otherwise $C(X) = F(X)$. If $V \subset U$ then $U = V \cup (U - V)$ is in u. If $V - U$ is non-empty let p be in $V - U$ and q in $X - V$. Define f by $f(x) = x$ if x is in $U \cap V$ or $X - (U \cup V)$, $f(x) = p$ if

$x \in U - V$, and $f(x) = q$ if $x \in V - U$. $U - V \subset X - V$ which is a finite set. Also, $V - U$ is a finite set. It follows that f is continuous since it is finite to one, and consequently $f^{-1}(V) = (U \cap V) \cup (U - V) = U \in \boldsymbol{u}$. Therefore, $U \in \boldsymbol{u}$ if and only if $X - U$ is finite.

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