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D. L. Fread

Terence E. Harbaugh
Missouri University of Science and Technology

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OPEN-CHANNEL PROFILES BY NEWTON'S ITERATION TECHNIQUE

D. L. FREAD and T. E. HARBAUGH

Civil Engineering Department, University of Missouri-Rolla, Missouri, U.S.A.

Abstract: Computation of surface profiles for steady gradually varied flow can be accomplished by use of Newton's Iteration Technique. The magnitude of error is controlled and the profile depth can be conveniently calculated at selected distances upstream or downstream from a control point. A Fortran IV program is provided for utilizing the technique in a trapezoidal channel.

Introduction

Computer applications to problems of simulating complex interrelated systems of channels, dams, bridges, and other hydraulic structures are increasing. This increased use of the computer to solve engineering problems poses new challenges to the designer. Computational methods which designers have employed for several years are under review concerning possible revisions to adapt them to rapid computation on high speed computers. Many of the existing design techniques may be conveniently computerized through use of numerical analysis techniques. One example of the use of numerical analysis occurs for the computation of steady gradually varied open-channel flow profiles.

The use of the present hand techniques in calculating open-channel profiles is generally accomplished by methods such as the direct step, and the method of direct integration[1]). These techniques provide values of distance for specified values of depth from known controls in the channel flow. It is often necessary to obtain the depth of flow for specified distances from control points in the channel particularly when simulating large systems involving bridges, culverts, etc. The use of direct step, and other hand techniques, require an interpolation scheme to provide this information.

The standard step method provides the value of depth at specified distances from a control point. Chow[1]) presents this method as a hand technique. Henderson[2]) describes a graphical procedure which is not amiable to computer application. Prasad[3]) discusses a numerical analysis technique of trapezoidal integration which is a convenient method for computer application.

This paper deals with a numerical analysis technique which will provide values of flow depth at specified distances without using existing design methods or interpolation schemes. This method is generally referred to as the Newton Iteration Technique and has an advantage over the trapezoidal method by being a more straightforward solution to the integration of the gradually varied flow equation and being computationally more efficient.

Gradually varied flow

The derivation of a gradually varied flow equation is presented in many texts[1,2,4]) and will not be repeated here. The gradually varied flow equation is generally expressed as a differential equation in the following form:

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 + \alpha\, d(v^2/2g)/dy} \tag{1}$$

where

y = depth of flow in feet
x = distance along channel in feet
v = average velocity in channel in feet per sec
S_0 = slope of channel in feet/feet
S_f = slope of the energy gradient in feet/feet
α = energy coefficient
g = gravity acceleration constant.

If Eq. (1) is expressed in difference form for an incremental reach of channel, it clearly represents an energy balance between section 1–1 and

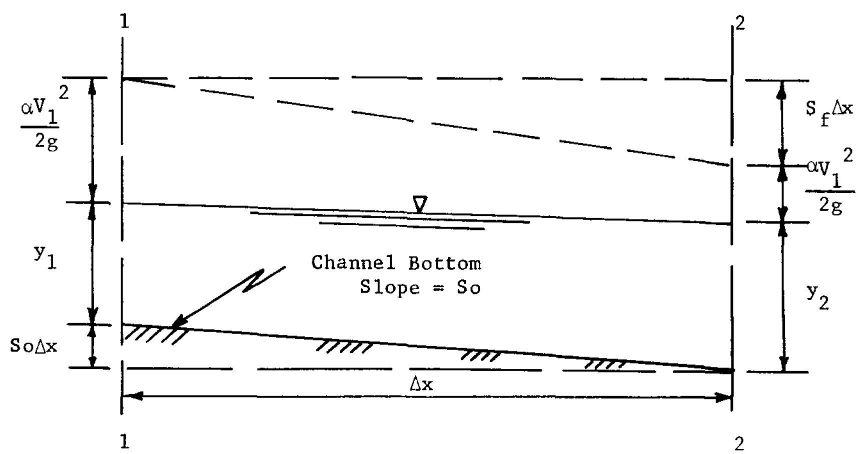

Fig. 1. Incremental reach of open channel.

section 2–2 (see Fig. 1).

$$S_0 \Delta x + \frac{\alpha V_1^2}{2g} + y_1 = y_2 + \frac{\alpha V_2^2}{2g} + \left(\frac{S_{f_1} + S_{f_2}}{2}\right) \Delta x \tag{2}$$

where Δx is the distance between sections 1–1 and 2–2. When the Manning formula is used, the energy gradient is given by

$$S_f = \frac{n^2 v^2}{2.208\ R^{\frac{4}{3}}}. \tag{3}$$

Thus, if the depth, y_2, at section 2–2 is known, the depth, y_1, at section 1–1 for a rectangular cross-section is obtained by substituting Eq. (3) into Eq. (2), remembering $Q = AV$, and rearranging. Thus,

$$y_1 + \frac{\alpha Q^2}{2gB^2 y_1^2} - \frac{n^2 Q^2 \Delta x}{4.416\ B^2 y_1^2} \left(\frac{B + 2y_1}{By_1}\right)^{\frac{4}{3}} + F = 0 \tag{4}$$

where

$$F = S_0 \Delta x - y_2 - \frac{\alpha Q^2}{2gB^2 y_2^2} - \frac{n^2 Q^2 \Delta x}{4.416\ B^2 y_2^2} \left(\frac{B + 2y_2}{By_2}\right)^{\frac{4}{3}}.$$

Equation (4) is nonlinear with respect to the unknown variable y_1, while the term F represents the sum of all known quantities of Eq. (2). Equation (4) may now be solved for y_1 by utilizing an iterative technique commonly referred to as Newton's Iteration Technique.

Newton's Iteration Technique

Suppose an expression $y = f(x)$, as shown in Fig. 2, is known and it is desired to find the value of x which will make $y = 0$. This value of x is denoted as X in Fig. 2. An iterative method, Newton's technique, may be used to find a solution to the equation, $f(x) = 0$. In this method, $f(x)$ is expanded in a Taylor Series about the point x_1, which is an initial approximation of X. Thus,

$$y = f(x) = f(x_1) + \frac{f'(x_1)(x - x_1)}{1!} + \frac{f''(x_1)(x - x_1)^2}{2!} + \cdots. \tag{5}$$

The prime (′) denotes a derivative, i.e., $f'(x_1) = \mathrm{d}f(x_1)/\mathrm{d}x_1$. Since $f(x) = 0$, and if second order terms of the Taylor Series expansion in Eq. (5) are neglected, then

$$f(x_1) + f'(x_1)(x - x_1) = 0. \tag{6}$$

Solving Eq. (6) for x yields

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{7}$$

This value of x is an approximation of X which is being sought since the second and higher terms of the Taylor Series were discarded. However, if x is now substituted for x_1 in Eq. (7), the resulting value of x will converge toward X. Thus, successive iterations of Eq. (7) will produce a value of x which is sufficiently close to X. A general iterative formula for this procedure is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{8}$$

where the subscript k denotes the number of iterations. After choosing an appropriate starting value, x_1, and applying Eq. (8) successively, the value

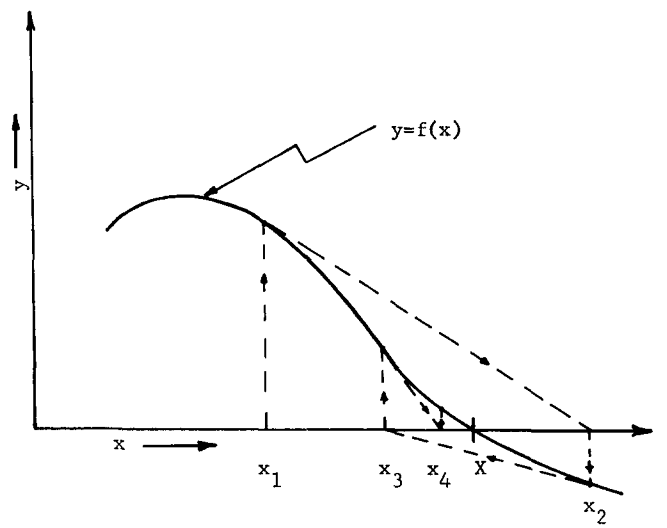

Fig. 2. Graph of a function, $y = f(x)$.

of x will be found such that $f(x)=0$. The initial approximation x_1 must be chosen sufficiently close to X. It can be shown, (5) that if x_1 satisfies the inequality

$$\frac{|f(x_1)\, f''(x_1)|}{[f'(x_1)]^2} < 1 \tag{9}$$

where the double prime superscript denotes the second derivative, then x_1 will converge quadratically to the root X of the equation $f(x)=0$ upon applying Eq. (8).

Convergence is attained when either

$$|x_{k+1} - x_k| < E_1 \tag{10}$$

or

$$|f(x_k)| < E_2 \tag{11}$$

where E_1 and E_2 are error tolerances, i.e., the desired absolute accuracy of the solution X of the equation $f(x)=0$.

Application of Newton's Iteration Technique

The energy balance as shown by Eq. (4) is in the form $f(x)=0$. Thus, application of Newton's Iteration Technique can readily be utilized to solve for the unknown, y_1. Thus,

$$y_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)} \tag{12}$$

where

$$f(y_k) = y_k + \frac{\alpha Q^2}{2gB^2 y_k^2} - \frac{n^2 Q^2 \Delta x (B + 2y_k)^{\frac{4}{3}}}{4.416\, B^2 y_k^2 (By_k)^{\frac{4}{3}}} + F$$

where

$$f'(y_k) = 1 - \frac{\alpha Q^2}{gB^2 y_k^3} + \frac{n^2 Q^2 \Delta x (6y_k + 5B)}{6.624\, B^3 y_k^4} \left(\frac{B + 2y_k}{By_k}\right)^{\frac{1}{3}}$$

and F is defined in Eq. (4).

The initial approximation for y_1 of Eq. (4) is simply taken as the known value of the depth, y_2, at section 2–2. Since Δx is chosen relatively small as compared to the total length of the reach of channel for which the surface profile is sought, the change in depth between sections 1–1 and 2–2 will also be relatively small. Thus, the initial approximation y_1 is quite close to the desired solution of Eq. (4), and Eq. (12) will rapidly converge to the desired solution.

The above procedure is then repeated in order to determine a new upstream depth at a new section 1′–1′ located at a Δx distance upstream from section 1–1. The initial approximation of y is taken as the previously calculated value of y_1. This procedure may be repeated for as many Δx reaches as desired. (The incremental reach length, Δx, need not be constant although it is treated as a constant in this paper.) When E_1 is chosen extremely small, say 0.000001, Eq. (12) will satisfy the inequality Eq. (10), after only a few iterations.

An accelerated convergence is easily obtained after the first Δx increment by improving the estimation of the initial value of the unknown depth. Rather than use the previously calculated value of depth for the initial approximation, an approximation of the form $y = y_1 - (y_2 - y_1)$ will allow convergence in two or three iterations, when E_1 is 0.000001.

Computations may proceed in an upstream or downstream direction depending on the particular situation for which the surface profile is desired. For calculations of profiles in a rectangular channel where the computations begin at an upstream control and proceed in a downstream direction, Eq. (4) becomes

$$y_2 + \frac{\alpha Q^2}{2gB^2y_2^2} + \frac{n^2Q^2\Delta x}{4.416\,B^2y_2^2}\left(\frac{B+2y_2}{By_2}\right)^{\frac{4}{3}} + F = 0 \tag{13}$$

where

$$F = -\,S_0\Delta x - y_1 - \frac{\alpha Q_2^2}{2gB^2y_1^2} + \frac{n^2Q^2\Delta x}{4.416\,B^2y_1^2}\left(\frac{B+2y_1}{By_1}\right)^{\frac{4}{3}}.$$

The equations for $f(y_k)$ and $f'(y_k)$ are identical with those associated with Eq. (12) except for a sign change in the third term of each equation.

If the cross-section of the previous example were trapezoidal with side slopes $1{:}Z_L$ and $1{:}Z_R$, for the left and right sides, respectively, as viewed in the upstream direction, and with a bottom width B, then the relationship for the depth, y, at section 1–1, would result in

$$f(y_k) = y_k + \frac{\alpha Q^2}{2g\left[By_k + \frac{1}{2}y_k^2(z_L+z_R)\right]^2} - \frac{n^2Q^2\Delta x\left[B + y_k(z_{HL}+z_{HR})\right]^{\frac{4}{3}}}{4.416\left[By_k + \frac{1}{2}y_k^2(z_L+z_R)\right]^{\frac{10}{3}}} + F \tag{14}$$

where

$$F = S_0\Delta x - \frac{\alpha Q^2}{2g\left[By_2 + \frac{1}{2}y_2^2(z_L+z_R)\right]^2} - \frac{n^2Q^2\Delta x\left[B + y_2(z_{HL}+z_{HR})\right]^{\frac{4}{3}}}{4.416\left[By_2 + \frac{1}{2}y_2^2(z_L+z_R)\right]^{\frac{10}{3}}} - y_2$$

and

$$f'(y_k) = 1 - \frac{\alpha Q^2\left[B + y_k(z_L+z_R)\right]}{g\left[By_k + \frac{1}{2}y_k^2(z_L+z_R)\right]^3} - \frac{n^2Q^2\Delta x\left[B + y_k(z_{HL}+z_{HR})\right]^{\frac{1}{3}}}{6.624\left[By_k + y_k^2(z_L+z_R)\right]^{\frac{13}{3}}}$$
$$\times \left\{2(z_{HL}+z_{HR})\left[By_k + \tfrac{1}{2}y_k^2(z_L+z_R)\right] - 5\left[B + y_k(z_L+z_R)\right]\left[B + y_k(z_{HL}+z_{HR})\right]\right\}$$

where

$$Z_{HL} = \sqrt{Z_L^2 + 1}$$

and

$$Z_{HR} = \sqrt{Z_R^2 + 1}\,.$$

A Fortran IV program is presented in Fig. 3 for determining the surface profile of a trapezoidal channel with a steady flow Q of 400 cfs. The depth is known to be 5.0 ft at a dam located 2400 ft downstream. It is desired to compute the depth and velocity at 50 ft stations along the 2400 ft reach of channel. E_t is chosen as 0.000001. Other pertinent data includes: $S_0 = 0.0016$; $B = 20.0$; $n = 0.025$; $\alpha = 1.10$; $z_L = 2.0$; and $z_R = 2.0$. This problem has been solved by Chow[1]) using the graphical integration technique and the direct step method. Both methods determine the distance along the channel associated with a specified change in depth. The surface profile obtained by the Newton Iteration Technique described in this paper produced results varying by less than 0.5 percent of the results shown by Chow[1]), see Fig. 4.

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  COMPUTATION OF STEADY GRADUALLY VARIED FLOW PROFILE
  DEPTH IS COMPUTED AT SPECIFIED LOCATIONS ALONG THE CHANNEL
       STARTING WITH A KNOWN DOWNSTREAM DEPTH ( YA ) AND PROCEDING
       UPSTREAM AT INCREMENTAL DISTANCES ( DX ) UNTIL THE TOTAL
       DISTANCE IS EQUAL TO TTL.
  Q IS THE DISCHARGE,CFS; RN IS MANNING'S N; SO IS THE CHANNEL
    SLOPE; B IS THE CHANNEL BOTTOM WIDTH; ALPHA IS THE VELOCITY
    DISTRIBUTION ENERGY COEFFICIENT; TTL IS THE LENGTH OF CHANNEL;
    YA IS THE KNOWN STARTING DEPTH; ER IS THE ERROR TOLERANCE FOR
    NEWTON'S ITERATION METHOD; M IS THE NUMBER + 1 OF INCREMENTAL
    REACHES OF LENGTH DX ALONG THE CHANNEL REACH; G IS THE GRAVITY
    ACCELERATION CONSTANT,32.2 FT/SEC/SEC; ZL IS THE LEFT SIDE
    SLOPE LOOKING UPSTREAM AND ZR IS THE RIGHT SIDE SLOPE.
  DIMENSIONS ARE AS FOLLOWS : LENGTH,FT; DEPTH,FT; WIDTH,FT; AND
    DISCHARGE,CUBIC FT PER SEC.
 DIMENSION X(49),Y(49),V(49)
 A(Y)=B*Y+Y*Y/2.*(ZL+ZR)
 P(Y)=B+Y*(ZHL+ZHR)
 AD(Y)=B+Y*(ZL+ZR)
 READ(1,200) Q,RN,SO,B,ALPHA,TTL
 READ(1,202) YA,ER,G,M,ZL,ZR
 WRITE(3,210) Q,RN,SO,B,ALPHA,TTL
 WRITE(3,212) YA,ER,M,ZL,ZR
 WRITE(3,214)
 DX=TTL/(M-1)
 ZHL=SQRT(ZL*ZL+1.)
 ZHR=SQRT(ZR*ZR+1.)
 F1=ALPHA*Q*Q/(2.*G)
 F2=RN*RN*Q*Q*DX/4.416
 F3=ALPHA*Q*Q/G
 F4=RN*RN*Q*Q*DX/6.624
 EF1=1./3.
```

Fig. 3a

Solution sensitivity

The Newton Iteration Technique will enable the computations to proceed either upstream or downstream *if and only if* the correct starting depth is used. For example, the computation of the M1 backwater curve can proceed downstream if a correct upstream depth is used to start the calculation. The general use of these equations, however, proceed from flow control points where the depth is known, i.e., critical depth, dams, sluicegates, etc. Table 1 shows the behavior of the Newton Iteration Technique for the trapezoidal channel with the solution proceeding both upstream and downstream. One can observe that for subcritical flow, the downstream solution procedure is valid if the correct depth at an upstream station is used as the starting value; however, if a 5% error in depth is introduced, the solution diverges to produce a 23% error at the dam. The proper use of the Newton Iteration

```
      Figure 3 (continued)
      EF4=4./3.
      EF10=10./3.
      EF13=13./3.
      X(M)=TTL
      Y(M)=YA
      V(M)=Q/A(YA)
      DSX=TTL
      DDY=0.0
      MM=M
       SOLVE NON-LINEAR EQUATION BY NEWTON'S ITERATION METHOD
    5 YAO=YA-DDY
      F=SO*DX-YA-F1/(A(YA)*A(YA))-F2*P(YA)**EF4/A(YA)**EF10
      DO 10 K=1,15
      YA1=YAO-(YAO+F1/(A(YAO)*A(YAO))-F2*P(YAO)**EF4/A(YAO)**EF10+F)/
     1(1.-F3*AD(YAO)/(A(YAO)*A(YAO)*A(YAO))-F4*P(YAO)**EF1/A(YAO)**EF13*
     2(2.*(ZHL+ZHR)*A(YAO)-5.*AD(YAO)*P(YAO)))
      IF(ABS(YA1-YAO)-ER) 12,10,10
   10 YAO=YA1
   12 DDY=YA-YA1
      YA=YA1
      M=M-1
      DSX=DSX-DX
      X(M)=DSX
      Y(M)=YA
      V(M)=Q/A(YA)
      IF(M-1) 20,20,5
   20 WRITE(3,215) (X(M),Y(M),V(M),M=1,MM)
      STOP
  200 FORMAT(7F10.6)
  202 FORMAT(3F10.6,I10,2F10.6)
  210 FORMAT(2X,'Q=',F10.2,5X,'RN=',F10.4,5X,'SO=',F10.6,5X,'B=',F10.2,
     15X,'ALPHA=',F10.2,5X,'TTL=',F10.2)
  212 FORMAT(2X,'YA=',F10.4,5X,'ER=',F10.6,5X,'M=',I5,5X,'ZL=',F10.2,5X,
     1'ZR=',F10.2)
  214 FORMAT(15X,'X=DISTANCE',10X,'DEPTH',9X,'VELOCITY')
  215 FORMAT(10X,3F15.3)
      END
```

Fig. 3b

Fig. 3. Fortran IV Newton's Iteration Technique for open-channel profiles.

Technique is to proceed upstream from the control point in subcritical flow. As can be noted in Table 1, introduction of starting depth errors at the downstream point will still allow convergence toward the true surface profile.

The supercritical data shown in Table 1 are for the trapezoidal channel example with a channel slope of 0.0036. Again note the fact that for error free starting depths, the solution may proceed either upstream or down-

Q=400.00 RN=0.0250 SO=0.001600 B=20.00 ALPHA=1.10 TTL=2400.00
YA=5.0000 ER=0.000001 M=49 ZL=2.00 ZR=2.00

X-Distance	Depth	Velocity
0.000	3.399	4.392
50.000	3.403	4.385
100.000	3.408	4.377
150.000	3.413	4.368
200.000	3.419	4.359
250.000	3.426	4.348
300.000	3.433	4.336
350.000	3.442	4.343
400.000	3.450	4.309
450.000	3.460	4.294
500.000	3.471	4.277
550.000	3.483	4.259
600.000	3.496	4.239
650.000	3.510	4.217
700.000	3.526	4.194
750.000	3.543	4.169
800.000	3.561	4.142
850.000	3.581	4.113
900.000	3.602	4.082
950.000	3.625	4.049
1000.000	3.650	4.015
1050.000	3.676	3.978
1100.000	3.704	3.940
1150.000	3.734	3.900
1200.000	3.766	3.858
1250.000	3.799	3.815
1300.000	3.834	3.770
1350.000	3.871	3.724
1400.000	3.910	3.677
1450.000	3.951	3.628
1500.000	3.993	3.579
1550.000	4.038	3.529
1600.000	4.083	3.478
1650.000	4.131	3.426
1700.000	4.180	3.374
1750.000	4.230	3.322
1800.000	4.282	3.270
1850.000	4.336	3.218
1900.000	4.391	3.165
1950.000	4.447	3.113
2000.000	4.504	3.062
2050.000	4.562	3.010
2100.000	4.622	2.959
2150.000	4.683	2.909
2200.000	4.744	2.859
2250.000	4.807	2.810
2300.000	4.870	2.761
2350.000	4.935	2.714
2400.000	5.000	2.667

Fig. 4. Output for trapezoidal channel.

TABLE 1

Effect of starting depth errors

Flow profile Flow regime	Direction of computations	Initial starting depth	Depth at end of the reach	% error beginning	% error end of reach
M-1 Subcritical	Upstream	5.000	3.399	0.0	0.0
M-1 Subcritical	Upstream	5.250	3.418	+5.0	+ 0.67
M-1 Subcritical	Upstream	4.750	3.385	−5.0	− 0.41
M-1 Subcritical	Downstream	3.399	5.000	0.0	0.0
M-1 Subcritical	Downstream	3.570	6.158	+5.0	+23.0
M-1 Subcritical	Downstream	3.170	Not computable	−5.0	–
M-3 Supercritical	Upstream	1.652	0.530	0.0	0.0
M-3 Supercritical	Upstream	1.730	0.608	+5.0	+15.6
M-3 Supercritical	Upstream	1.570	0.440	−5.0	−18.0
M-3 Supercritical	Downstream	0.530	1.652	0.0	0.0
M-3 Supercritical	Downstream	0.580	1.701	+9.4	+2.96
M-3 Supercritical	Downstream	0.480	1.606	−9.4	−2.79

stream. However, errors in the starting depth of supercritical flow, amplify when proceeding upstream, and dampen when proceeding downstream.

Conclusions

Steady gradually varied surface profiles can be efficiently and conveniently obtained by digital computers utilizing the Newton Iteration Technique of solving the difference form of the gradually varied flow equation. The accuracy of this procedure is dependent upon the accuracy of the starting depth and/or the direction (upstream and downstream) of the computations. These characteristics are the same as those displayed by the conventional step methods, since both are merely variations in the technique of solving an energy balance equation of the general form of Eq. (4).

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