Rank two bundles on P^n with isolated cohomology

Original
Rank two bundles on P^n with isolated cohomology / Malaspina, F.; Rao, A. P.. - In: BULLETIN OF THE LONDON MATHEMATICAL SOCIETY. - ISSN 0024-6093. - (2023), pp. 1-12. [10.1112/blms.12877]

Availability:
This version is available at: 11583/2979703.5 since: 2023-06-30T15:15:05Z

Publisher:
Wiley

Published
DOI:10.1112/blms.12877

Received: 1 February 2022 | Revised: 24 January 2023 | Accepted: 19 May 2023

DOI: 10.1112/blms.12877

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Rank two bundles on $\mathbf{P}^n$ with isolated cohomology

F. Malaspina[1] | **A.P. Rao**[2]

[1]Politecnico di Torino, Torino, Italy

[2]University of Missouri-St.Louis, St.Louis, Missouri, USA

Correspondence
F. Malaspina, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.
Email: francesco.malaspina@polito.it

Funding information
INdAM

Abstract

The purpose of this paper is to study minimal monads associated to a rank two vector bundle $\mathcal{E}$ on $\mathbf{P}^n$. In particular, we study situations where $\mathcal{E}$ has $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$, except for one pair of values $(k, n-k)$. We show that on $\mathbf{P}^8$, if $H^3_*(\mathcal{E}) = H^4_*(\mathcal{E}) = 0$, then $\mathcal{E}$ must be decomposable. More generally, we show that for $n \geqslant 4k$, there is no indecomposable bundle $\mathcal{E}$ for which all intermediate cohomology modules except for $H^1_*, H^k_*, H^{n-k}_*, H^{n-1}_*$ are zero.

MSC 2020

14F06, 14J60 (primary)

1 | INTRODUCTION

It has been difficult to disprove the existence of an indecomposable rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$ for large n. Most known results have been obtained by imposing other conditions on $\mathcal{E}$ to show that $\mathcal{E}$ cannot exist or must be split. For example, the so-called Babylonian condition which requires $\mathcal{E}$ to be extendable to $\mathbf{P}^{n+m}$ for every m has been studied by a number of people including Barth and van de Ven [2] and Coanda and Trautmann [3]. Numerical criteria that force splitting are found again in Barth and van de Ven, where for a normalized rank two bundle with second Chern class a and with splitting type $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(b)$ on the general line l, a function $f(a, b)$ is found such that if $n > f(a, b)$, then a bundle on $\mathbf{P}^n$ with these invariants must be split.

Cohomological criteria for forcing the splitting of $\mathcal{E}$ start with Horrocks [7]. If S is the polynomial ring corresponding to $\mathbf{P}^n$, then $H^i_*(\mathcal{E})$ (defined as $\oplus_\nu H^i(\mathbf{P}^n, \mathcal{E}(\nu))$) is an S-module. The intermediate cohomology modules $H^i_* sE), 1 \leqslant i \leqslant n-1$ are all graded modules of finite length and there is a strong relationship between $\mathcal{E}$ and its intermediate cohomology modules. He shows that if $H^i_*(\mathcal{E}) = 0$ for all i with $i \leqslant i \leqslant n-1$, then $\mathcal{E}$ is split. Moreover, Horrocks in [7] established that a vector bundle on $\mathbf{P}^n$ is determined up to isomorphism and up to a sum of line bundles (i.e., up to stable equivalence) by its collection of intermediate cohomology modules and also a certain collection of extension classes involving these modules. This correspondence has been generalized to any Arithmetically Cohen-Macaulay (ACM) varieties in [12]. The Syzygy Theorem ([5, 6]) shows that for a rank two bundle $\mathcal{E}$, it is enough to know that $H^1_*(\mathcal{E}) = 0$ to force splitting. In [14], it is shown that for a indecomposable rank two bundle on $\mathbf{P}^n$, in addition to $H^1_*(\mathcal{E})$ and $H^{n-1}_*(\mathcal{E})$ being nonzero, some intermediate cohomology module $H^k_*(\mathcal{E})$ $(1 < k < n-1)$ (and hence also $H^{n-k}_*(\mathcal{E})$) must be nonzero. Various calculations in [13] and [14] show that there are limitations on the module structure of $H^1_*(\mathcal{E})$ and $H^2_*(\mathcal{E})$ for some values of n.

In this paper, we study situations where a rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$ has $H^i_*(\mathcal{E}) = 0$ for $1 < i < n-1$, except for one pair of values $(k, n-k)$. We describe the minimal monads associated to $\mathcal{E}$. We show that on $\mathbf{P}^8$, if $H^3_*(\mathcal{E}) = H^4_*(\mathcal{E}) = 0$, then $\mathcal{E}$ must be decomposable. More generally, we show that for $n \geqslant 4k$, there is no indecomposable bundle $\mathcal{E}$ for which all intermediate cohomology modules except for $H^1_*, H^k_*, H^{n-k}_*, H^{n-1}_*$ are zero. The proof utilizes the space between k and $n-k$ when $n \geqslant 4k$ for making cohomological computations.

2 | MONADS FOR RANK TWO VECTOR BUNDLES ON $\mathbf{P}^n$

Let $\mathcal{E}$ be an indecomposable rank two vector bundle on $\mathbf{P}^n$ over an algebraically closed field of characteristic different from two. If S is the polynomial ring on $n+1$ variables, let $N_i = H^i_*(\mathcal{E}) = \oplus_\nu H^i(\mathcal{E}(\nu))$ be the finite length graded S-module over S, for $1 \leqslant i \leqslant n-1$. By the Syzygy Theorem, both N_1 and N_{n-1} are nonzero modules. Horrocks ([8]) gives a brief description of the construction of a minimal monad for a bundle $\mathcal{E}$ of any rank on $\mathbf{P}^n$ by "killing both H^1_* and H^{n-1}_*." Barth and Hulek ([1]) use this idea to construct (with more detail) l-m minimal monads for a bundle $\mathcal{E}$ on $\mathbf{P}^n$, where only $H^1_{\geqslant l}(\mathcal{E})$ and $H^{n-1}_{<-m-n}(\mathcal{E})$ are killed. Horrocks' construction, which we use below, is obtained when both l and m are very negative.

The monad is a complex

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \to 0,$$

where $\mathcal{P}$ is a bundle with $H^i_*(\mathcal{P}) = 0$ for $i = 1$ and $i = n-1$, and where $\mathcal{A}, \mathcal{B}$ are free bundles. Let $\mathcal{G}$ be kernel β. We have two sequences

$$\begin{gathered} 0 \to \mathcal{G} \to \mathcal{P} \to \mathcal{B} \to 0, \\ 0 \to \mathcal{A} \to \mathcal{G} \to \mathcal{E} \to 0, \end{gathered} \tag{1}$$

from which we see that $H^i_*(\mathcal{E}) = H^i_*(\mathcal{G})$ for $1 \leqslant i \leqslant n-2$, while $H^{n-1}_*(\mathcal{G}) = 0$, and $H^i_*(\mathcal{E}) = H^i_*(\mathcal{P})$ for $2 \leqslant i \leqslant n-2$, while $H^1_*(\mathcal{P}) = H^{n-1}_*(\mathcal{P}) = 0$.

The minimality of the complex means that the rank of $\mathcal{B}$ equals the number of generators of $H^1_*(\mathcal{G}) = N_1$ and the rank of $\mathcal{A}^\vee$ equals the number of generators of $H^1_*(\mathcal{E}^\vee)$. When the rank of the bundle $\mathcal{E}$ equals 2, we find that $\mathcal{A}$ and $\mathcal{B}$ have the same rank.

The two sequences give rise to

$$\begin{gathered} 0 \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{P} \to \mathcal{B} \otimes \mathcal{P} \to S^2\mathcal{B} \to 0, \\ 0 \to S^2\mathcal{A} \to \mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{E} \to 0. \end{gathered} \tag{2}$$

(Since the characteristic of the field is different from 2, we can assume that S^2 commutes with duals.)

Lemma 2.1. *If $H^2_*(\mathcal{E}) = 0$, then $H^1_*(\wedge^2\mathcal{P})$ and $H^{n-1}_*(\wedge^2\mathcal{P})$ are nonzero. If $H^l_*(\mathcal{E}) = 0$ for some l, with $2 \leqslant l \leqslant n-2$, then $H^l_*(\wedge^2\mathcal{P}) = 0$.*

Proof. See [11, Theorem 2.2] for the first part. Next, suppose $H^l_*(\mathcal{E}) = 0$ for some l, with $2 \leqslant l \leqslant n-2$. So, $N_l = N_{n-l} = 0$ by Serre duality. In particular, $\mathcal{G}$ and $\mathcal{P}$ have $H^l_* = 0$ as well. It follows from Equation (2), that $H^l_*(\wedge^2\mathcal{G}) = 0$ and hence $H^l_*(\wedge^2\mathcal{P}) = 0$. □

Lemma 2.2. *Let $2 \leqslant t \leqslant n-2$. Let $A = H^0_*(\mathcal{A}), B = H^0_*(\mathcal{B})$. There is an exact sequence*

$$A \otimes N_t \to H^t_*(\wedge^2\mathcal{P}) \to B \otimes N_t,$$

which is injective on the left if $t \geqslant 3$ and $N_{t-1} = 0$, and is surjective on the right if $t \leqslant n-3$ and $N_{t+1} = 0$.

Proof. Break up the first sequence in 2 as $0 \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{P} \to \mathcal{D} \to 0$, $0 \to \mathcal{D} \to \mathcal{B} \otimes \mathcal{P} \to S^2\mathcal{B} \to 0$. We get long exact sequences

$$H^{t-1}_*(\mathcal{D}) \to H^t_*(\wedge^2\mathcal{G}) \to H^t_*(\wedge^2\mathcal{P}) \to H^t_*(\mathcal{D}) \to H^{t+1}_*(\wedge^2\mathcal{G}),$$

where $H^t_*(\mathcal{D}) \cong B \otimes N_t$ (always) and $H^{t-1}_*(\mathcal{D}) \cong B \otimes N_{t-1}$ provided $t \geqslant 3$. Likewise break up the second sequence as $0 \to S^2\mathcal{A} \to \mathcal{A} \otimes \mathcal{G} \to \mathcal{C} \to 0$, $0 \to \mathcal{C} \to \wedge^2\mathcal{G} \to \wedge^2\mathcal{E} \to 0$. We see that $H^i_*(\wedge^2\mathcal{G}) \cong H^i_*(\mathcal{C})$ for $i = t, t+1$, $H^t_*(\mathcal{C}) \cong A \otimes N_t$ and when $t \leqslant n-3$, $H^{t+1}_*(\mathcal{C}) \cong A \otimes N_{t+1}$. □

Moreover, when $H^l_*(\mathcal{E}) = 0$ for some l, with $2 \leqslant l \leqslant n-3$, from Equation (2) since $H^{l+1}_*(S^2\mathcal{A}) = H^{l+2}_*(\mathcal{A} \otimes \mathcal{G}) = H^l_*(\wedge^2\mathcal{E}) = H^{l+1}_*(\wedge^2\mathcal{E}) = 0$, we get $H^{l+1}_*(\wedge^2\mathcal{G}) \cong H^{l+1}_*(\mathcal{A} \otimes \mathcal{G})$. Since $H^{l-1}_*(S^2\mathcal{B}) = H^l_*(\mathcal{B} \otimes \mathcal{P}) = 0 H^{l+1}_*(\wedge^2\mathcal{G}) \hookrightarrow H^{l+1}_*(\wedge^2\mathcal{P})$. From

$$0 \to \mathcal{D} \to \mathcal{B} \otimes \mathcal{P} \to S^2\mathcal{B} \to 0,$$

we obtain $H^{l+1}_*(\mathcal{B} \otimes \mathcal{P}) \cong H^{l+1}_*(\mathcal{D})$ giving an exact sequence

$$0 \to A \otimes N_{l+1} \to H^{l+1}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l+1}, \tag{3}$$

where $A = H^0_*(\mathcal{A}), B = H^0_*(\mathcal{B})$. Notice that the sequence is exact on the right if $N_{l+2} = 0$.

Likewise, if $H^l_*(\mathcal{E}) = 0$ for some l, with $3 \leqslant l \leqslant n-2$, we get the exact sequence

$$A \otimes N_{l-1} \to H^{l-1}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l-1} \to 0. \tag{4}$$

Notice that the sequence is exact on the right if $N_{l-2} = 0$.

The following proposition is a typical one that shows that a minimal monad for a rank two bundle is built very minimally out of the cohomological data for $\mathcal{E}$. Other examples of such a result can be found in [15] and [13]. Decker ([4]) has conjectured such a minimality for rank two bundles on $\mathbf{P}^4$.

Proposition 2.3. *Suppose that $\mathcal{E}$ is a nonsplit rank two bundle on $\mathbf{P}^n$ $(n \geqslant 6)$, with $H^l_*(\mathcal{E}) = 0$ for some l with $2 \leqslant l \leqslant n-2$. Then in the minimal monad for $\mathcal{E}$, the bundle $\mathcal{P}$ has no line bundle summands.*

Proof. Note that the statement is vacuous for $n = 4, 5$, since $\mathcal{E}$ will be split by [14]. So, assume that $n \geqslant 6$ and that $\mathcal{E}$ satisfies $H^l_*(\mathcal{E}) = 0$ for some $2 \leqslant l \leqslant n-2$. By [14], there must also be a j such that $H^j_*(\mathcal{E}) \neq 0$ for some $2 \leqslant j \leqslant n-2$.

We may choose l to be the lowest value with $H^l_*(\mathcal{E}) = 0$ and let us suppose that $l \geqslant 3$. Then $H^{l-1}_*(\mathcal{E}) = N_{l-1} \neq 0$. Consider the exact sequence using Lemma 2.2 (with $t = l-1$)

$$A \otimes N_{l-1} \to H^{l-1}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l-1} \to 0.$$

Now if $\mathcal{P} \cong \mathcal{Q} \oplus \mathcal{O}_{\mathbf{P}}(a)$, then $H^{l-1}_*(\wedge^2\mathcal{P}) \cong H^{l-1}_*(\wedge^2\mathcal{Q}) \oplus [S(a) \otimes N_{l-1}]$, where $N_{l-1} \neq 0$. The map $S(a) \otimes N_{l-1} \to B \otimes N_{l-1}$ in the sequence is induced by the map $\mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}_k \xrightarrow{\beta_2 \otimes I} \mathcal{B} \otimes \mathcal{P}_k$, where $\beta = [\beta_1, \beta_2]$ in the monad for $\mathcal{E}$.

The map $A \otimes N_{l-1} \to S(a) \otimes N_{l-1}$ is induced by the map $\mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{G} \hookrightarrow \wedge^2\mathcal{P} \twoheadrightarrow \mathcal{O}_{\mathbf{P}}(a) \otimes \mathcal{P}$, hence by $\mathcal{A} \otimes \mathcal{P} \xrightarrow{\alpha_2 \otimes I} \mathcal{L} \otimes \mathcal{P}$ if $\alpha = [\alpha_1, \alpha_2]^T$ in the monad.

The sequence above now reads

$$A \otimes N_{l-1} \xrightarrow{\begin{bmatrix} * \\ \alpha_2 \otimes I \end{bmatrix}} H^{l-1}_*(\wedge^2\mathcal{Q}) \oplus [S(a) \otimes N_{l-1}] \xrightarrow{\left[*, \beta_2 \otimes I\right]} B \otimes N_{l-1} \to 0.$$

If we tensor the sequence by the quotient $k = S/(X_0, \dots, X_{n+1})$, since the matrix β_2 is a minimal matrix, $(\beta_2 \otimes I) \otimes k = 0$, hence $[S(a) \otimes N_{l-1} \otimes k]$ is inside the kernel of $\left[*, \beta_2 \otimes I\right] \otimes k$. By exactness, $S(a) \otimes N_{l-1} \otimes k$ is inside the image of $(\alpha_2 \otimes I) \otimes k$, which is not possible since α_2 is also a minimal matrix.

It remains to study the case where $l = 2$. There is a value l' between 3 and $n-3$ for which $H^{l'}_*(\mathcal{E}) = N_{l'} \neq 0$ and $H^{l'+1}_*(\mathcal{E}) = 0$. We now have an exact sequence of nonzero S-modules

$$A \otimes N_{l'} \to H^{l'}_*(\wedge^2\mathcal{P}) \to B \otimes N_{l'} \to 0,$$

and we repeat the earlier argument to get a contradiction. ☐

Definition 2.4. A rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$, $n \geqslant 6$, will be said to have isolated cohomology of type (n, k) if there exists an integer k, $1 < k \leqslant \frac{n}{2}$, with $H^k_*(\mathcal{E})$ and $H^{n-k}_*(\mathcal{E})$ nonzero modules, and $H^i_*(\mathcal{E}) = 0$ for $i \neq 1, k, n-k, n-1$.

Remark 2.5. By Lemma 2.1, we get that if $\mathcal{E}$ has isolated cohomology of type (n, k), then $H^i_*(\wedge^2\mathcal{P}) = 0$ for $i \neq 1, k, n-k, n-1$.

A special case in the definition is when the middle cohomology is not zero, that is, of type (n, k), where n is even, equal to $2k$, and the only nonzero cohomology modules are $H^1_*(\mathcal{E}), H^k_*(\mathcal{E}), H^{n-1}_*(\mathcal{E})$.

Note that the conditions that $H^1_*(\mathcal{E}), H^{n-1}_*(\mathcal{E})$ are both nonzero for an indecomposable rank two bundle follow from the Syzygy Theorem. In [14], it is proved that for an indecomposable rank two bundle on $\mathbf{P}^n$, $n \geqslant 4$, at least one cohomology module $H^l_*(\mathcal{E})$ must be nonzero with $1 < l < n-1$. The reason n is chosen to be $\geqslant 6$ in the definition is that first, the definition is vacuous for $n = 2, 3$ and second, for $n = 4, 5$, k must be 2, and the definition made is always satisfied by any possible indecomposable rank two bundle on $\mathbf{P}^4$ or $\mathbf{P}^5$, and hence imposes no restrictions.

Let $\mathcal{P}_k(N)$ be the kth syzygy bundle of the finite length module N. By this, we mean that in a minimal free resolution for N over the polynomial ring S:

$$0 \to L_{n+1} \xrightarrow{f_{n+1}} L_n \to \cdots \to L_{k+1} \xrightarrow{f_{k+1}} L_k \to \cdots \to L_1 \xrightarrow{f_1} L_0 \to N \to 0.$$

$P_k(N)$ will denote the image of f_{k+1} and $\mathcal{P}_k(N)$ will denote the sheafification of $P_k(N)$. Hence, $H^k_*(\mathcal{P}_k(N)) = N$, with $H^i_*(\mathcal{P}_k(N)) = 0$ when $i \neq 0, k, n$. According to [7], if $\mathcal{P}$ is any bundle on $\mathbf{P}^n$ with the property that $H^k_*(\mathcal{P}) = N$ and $H^i_*(\mathcal{P}) = 0$ when $i \neq 0, k, n$, then $\mathcal{P} \cong \mathcal{P}_k(N) \oplus \mathcal{F}$ where $\mathcal{F}$ is a direct sum of line bundles.

Lemma 2.6. *Let $\mathcal{P}$ be a vector bundle on $\mathbf{P}^n$ with nonzero cohomology modules $H^k_*(\mathcal{P}) = N$, $H^l_*(\mathcal{P}) = M$ for $1 \leqslant k < l \leqslant n-1$, and with $H^i_*(\mathcal{P}) = 0$ when $i \neq 0, k, l, n$. Then there is an exact sequence*

$$0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_l(M) \to 0,$$

where $\mathcal{F}$ is some free bundle.

Proof. This too follows from [7]. Letting P denote $H^0_*(\mathcal{P})$, form an exact sequence (by partially resolving $P^\vee$)

$$0 \to P \to L_k \to L_{k-1} \to \dots L_1 \to A \to N \to 0,$$

where A is not a free module. Compare this with a truncated minimal free resolution of N:

$$0 \to P_k(N) \to L'_k \to L'_{k-1} \to \dots L'_1 \to L'_0 \to N \to 0.$$

The induced map $P_k(N) \to P$ gives a map $\mathcal{P}_k(N) \to \mathcal{P}$ that is an isomorphism at the cohomology level H^k_*. Minimally add a free module F to P to force a surjection $P^\vee \oplus F^\vee \to P_k(N)^\vee$. This gives an inclusion of bundles $\mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F}$ whose cokernel is $\mathcal{P}_l(M) \oplus \mathcal{F}'$ where $\mathcal{F}'$ is a free bundle (since it has only H^l_* intermediate cohomology). We notice that both for $k = 1$ and for $k > 1$, the map $H^1_*(\mathcal{P}_k(N)) \to H^1_*(\mathcal{P} \oplus \mathcal{F})$ is an isomorphism, so we get a surjection from $H^0_*(\mathcal{P} \oplus \mathcal{F})$ to $H^0_*(\mathcal{P}_l(M) \oplus \mathcal{F}')$. By the minimality of F, we may conclude that $\mathcal{F}' = 0$ □

Summarizing this below, we get the following.

Proposition 2.7. *Let $\mathcal{E}$ be a rank two bundle on $\mathbf{P}^n$, $n \geqslant 6$ with isolated cohomology of type (n, k) with $H^k_*(\mathcal{E}) = N$, for some k strictly between 1 and $\frac{n}{2}$. Then $\mathcal{E}$ has the monad*

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P} \xrightarrow{\beta} \mathcal{B} \to 0,$$

where

- $\mathcal{P}$ *satisfies an exact sequence* $0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_{n-k}(M) \to 0$, *where* $\mathcal{F}$ *is some free bundle,* $M = H^{n-k}_*(\mathcal{E})$ *(which can be identified with* $N^\vee$ *up to twist).*
- $H^i_*(\wedge^2\mathcal{P}) = 0$ *for* $i \neq 1, k, n-k, n-1$.
- $H^1_*(\wedge^2\mathcal{P})$ *and* $H^{n-1}_*(\wedge^2\mathcal{P})$ *are nonzero if* $k \neq 2$.

In the case left out in the above proposition, where $\mathcal{E}$ has isolated middle cohomology with $n = 2k$ and with $H^k_*(\mathcal{E}) = N \neq 0$ equal to the only nonzero cohomology module in the range $1 < i < n-1$, the monad for $\mathcal{E}$ has the form

$$0 \to \mathcal{A} \to \mathcal{P}_k(N) \to \mathcal{B} \to 0.$$

Also, there is a short exact sequence

$$0 \to A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k(N)) \to B \otimes N \to 0.$$

Thus,

Proposition 2.8. *Let* $\mathcal{E}$ *be a rank two bundle on* $\mathbf{P}^n$, $n = 2k, n \geqslant 6$, *with* $H^k_*(\mathcal{E}) = N$, $H^i_*(\mathcal{E}) = 0, i \neq 1, k, n$. *Let* $\mathcal{P}_k$ *be the kth syzygy bundle of N where* $\mathcal{P}_k$ *is the sheafification of* P_k *with* $P_k =$ *Image of* $(f_{k+1} : L_{k+1} \to L_k)$ *in a minimal free resolution of N. Then* $\mathcal{E}$ *has the monad*

$$0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{P}_k \xrightarrow{\beta} \mathcal{B} \to 0,$$

where $\mathcal{A}, \mathcal{B}$ *are sheafifications of free summands* A, B *of* L_{k+1} *and* L_k, *respectively, and where* α, β *are induced by* f_{k+1}. *Furthermore,*

- $H^i_*(\wedge^2\mathcal{P}_k) = 0$ *for* $i \neq 1, k, n-1$,
- *the induced sequence* $0 \to A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k) \to B \otimes N \to 0$ *is exact,*
- $H^1_*(\wedge^2\mathcal{P}_k)$ *and* $H^{n-1}_*(\wedge^2\mathcal{P}_k)$ *are nonzero.*

Proof. The only item to verify is that $\mathcal{A}, \mathcal{B}$ are sheafifications of free summands A, B of L_{k+1} and L_k, respectively, and that α, β are induced by f_{k+1}. Since $L_{k+1} \to P_k$ is surjective, $\alpha : A \to P_k$ factors through $\tilde{\alpha} : A \to L_{k+1}$. Likewise, since $L_k^\vee \to P_k^\vee$ is surjective, $\beta^\vee : B^\vee \to P_k^\vee$ factors through $\tilde{\beta}^\vee : B^\vee \to L_k^\vee$. It remains to show that the matrices $\tilde{\alpha}, \tilde{\beta}$ have full rank when tensored by k.

The map $H^k_*(\wedge^2\mathcal{P}_k) \to B \otimes N \to 0$ in the short sequence above is obtained from $\wedge^2\mathcal{P}_k \to \mathcal{B} \otimes \mathcal{P}_k$ where $p \wedge q$ maps to $\beta(p) \otimes q - \beta(q) \otimes p$. This factors through $\mathcal{L}_k \otimes \mathcal{P}_k$ via the lift $\tilde{\beta}$. In particular, the map $L_k \otimes N \to B \otimes N$, given by $\tilde{\beta} \otimes I$, is onto. Hence so is $(\tilde{\beta} \otimes k) \otimes I$, a map of vector spaces. Hence, the matrix $\tilde{\beta} \otimes k$ has rank equal to the rank of B. So, B is a direct summand of L_k.

The map $0 \to A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k)$ is obtained from $H^k_*(\mathcal{A} \otimes \mathcal{G}) \cong H^k_*(\wedge^2\mathcal{G}) \hookrightarrow H^k_*(\wedge^2\mathcal{P}_k)$, which, in turn, is obtained from $\mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{G} \hookrightarrow \wedge^2\mathcal{P}_k$, where $a \otimes g$ maps to $\alpha(a) \wedge g$ in $\wedge^2\mathcal{P}_k$. This map $\mathcal{A} \otimes \mathcal{G} \to \wedge^2\mathcal{P}_k$ factors through $\mathcal{L}_{k+1} \otimes \mathcal{G}$, vial the lift $\tilde{\alpha}$.

It follows that the injection $A \otimes N \to H^k_*(\wedge^2\mathcal{P}_k)$ factors through $A \otimes N \to L_{k+1} \otimes N$, by the map $\tilde{\alpha} \otimes I$. This must also be injective. Choose a socle element n in N (an element that is annihilated by all linear forms in S). The submodule generated by n, $\langle n \rangle$, is a one-dimensional vector space and $A \otimes \langle n \rangle$ is mapped injectively by $\tilde{\alpha} \otimes I$ to $L_{k+1} \otimes N$. Since the image of $\tilde{\alpha} \otimes I$ on $A \otimes \langle n \rangle$ is the same as the image of $(\tilde{\alpha} \otimes k) \otimes I$ on $(A \otimes k) \otimes \langle n \rangle$, it follows that the rank of the matrix $\tilde{\alpha} \otimes k$ has rank equal to the rank of A. Thus, A is a direct summand of L_{k+1}. ☐

We now review a result of Jyotilingam [9] about cohomology modules of tensor products, applying it to the special case of syzygy bundles for our purposes. In the theorem below, N and M will be graded finite length S-modules where $S = k[X_0, X_1, \ldots, X_n]$ corresponding to $\mathbf{P}^n$. $\mathcal{P}_k(N)$ and $\mathcal{Q}_l(M)$ will indicate syzygy bundles obtained from minimal free resolutions of N and M. Note that in the minimal free resolution,

$$0 \to L_{n+1} \to L_n \to \cdots \to L_1 \to L_0 \to N \to 0,$$

when we tensor by M, the map $L_{n+1} \otimes M \to L_n \otimes M$, cannot be injective since M has finite length, hence $\mathrm{Tor}^S_{n+1}(N, M) \neq 0$, and by Lichtenbaum's theorem [10] $\mathrm{Tor}^S_i(N, M) \neq 0$ for all $i \leqslant n+1$.

Theorem 2.9. *Let N be a finite S-module and let $\mathcal{P}_k$ be its kth syzygy bundle on $\mathbf{P}^n$, with $k \geqslant 1$. Let $\mathcal{Q}$ be a bundle on $\mathbf{P}^n$ with $H^l_*(\mathcal{Q}) = M \neq 0$, with $k \leqslant l \leqslant n-2$, and with $H^i_*(\mathcal{Q}) = 0$ for $i = l-1, l-2, \ldots, l-k+2$. Then $H^{l+1}_*(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$.*

Proof. The cases $k=1$ and $k=2$ require no conditions on $H^{l-1}_*(\mathcal{Q})$. When $k=1$, we get the sequence $H^l_*(\mathcal{L}_1 \otimes \mathcal{Q}) \to H^l_*(\mathcal{L}_0 \otimes \mathcal{Q}) \to H^{l+1}_*(\mathcal{P}_1 \otimes \mathcal{Q}) \to 0$ and the map $L_1 \otimes M \to L_0 \otimes M$ can never be surjective. When $k > 1$, consider the diagram obtained from the sequences $0 \to \mathcal{P}_i \otimes \mathcal{Q} \to \mathcal{L}_i \otimes \mathcal{Q} \to \mathcal{P}_{i-1} \otimes \mathcal{Q} \to 0$, $i = k, k-1, k-2$ (with $\mathcal{P}_j = 0$ if $j < 0$ and $\mathcal{P}_0 = \mathcal{L}_0$):

$$\begin{array}{ccccc}
L_k \otimes M & = & L_k \otimes M & & H^{l-1}_*(\mathcal{P}_{k-3} \otimes \mathcal{Q}) \\
\downarrow & & \downarrow \gamma & & \downarrow \\
H^l_*(\mathcal{P}_{k-1} \otimes \mathcal{Q}) & \xrightarrow{\alpha} & L_{k-1} \otimes M & \xrightarrow{\beta} & H^l_*(\mathcal{P}_{k-2} \otimes \mathcal{Q}) \\
\downarrow \mu & & \downarrow \delta & & \downarrow \\
H^{l+1}_*(\mathcal{P}_k \otimes \mathcal{Q}) & & L_{k-2} \otimes M & = & L_{k-2} \otimes M
\end{array}$$

The vanishing conditions on $H^i_*(\mathcal{Q})$ show that $H^{l-1}_*(\mathcal{P}_{k-3} \otimes \mathcal{Q}) = H^{l-2}_*(\mathcal{P}_{k-4} \otimes \mathcal{Q}) = \cdots = H^{l-k+2}_*(\mathcal{L}_0 \otimes \mathcal{Q}) = 0$. So, $\ker \delta = \operatorname{im} \alpha$ and the diagram induces a surjection $\operatorname{im} \mu \to \mathrm{Tor}_{k-1}(N, M)$. By Lichtenbaum's theorem, $H^{l+1}_*(\mathcal{P}_k \otimes \mathcal{Q}) \neq 0$. □

3 | ISOLATED COHOMOLOGY OF TYPE (n, k), WITH $n \geqslant 4k$

In this section, we will prove that there are no indecomposable rank two bundles on $\mathbf{P}^n$ with isolated cohomology of type (n, k), where $n \geqslant 4k$. We study the sequence $0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_{n-k}(M) \to 0$ of Proposition 2.7. We will need to pay special attention to the case where N is a cyclic module. Hence the following lemma.

Lemma 3.1. *Let N be a graded cyclic S-module. For the corresponding syzygy bundle $\mathcal{P}_2(N)$ on $\mathbf{P}^n$, $H^3_*(S^2\mathcal{P}_2(N)) = 0$ and $H^3_*(\wedge^2\mathcal{P}_2(N)) \neq 0$.*

Proof. From the sequence $0 \to \mathcal{P}_2 \to \mathcal{L}_2 \to \mathcal{P}_1 \to 0$ obtained from a minimal resolution of N, it suffices to show that the map $H^1_*(\mathcal{L}_2 \otimes \mathcal{P}_1) \to H^1_*(\wedge^2\mathcal{P}_1)$ is surjective to prove that

$H^3_*(S^2\mathcal{P}_2(N)) = 0$. This map can be studied using the natural commuting diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{L}_2 \otimes \mathcal{P}_1 & \to & \mathcal{L}_2 \otimes \mathcal{L}_1 & \to & \mathcal{L}_2 \otimes \mathcal{L}_0 & \to 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \to & \wedge^2\mathcal{P}_1 & \to & \wedge^2\mathcal{L}_1 & \to & \mathcal{L}_1 \otimes \mathcal{L}_0 & \to & S^2\mathcal{L}_0 & \end{array}$$

It simplifies when $\mathcal{L}_0$ has rank one, where without loss of generality, we can take $\mathcal{L}_0$ to be $\mathcal{O}_{\mathbf{P}^n}$, yielding

$$\begin{array}{ccccccc} 0 \to & \mathcal{L}_2 \otimes \mathcal{P}_1 & \to & \mathcal{L}_2 \otimes \mathcal{L}_1 & \to & \mathcal{L}_2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & . \\ 0 \to & \wedge^2\mathcal{P}_1 & \to & \wedge^2\mathcal{L}_1 & \to & \mathcal{P}_1 & \to 0 \end{array}$$

Since $\mathcal{L}_2$ surjects onto the global sections of $\mathcal{P}_1$, it follows from the diagram of long exact sequences of cohomology modules that $H^1_*(\mathcal{L}_2 \otimes \mathcal{P}_1) \to H^1_*(\wedge^2\mathcal{P}_1)$ is onto.

For the second part, we will show that $H^3_*(\mathcal{P}_2 \otimes \mathcal{P}_2) \neq 0$. (This argument will be repeated later in a slightly different setting.) With $H^3_*(S^2\mathcal{P}_2) = 0$, since $H^3_*(\mathcal{P}_2 \otimes \mathcal{P}_2) = H^3_*(S^2\mathcal{P}_2) \oplus H^3_*(\wedge^2\mathcal{P}_2)$, the conclusion of the lemma follows.

Consider $0 \to \mathcal{P}_2 \otimes \mathcal{P}_2 \to \mathcal{L}_2 \otimes \mathcal{P}_2 \to \mathcal{L}_1 \otimes \mathcal{P}_2 \to \mathcal{L}_0 \otimes \mathcal{P}_2 \to 0$. From $0 \to \mathcal{P}_1 \otimes \mathcal{P}_2 \to \mathcal{L}_1 \otimes \mathcal{P}_2 \to \mathcal{L}_0 \otimes \mathcal{P}_2 \to 0$, we get

$$H^2_*(\mathcal{P}_1 \otimes \mathcal{P}_2) = \ker(L_1 \otimes N \to L_0 \otimes N) = L_1 \otimes N$$

since N is cyclic. Hence, we get

$$H^3_*(\mathcal{P}_2 \otimes \mathcal{P}_2) = \text{ coker } (L_2 \otimes N \to L_1 \otimes N),$$

which is clearly nonzero. □

Proposition 3.2. *Suppose that $\mathcal{E}$ on $\mathbf{P}^n$ is a rank two bundle of type (n, k) with $n \geqslant 7$, k strictly less than $\frac{n}{2}$. Then the sequence $0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_l(M) \to 0$, in Proposition 2.7, is not-split.*

Proof. Suppose $\mathcal{P} \oplus \mathcal{F} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$. Neither $\mathcal{P}_k(N)$ nor $\mathcal{P}_{n-k}(M)$ has any line bundle summands, hence $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$. So, $\wedge^2\mathcal{P}$ has summands $\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M)$ and $\wedge^2\mathcal{P}_k(N)$. If $k > 2$, then using Proposition 2.9, $H^{n-k+1}_*(\mathcal{P}_k(N) \otimes \mathcal{P}_{n-k}(M))$ is nonzero which contradicts the requirement in Proposition 2.7 that $H^{n-k+1}_*(\wedge^2\mathcal{P}) = 0$.

If $k = 2$, there are two cases: if N is cyclic, then $H^3_*(\wedge^2\mathcal{P}_2(N)) \neq 0$ by Lemma 3.1, which contradicts Proposition 2.7 since $n - k > 3$ when $n \geqslant 6$.

If N is noncyclic, then from the sequences $0 \to \mathcal{P}_2(N) \to \mathcal{L}_2 \to \mathcal{P}_1(N) \to 0$ and $0 \to \mathcal{P}_1 \to \mathcal{L}_1 \to \mathcal{L}_0 \to 0$, we get $H^4_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. This a contradiction to Proposition 2.7 when $n \geqslant 7$. □

Remark 3.3. The case $n = 6, k = 2$ is not answered above. A weaker argument can be made here that even though $\mathcal{P} = \mathcal{P}_k(N) \oplus \mathcal{P}_{n-k}(M)$, N itself is neither cyclic nor a direct sum of submodules $N_1 \oplus N_2$.

Theorem 3.4. *Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^8$ with $H^3_*(\mathcal{E}) = H^4_*(\mathcal{E}) = 0$, then $\mathcal{E}$ splits.*

Proof. Let $N = H^2_*(\mathcal{E})$ and $M = H^6_*(\mathcal{E})$. Both are nonzero unless $\mathcal{E}$ splits. By Proposition 3.2 (with $k = 2$), we know that the sequence below is nonsplit.

$$0 \to \mathcal{P}_2(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_6(M) \to 0. \tag{5}$$

The proof will analyze the consequences of the two sequences below obtained from sequence.

$$0 \to S^2\mathcal{P}_2(N) \to \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \wedge^2\mathcal{P}_6(M) \to 0, \tag{6}$$

$$0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_6(M) \to 0. \tag{7}$$

Case 1 If N is cyclic, we look at the sequence (6).
It breaks into

$$\begin{aligned} 0 \to S^2\mathcal{P}_2(N) \to \mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}] \to \mathcal{D} \to 0, \\ 0 \to \mathcal{D} \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \wedge^2\mathcal{P}_6(M) \to 0, \end{aligned} \tag{8}$$

$H^3_*(\mathcal{P}_2(N) \otimes [\mathcal{P} \oplus \mathcal{F}]) \neq 0$ by the same argument in the second part of the proof of Lemma 3.1, and by the same lemma, $H^3_*(S^2\mathcal{P}_2(N)) = 0$. Hence, $H^3_*(\mathcal{D}) \neq 0$ from the first sequence in (8).

In the second sequence in (8), $H^3_*(\mathcal{P}) = 0$. Hence so is $H^3_*(\wedge^2\mathcal{P})$. Finally, $\mathcal{P}_6(M)$ fits into a sequence with free bundles

$$0 \to \mathcal{L}'_9 \to \mathcal{L}'_8 \to \mathcal{L}'_7 \to \mathcal{P}_6 \to 0.$$

This yields two exact sequences

$$\begin{aligned} 0 \to S^2\mathcal{P}_7 \to S^2\mathcal{L}'_7 \to \mathcal{L}'_7 \otimes \mathcal{P}_6 \to \wedge^2\mathcal{P}_6 \to 0, \\ 0 \to \wedge^2\mathcal{L}'_9 \to \wedge^2\mathcal{L}'_8 \to \mathcal{L}'_8 \otimes \mathcal{P}_7 \to S^2\mathcal{P}_7 \to 0. \end{aligned} \tag{9}$$

From these, we can chase down $H^2_*(\wedge^2\mathcal{P}_6)$ to be equal to zero since $H^2_*(\mathcal{P}_6) = 0, H^4_*(\mathcal{P}_7) = 0, H^6_*(\wedge^2\mathcal{L}'_9) = 0$. Hence, $H^3_*(\mathcal{D})$ is both zero and nonzero, a contradiction.

Case 2 If N is noncyclic, we look at the sequence (7)

$$0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_6(M) \to 0.$$

It breaks into

$$\begin{aligned} 0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{D} \to 0, \\ 0 \to \mathcal{D} \to \mathcal{P}_6(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_6(M) \to 0. \end{aligned} \tag{10}$$

From

$$0 \to S^2\mathcal{P}_1(N) \to S^2\mathcal{L}_1 \to \mathcal{L}_1 \otimes \mathcal{L}_0 \to \wedge^2\mathcal{L}_0 \to 0,$$

$$0 \to \wedge^2\mathcal{P}_2(N) \to \wedge^2\mathcal{L}_2 \to \mathcal{L}_2 \otimes \mathcal{L}_1 \to S^2\mathcal{P}_1(N) \to 0,$$

we get $H^2_*(S^2\mathcal{P}_1(N)) \neq 0$ and $H^4_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. Since $H^4_*(\mathcal{P})$ and $H^4_*(\wedge^2\mathcal{P})$ are zero, we obtain $H^3_*(\mathcal{D}) \neq 0$.

Again, in the second sequence in (10), $H^3_*(\mathcal{P}_6(M) \otimes \mathcal{F}) = 0$ and $H^3_*(\mathcal{P}_6(M) \otimes \mathcal{P})$ can be studied using a resolution for $\mathcal{P}_6(M)$ and tensoring with $\mathcal{P}$.

$$0 \to \mathcal{L}'_9 \otimes \mathcal{P} \to \mathcal{L}'_8 \otimes \mathcal{P} \to \mathcal{L}'_7 \otimes \mathcal{P} \to \mathcal{P}_6(M) \otimes \mathcal{P} \to 0.$$

Then $H^3_*(\mathcal{P}_6(M) \otimes \mathcal{P}) = 0$ since $H^3_*(\mathcal{P}), H^4_*(\mathcal{P}), H^5_*(\mathcal{P})$ are all zero.

We compute $H^2_*(S^2\mathcal{P}_6(M))$, breaking up the resolution of $\mathcal{P}_6$ (suppressing the letter M) into short exact sequences:

$$\begin{aligned} &0 \to \wedge^2\mathcal{P}_7 \to \wedge^2\mathcal{L}'_7 \to \mathcal{L}'_7 \otimes \mathcal{P}_6 \to S^2\mathcal{P}_6 \to 0, \\ &0 \to S^2\mathcal{L}'_9 \to S^2\mathcal{L}'_8 \to \mathcal{L}'_8 \otimes \mathcal{P}_7 \to \wedge^2\mathcal{P}_7 \to 0. \end{aligned} \tag{11}$$

$H^2_*(S^2\mathcal{P}_6(M))$ will vanish since $H^2_*(\mathcal{P}_6)$, $H^4_*(\mathcal{P}_7)$ and $H^6_*(S^2\mathcal{L}'_9)$ are all zero. □

Corollary 3.5. *Let $n \geqslant 8$. Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^n$ with $H^i_*(\mathcal{E}) = 0$ for $i = 3, \dots n-3$. Then $\mathcal{E}$ splits.*

Proof. Use induction on n. The case $n = 8$ is proved in the above theorem. Assume the result for $n-1$. Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^n$ with $H^i_*(\mathcal{E}) = 0$ for $i = 3, \dots n-3$. For a hyperplane H, by the restriction sequence in cohomology,

$$H^i_*(\mathcal{E}) \to H^i_*(\mathcal{E}_H) \to H^{i+1}_*(\mathcal{E}(-1)),$$

we get that $H^i_*(\mathcal{E}_H) = 0$ for $i = 3, \dots n-4$ on $\mathbf{P}^{n-1}$. So, $\mathcal{E}_H$ splits and hence also $\mathcal{E}$. □

The theorem above can be generalized to arbitrary k using the similar calculations.

Theorem 3.6. *Let $n \geqslant 4k$, with $k > 1$. Then there cannot exist a rank two bundle $\mathcal{E}$ on $\mathbf{P}^n$, for which the only nonzero intermediate cohomology modules are $H^1_*(\mathcal{E})$, $H^k_*(\mathcal{E}) = N$, $H^{n-k}_*(\mathcal{E}) = M$, and $H^{n-1}_*(\mathcal{E})$.*

Proof. The case $k = 2$ was done in the corollary above. So, we assume that $k > 2$. The proof will analyze the consequences of the sequence

$$0 \to \mathcal{P}_k(N) \to \mathcal{P} \oplus \mathcal{F} \to \mathcal{P}_{n-k}(M) \to 0, \tag{12}$$

which is nonsplit by Proposition 3.2. We get the collateral sequence:

$$0 \to \wedge^2\mathcal{P}_k(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_{n-k}(M) \to 0. \tag{13}$$

We will prove it using several cases.

Case 1 The case where N is cyclic, k is even and > 2.

We look at the sequence (13) which breaks into

$$\begin{aligned} &0 \to \wedge^2\mathcal{P}_k(N) \to \wedge^2\mathcal{P} \oplus [\mathcal{P} \otimes \mathcal{F}] \oplus \wedge^2\mathcal{F} \to \mathcal{D} \to 0, \\ &0 \to \mathcal{D} \to \mathcal{P}_{n-k}(M) \otimes [\mathcal{P} \oplus \mathcal{F}] \to S^2\mathcal{P}_{n-k}(M) \to 0, \end{aligned} \tag{14}$$

$H^3_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. This yields $H^{2k-1}_*(\wedge^2\mathcal{P}_k(N)) \neq 0$, since $n > 2k-1$. On the other hand, $H^{2k-1}_*(\mathcal{P})$ and $H^{2k-1}_*(\wedge^2\mathcal{P})$ are zero, since $k < 2k-1 < n-k$ when $n \geqslant 4k$. Hence $H^{2k-2}_*(\mathcal{D}) \neq 0$ using the first short exact sequence in (14).

In the second sequence in (14), $H^{2k-2}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{F}) = 0$ since $2k-2 \neq n-k$. $H^{2k-2}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{P})$ can be studied using a resolution for $\mathcal{P}_{n-k}(M)$ and tensoring with $\mathcal{P}$.

$$0 \to \mathcal{L}'_{n+1} \otimes \mathcal{P} \to \mathcal{L}'_n \otimes \mathcal{P} \to \dots \mathcal{L}'_{n-k+2} \otimes \mathcal{P} \to \mathcal{L}'_{n-k+1} \otimes \mathcal{P} \to \mathcal{P}_{n-k}(M) \otimes \mathcal{P} \to 0.$$

Then $H^{2k-2}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{P}) = 0$ provided $H^{2k-2}_*(\mathcal{P}), H^{2k-1}_*(\mathcal{P}), \dots, H^{3k-2}_*(\mathcal{P})$ are all zero. Since $n \geqslant 4k$, $n-k > 3k-2$ and since $k > 2$, $k < 2k-2$. Hence, these vanishings hold.

We compute $H^{2k-3}_*(S^2\mathcal{P}_{n-k}(M))$, breaking up the resolution of $\mathcal{P}_{n-k}$ (suppressing the letter M) into short exact sequences:

$$\begin{aligned} 0 \to \wedge^2\mathcal{P}_{n-k+1} &\to \wedge^2\mathcal{L}'_{n-k+1} \to \mathcal{L}'_{n-k+1} \otimes \mathcal{P}_{n-k} \to S^2\mathcal{P}_{n-k} \\ 0 \to S^2\mathcal{P}_{n-k+2} &\to \wedge^2\mathcal{L}'_{n-k+2} \to \mathcal{L}'_{n-k+2} \otimes \mathcal{P}_{n-k+1} \to \wedge^2\mathcal{P}_{n-k+1} \\ 0 \to \wedge^2\mathcal{P}_{n-k+3} &\to \wedge^2\mathcal{L}'_{n-k+3} \to \mathcal{L}'_{n-k+3} \otimes \mathcal{P}_{n-k+2} \to S^2\mathcal{P}_{n-k+2} \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ 0 \to S^2\mathcal{L}'_{n+1} &\to S^2\mathcal{L}'_n \to \mathcal{L}'_n \otimes \mathcal{P}_{n-1} \to \wedge^2\mathcal{P}_{n-1}. \end{aligned} \tag{15}$$

$H^{2k-3}_*(S^2\mathcal{P}_{n-k}(M))$ will vanish provided $H^{2k-3}_*(\mathcal{P}_{n-k})$, $H^{2k-1}_*(\mathcal{P}_{n-k+1})$, …, $H^{4k-5}_*(\mathcal{P}_{n-1})$ and $H^{4k-3}_*(S^2\mathcal{L}'_{n+1})$ are all zero. $H^{4k-3}_*(S^2\mathcal{L}'_{n+1}) = 0$ since $n > 4k-3$. For the others, $H^{2k-3+2i}_*(\mathcal{P}_{n-k+i}) = 0$ since $n-k+i > 2k-3+2i$ when $0 \leqslant i \leqslant k-1$. We have concluded that $H^{2k-2}_*(\mathcal{D}) = 0$ from the second sequence, contradicting the earlier result of being nonzero.

Case 2 The case where N is noncyclic, $k > 2$ is even.

This is very similar to Case 1. We use the same sequence (13). Now $H^4_*(\wedge^2\mathcal{P}_2(N)) \neq 0$. Hence, $H^{2k}_*(\wedge^2\mathcal{P}_k(N)) \neq 0$, since $n > 2k$. $H^{2k}_*(\mathcal{P})$ and $H^{2k}_*(\wedge^2\mathcal{P})$ are zero, since $k < 2k < n-k$, hence $H^{2k-1}_*(\mathcal{D}) \neq 0$.

Again, $H^{2k-1}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{F}) = 0$ since $2k-1 \neq n-k$ and $H^{2k-1}_*(\mathcal{P}_{n-k}(M) \otimes \mathcal{P}) = 0$ since $n-k > 3k-1$ and $k < 2k-1$. Lastly, $H^{2k-2}_*(S^2\mathcal{P}_{n-k}(M)) = 0$ since $n > 4k-2$ and $n-k+i > 2k-2+2i$ when $0 \leqslant i \leqslant k-1$. Hence, $H^{2k-1}_*(\mathcal{D})$ is also equal to 0.

Case 3 The case where k is odd.

$H^{2k}_*(\wedge^2\mathcal{P}_k(N)) \neq 0$ as in Case 2. We use sequence (13) and copy the proof in Case 2. □

ACKNOWLEDGEMENTS

The first author is members of the GNSAGA group of INdAM. The second author would like to thank the first author for the warm hospitality during his stay at Politecnico di Torino.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. W. Barth and K. Hulek, *Monads and moduli of vector bundles*, Manuscripta Math. **25** (1978), 323–347.
2. W. Barth and A. Van de Ven, *A decomposability criterion for algebraic 2-bundles on projective spaces*, Invent. Math. **25** (1974), 91–106.
3. I. Coandă and G. Trautmann, *The splitting criterion of Kempf and the Babylonian tower theorem*, Comm. Algebra **34** (2006), no. 7, 2485–2488.
4. W. Decker, *Monads and cohomology modules of rank two bundles*, Comp. Math. **76** (1990), 7–17.
5. L. Ein, *An analogue of Max Noether's theorem*, Duke Math. J. **52** (1985), 689–706.
6. G. Evans and P. Griffiths, *The Syzygy problem*, Annals of Math. **114** (1981), 323–353.
7. G. Horrocks, *Vector bundles on the punctured spectrum of a ring*, Proc. London Math. Soc. (3) **14** (1964), 689–713.
8. G. Horrocks, *Construction of bundles on* $\mathbf{P}^n$, Astérisque **71–72** (1980), 197–204.
9. P. Jothilingam, *A note on grade*, Nagoya Math J. **59** (1975), 149–152.
10. S. Lichtenbaum, *On the vanishing of Tor in regular local rings*, Illinois J. Math. **10** (1966), 220–226.
11. F. Malaspina, *Monads and rank 3 vector bundles on quadrics*, Ann. Mat. Pura Appl. **188** (2009), 455–465.
12. F. Malaspina and P. Rao, *Horrocks correspondence on ACM varieties*, Algebra Number Theory **9** (2016), no. 4, 981–1003.
13. N. Mohan Kumar and P. Rao, *Buchsbaum bundles on* $\mathbf{P}^n$, J. Pure Appl. Algebra **152** (2000), 195–199.
14. N. Mohan Kumar, C. Peterson, and P. Rao, *Monads on projective spaces*, Manuscripta Math. **112** (2003), 183–189.
15. P. Rao, *A note on cohomology modules of rank two bundles*, J. Algebra **86** (1984), 23–34.