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**ELEMENTS OF HIGHER HOMOTOPY GROUPS  
UNDETECTABLE BY POLYHEDRAL APPROXIMATION**

JOHN K. ACETI AND JEREMY BRAZAS

# ELEMENTS OF HIGHER HOMOTOPY GROUPS UNDETECTABLE BY POLYHEDRAL APPROXIMATION

JOHN K. ACETI AND JEREMY BRAZAS

When nontrivial local structures are present in a topological space  $X$ , a common approach to characterizing the isomorphism type of the  $n$ -th homotopy group  $\pi_n(X, x_0)$  is to consider the image of  $\pi_n(X, x_0)$  in the  $n$ -th Čech homotopy group  $\check{\pi}_n(X, x_0)$  under the canonical homomorphism  $\Psi_n : \pi_n(X, x_0) \rightarrow \check{\pi}_n(X, x_0)$ . The subgroup  $\ker(\Psi_n)$  is the obstruction to this tactic as it consists of precisely those elements of  $\pi_n(X, x_0)$ , which cannot be detected by polyhedral approximations to  $X$ . In this paper, we use higher dimensional analogues of Spanier groups to characterize  $\ker(\Psi_n)$ . In particular, we prove that if  $X$  is paracompact, Hausdorff, and  $LC^{n-1}$ , then  $\ker(\Psi_n)$  is equal to the  $n$ -th Spanier group of  $X$ . We also use the perspective of higher Spanier groups to generalize a theorem of Kozłowski–Segal, which gives conditions ensuring that  $\Psi_n$  is an isomorphism.

## 1. Introduction

When nontrivial local structures are present in a topological space  $X$ , a common approach to characterizing the isomorphism type of  $\pi_n(X, x_0)$  is to consider the image of  $\pi_n(X, x_0)$  in the  $n$ -th Čech (shape) homotopy group  $\check{\pi}_n(X, x_0)$  under the canonical homomorphism  $\Psi_n : \pi_n(X, x_0) \rightarrow \check{\pi}_n(X, x_0)$ . The  $n$ -th shape kernel  $\ker(\Psi_n)$  is the obstruction to this tactic as it consists of precisely those elements of  $\pi_n(X, x_0)$ , which cannot be detected by polyhedral approximations to  $X$ . This method has proved successful in many situations for both the fundamental group [Cannon and Conner 2006; Eda and Kawamura 1998; Fischer and Guilbault 2005; Fischer and Zastrow 2005] and higher homotopy groups [Brazas 2021; Eda and Kawamura 2000a; 2010; Eda et al. 2013; Kawamura 2003]. In this paper, we study the map  $\Psi_n$  and give a characterization the  $n$ -th shape kernel in terms of higher-dimensional analogues of Spanier groups.

The subgroups of fundamental groups, which are now commonly referred to as “Spanier groups,” first appeared in E.H. Spanier’s unique approach [1966] to

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*MSC2020:* primary 55P55, 55Q07, 55Q52; secondary 54C56.

*Keywords:* shape homotopy group, higher Spanier group,  $\pi_n$ -shape injective,  $n$ -dimensional earring space.

covering space theory. If  $\mathcal{U}$  is an open cover of a topological space  $X$  and  $x_0 \in X$ , then the *Spanier group with respect to  $\mathcal{U}$*  is the subgroup  $\pi_1^{Sp}(\mathcal{U}, x_0)$  of  $\pi_1(X, x_0)$  generated by path-conjugates  $[\alpha][\gamma][\alpha]^{-1}$  where  $\alpha$  is a path starting at  $x_0$  and  $\gamma$  is a loop based at  $\alpha(1)$  with image being contained in some element of  $\mathcal{U}$ . These subgroups are particularly relevant to covering space theory since, when  $X$  is locally path-connected, a subgroup  $H \leq \pi_1(X, x_0)$  corresponds to a covering map  $p: (Y, y_0) \rightarrow (X, x_0)$  if and only if  $\pi_1^{Sp}(\mathcal{U}, x_0) \leq H$  for some open cover  $\mathcal{U}$  [Spanier 1966, 2.5.12]. The intersection  $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathcal{U}} \pi_1^{Sp}(\mathcal{U}, x_0)$  is called the *Spanier group of  $(X, x_0)$*  [Fischer et al. 2011]. The inclusion  $\pi_1^{Sp}(X, x_0) \subseteq \ker(\Psi_1)$  always holds [Fischer and Zastrow 2007, Proposition 4.8]. It is proved in [Brazas and Fabel 2014, Theorem 6.1] that  $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$  whenever  $X$  is paracompact Hausdorff and locally path connected. The upshot of this equality is having a description of level-wise generators (for each open cover  $\mathcal{U}$ ) whereas there may be no readily available generating set for the kernel of a homomorphism induced by a canonical map from  $X$  to the nerve  $|N(\mathcal{U})|$ . Indeed, 1-dimensional Spanier groups have proved useful in persistence theory [Virk 2020]. Since much of applied topology is based on a geometric refinement of polyhedral approximation from shape theory, there seems potential for higher dimensional analogues to be useful as well.

Higher dimensional analogues of Spanier groups recently appeared in [Bahredar et al. 2021] and are defined in a similar way:  $\pi_n^{Sp}(\mathcal{U}, x_0)$  is the subgroup of  $\pi_n(X, x_0)$  consisting of homotopy classes of path-conjugates  $\alpha * f$  where  $\alpha$  is a path starting at  $x_0$  and  $f: S^n \rightarrow X$  is based at  $\alpha(1)$  with image being contained in some element of  $\mathcal{U}$ . Then  $\pi_n^{Sp}(X, x_0)$  is the intersection of these subgroups. In this paper, we prove a higher-dimensional analogue of the 1-dimensional equality  $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$  from [Brazas and Fabel 2014].

A space  $X$  is  $LC^n$  if for every neighborhood  $U$  of a point  $x \in X$ , there is a neighborhood  $V$  of  $x$  in  $U$  such that every map  $f: S^k \rightarrow V$ ,  $0 \leq k \leq n$  is null-homotopic in  $U$ . When a space is  $LC^n$ , “small” maps on spheres of dimension  $\leq n$  contract by null-homotopies of relatively the same size. Certainly, every locally  $n$ -connected space is  $LC^n$ . However, when  $n \geq 1$ , the converse is not true even for metrizable spaces. Our main result is the following.

**Theorem 1.1.** *Let  $n \geq 1$  and  $x_0 \in X$ . If  $X$  is paracompact, Hausdorff, and  $LC^{n-1}$ , then  $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$ .*

This result confirms that higher Spanier groups, like their 1-dimensional counterparts, often identify precisely those elements of  $\pi_n(X, x_0)$  which can be detected by polyhedral approximations to  $X$ . More precisely, under the hypotheses of Theorem 1.1,  $g \in \pi_n^{Sp}(X, x_0)$  if and only if  $f_{\#}(g) = 0$  for every map  $f: X \rightarrow K$  to a polyhedron  $K$ . A first countable path-connected space is  $LC^0$  if and only if it

is locally path connected. Hence, in dimension  $n = 1$ , [Theorem 1.1](#) only expands [[Brazas and Fabel 2014](#), [Theorem 6.1](#)] to some nonfirst countable spaces.

Regarding the proof of [Theorem 1.1](#), the inclusion  $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$  was first proved for  $n = 1$  in [[Fischer and Zastrow 2007](#), [Proposition 4.8](#)] and for  $n \geq 2$  in [[Bahredar et al. 2021](#), [Theorem 4.14](#)]. We include this proof for the sake of completion ([Corollary 3.11](#)). The proof of the inclusion  $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$  appears in [Section 5](#) and is more intricate, requiring a carefully chosen sequence of open cover refinements using the  $LC^{n-1}$  property. These refinements allow one to recursively extend maps on simplicial complexes skeleton-wise. These extension methods, established in [Section 4](#), are similar to methods found in [[Kozłowski and Segal 1977](#); [1978](#)].

We also put these extension methods to work in [Section 6](#) where we identify conditions that imply  $\Psi_n$  is an isomorphism. Kozłowski and Segal [[1978](#)], proved that if  $X$  is paracompact Hausdorff and  $LC^n$ , then  $\Psi_n$  is an isomorphism. Fischer and Zastrow [[2007](#)], generalized this result in dimension  $n = 1$  by replacing “ $LC^1$ ” with “locally path connected and semilocally simply connected.” Similar, to the approach of Fischer and Zastrow, our use of Spanier groups shows that the existence of *small* null-homotopies of small maps  $S^n \rightarrow X$  (specifically in dimension  $n$ ) is not necessary to prove that  $\Psi_n$  is injective. We say a space  $X$  is *semilocally  $\pi_n$ -trivial* if for every  $x \in X$  there exists an open neighborhood  $U$  of  $x$  such that every map  $S^n \rightarrow U$  is null-homotopic in  $X$ . This definition is independent of lower dimensions but certainly  $LC^n \Rightarrow (LC^{n-1}$  and semilocally  $\pi_n$ -trivial). Our second result proves Kozłowski–Segal’s theorem under a weaker hypothesis and is stated as follows.

**Theorem 1.2.** *Let  $n \geq 1$  and  $x_0 \in X$ . If  $X$  is paracompact, Hausdorff,  $LC^{n-1}$ , and semilocally  $\pi_n$ -trivial, then  $\Psi_n : \pi_n(X, x_0) \rightarrow \check{\pi}_n(X, x_0)$  is an isomorphism.*

The hypotheses in [Theorem 1.2](#) are the homotopical versions of the hypotheses used in [[Mardešić 1959](#)] to ensure that the canonical homomorphism  $\varphi_* : H_n(X) \rightarrow \check{H}_n(X)$  is an isomorphism; see also [[Eda and Kawamura 2000b](#)] regarding the surjectivity of  $\varphi_*$ . Examples show that  $\Psi_n$  can fail to be an isomorphism if  $X$  is semilocally  $\pi_n$ -trivial but not  $LC^{n-1}$  ([Example 7.4](#)) or if  $X$  is  $LC^{n-1}$  but not semilocally  $\pi_n$ -trivial ([Example 7.5](#)).

The authors are grateful to the referee for many suggestions, which substantially improved the exposition of this paper.

## 2. Preliminaries and notation

Throughout this paper,  $X$  is assumed to be a path-connected topological space with basepoint  $x_0$ . The unit interval is denoted  $I$  and  $S^n$  is the unit  $n$ -sphere with basepoint  $d_0 = (1, 0, \dots, 0)$ . The  $n$ -th homotopy group of  $(X, x_0)$  is denoted

$\pi_n(X, x_0)$ . If  $f : (X, x_0) \rightarrow (Y, y_0)$  is a based map, then  $f_{\#} : \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$  is the induced homomorphism.

A *path* in a space  $X$  is a map  $\alpha : I \rightarrow X$  from the unit interval. The *reverse* of  $\alpha$  is the path given by  $\alpha^{-}(t) = \alpha(1 - t)$  and the concatenation of two paths  $\alpha, \beta$  with  $\alpha(1) = \beta(0)$  is denoted  $\alpha \cdot \beta$ . Similarly, if  $f, g : S^n \rightarrow X$  are maps based at  $x \in X$ , then  $f \cdot g$  denotes the usual  $n$ -loop concatenation and  $f^{-}$  denotes the reverse map. We may write  $\prod_{i=1}^m f_i$  to denote an  $m$ -fold concatenation  $f_1 \cdot f_2 \cdot \dots \cdot f_m$ .

**2.1. Simplicial complexes.** We make heavy use of standard notation and theory of abstract and geometric simplicial complexes, which can be found in texts such as [Mardešić and Segal 1982; Munkres 1984]. We briefly recall relevant notation.

For an abstract (geometric) simplicial complex  $K$  and integer  $r \geq 0$ ,  $K_r$  denotes the  $r$ -skeleton of  $K$ . If  $K$  is abstract,  $|K|$  denotes the geometric realization of  $K$  with the weak topology. If  $K$  is geometric, then  $\text{sd}^m K$  denotes the  $m$ -th barycentric subdivision of  $K$  and if  $v$  is a vertex of  $K$ , then  $\text{st}(v, K)$  denotes the open star of the vertex  $v$ . When  $L \subseteq K$  is a subcomplex,  $\text{sd}^m L$  is a subcomplex of  $\text{sd}^m K$ . If  $\sigma = \{v_0, v_1, \dots, v_r\}$  is a  $r$ -simplex of  $K$ , then  $[v_0, v_1, \dots, v_r]$  denotes the  $r$ -simplex of  $|K|$  with the indicated orientation.

We frequently make use of the standard  $n$ -simplex  $\Delta_n$  in  $\mathbb{R}^n$  spanned by the origin  $\mathbf{o}$  and standard unit vectors. Since the boundary  $\partial \Delta_n = (\Delta_n)_{n-1}$  is homeomorphic to  $S^{n-1}$ , we fix a based homeomorphism  $\partial \Delta_n \cong S^{n-1}$  that allows us to represent elements of  $\pi_n(X, x_0)$  by maps  $(\partial \Delta_{n+1}, \mathbf{o}) \rightarrow (X, x_0)$ .

**2.2. The Čech expansion and shape homotopy groups.** We now recall the construction of the first shape homotopy group  $\check{\pi}_1(X, x_0)$  via the Čech expansion. For more details; see [Mardešić and Segal 1982].

Let  $\mathcal{O}(X)$  be the set of open covers of  $X$  directed by refinement; we write  $\mathcal{V} \succeq \mathcal{U}$  when  $\mathcal{V}$  refines  $\mathcal{U}$ . Similarly, let  $\mathcal{O}(X, x_0)$  be the set of open covers with a distinguished element containing  $x_0$ , i.e., the set of pairs  $(\mathcal{U}, U_0)$  where  $\mathcal{U} \in \mathcal{O}(X)$ ,  $U_0 \in \mathcal{U}$ , and  $x_0 \in U_0$ . We say  $(\mathcal{V}, V_0)$  refines  $(\mathcal{U}, U_0)$  if  $\mathcal{V} \succeq \mathcal{U}$  and  $V_0 \subseteq U_0$ .

The nerve of a cover  $(\mathcal{U}, U_0) \in \mathcal{O}(X, x_0)$  is the abstract simplicial complex  $N(\mathcal{U})$  whose vertex set is  $N(\mathcal{U})_0 = \mathcal{U}$  and vertices  $A_0, \dots, A_n \in \mathcal{U}$  span an  $n$ -simplex if  $\bigcap_{i=0}^n A_i \neq \emptyset$ . The vertex  $U_0$  is taken to be the basepoint of the geometric realization  $|N(\mathcal{U})|$ . Whenever  $(\mathcal{V}, V_0)$  refines  $(\mathcal{U}, U_0)$ , we can construct a simplicial map  $p_{\mathcal{U}\mathcal{V}} : N(\mathcal{V}) \rightarrow N(\mathcal{U})$ , called a *projection*, by sending a vertex  $V \in N(\mathcal{V})$  to a vertex  $U \in \mathcal{U}$  such that  $V \subseteq U$ . In particular, we make a convention that  $p_{\mathcal{U}\mathcal{V}}(V_0) = U_0$ . Any such assignment of vertices extends linearly to a simplicial map. Moreover, the induced map  $|p_{\mathcal{U}\mathcal{V}}| : |N(\mathcal{V})| \rightarrow |N(\mathcal{U})|$  is unique up to based homotopy. Thus the homomorphism  $p_{\mathcal{U}\mathcal{V}\#} : \pi_1(|N(\mathcal{V})|, V_0) \rightarrow \pi_1(|N(\mathcal{U})|, U_0)$  induced on fundamental groups is (up to coherent isomorphism) independent of the choice of simplicial map.

Recall that an open cover  $\mathcal{U}$  of  $X$  is normal if it admits a partition of unity subordinated to  $\mathcal{U}$ . Let  $\Lambda$  be the subset of  $\mathcal{O}(X, x_0)$  (also directed by refinement) consisting of pairs  $(\mathcal{U}, U_0)$  where  $\mathcal{U}$  is a normal open cover of  $X$  and such that there is a partition of unity  $\{\phi_U\}_{U \in \mathcal{U}}$  subordinated to  $\mathcal{U}$  with  $\phi_{U_0}(x_0) = 1$ . It is well-known that every open cover of a paracompact Hausdorff space  $X$  is normal. Moreover, if  $(\mathcal{U}, U_0) \in \mathcal{O}(X, x_0)$ , it is easy to refine  $(\mathcal{U}, U_0)$  to a cover  $(\mathcal{V}, V_0)$  such that  $V_0$  is the only element of  $\mathcal{V}$  containing  $x_0$  and therefore  $(\mathcal{V}, V_0) \in \Lambda$ . Thus, for paracompact Hausdorff  $X$ ,  $\Lambda$  is cofinal in  $\mathcal{O}(X, x_0)$ .

The  $n$ -th shape homotopy group is the inverse limit

$$\check{\pi}_n(X, x_0) = \varprojlim (\pi_n(|N(\mathcal{U})|, U_0), p_{\mathcal{U}\mathcal{V}\#}, \Lambda).$$

This group is also referred to as the  $n$ -th Čech homotopy group.

Given an open cover  $(\mathcal{U}, U_0) \in \mathcal{O}(X, x_0)$ , a map  $p_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$  is a (*based*) *canonical map* if  $p_{\mathcal{U}}^{-1}(\text{st}(U, N(\mathcal{U}))) \subseteq U$  for each  $U \in \mathcal{U}$  and  $p_{\mathcal{U}}(x_0) = U_0$ . Such a canonical map is guaranteed to exist if  $(\mathcal{U}, U_0) \in \Lambda$ : find a locally finite partition of unity  $\{\phi_U\}_{U \in \mathcal{U}}$  subordinated to  $\mathcal{U}$  such that  $\phi_{U_0}(x_0) = 1$ . When  $U \in \mathcal{U}$  and  $x \in U$ , determine  $p_{\mathcal{U}}(x)$  by requiring its barycentric coordinate belonging to the vertex  $U$  of  $|N(\mathcal{U})|$  to be  $\phi_U(x)$ . According to this construction, the requirement  $\phi_{U_0}(x_0) = 1$  gives  $p_{\mathcal{U}}(x_0) = U_0$ .

A canonical map  $p_{\mathcal{U}}$  is unique up to based homotopy and whenever  $(\mathcal{V}, V_0)$  refines  $(\mathcal{U}, U_0)$ , the compositions  $p_{\mathcal{U}\mathcal{V}} \circ p_{\mathcal{V}}$  and  $p_{\mathcal{U}}$  are homotopic as based maps. Hence, for  $n \geq 1$ , the homomorphisms

$$p_{\mathcal{U}\#} : \pi_n(X, x_0) \rightarrow \pi_n(|N(\mathcal{U})|, U_0)$$

satisfy  $p_{\mathcal{U}\mathcal{V}\#} \circ p_{\mathcal{V}\#} = p_{\mathcal{U}\#}$ . These homomorphisms induce the following canonical homomorphism to the limit, which is natural in the continuous maps of based spaces:

$$\Psi_n : \pi_n(X, x_0) \rightarrow \check{\pi}_n(X, x_0) \quad \text{given by } \Psi_n([f]) = ([p_{\mathcal{U}} \circ f]).$$

The subgroup  $\ker(\Psi_n)$ , which we refer to as the  $n$ -th shape kernel is, in a rough sense, an algebraic measure of the  $n$ -dimensional homotopical information lost when approximating  $X$  by polyhedra. Since  $(p_{\mathcal{U}})$  forms an HPol-expansion of  $X$  [Mardešić and Segal 1982, Appendix 1, Sectin 3.2, Theorem 8], we have  $[f] \in \pi_n(X, x_0) \setminus \ker(\Psi_n)$  if and only if there exist a polyhedron  $K$  and a map  $p : (X, x_0) \rightarrow (K, k_0)$  such that  $p_{\#}([f]) \neq 0$  in  $\pi_n(K, k_0)$ . Of utmost importance is the situation when  $\ker(\Psi_n) = 0$ . In this case,  $\pi_n(X, x_0)$  can be understood as a subgroup of  $\check{\pi}_n(X, x_0)$ , that is, the  $n$ -th shape group retains all the data in the  $n$ -th homotopy group of  $X$ . A space for which  $\ker(\Psi_n) = 0$  is said to be  $\pi_n$ -shape injective.

### 3. Higher Spanier groups

To define higher Spanier groups as in [Bahredar et al. 2021], we briefly recall the action of the fundamental groupoid on the higher homotopy groups of a space. Fix a retraction  $R : S^n \times I \rightarrow S^n \times \{0\} \cup \{d_0\} \times I$ . Given a map  $f : (S^n, d_0) \rightarrow (X, y_0)$  and a path  $\alpha : I \rightarrow X$  with  $\alpha(0) = x_0$  and  $\alpha(1) = y_0$ , define  $F : S^n \times \{0\} \cup \{d_0\} \times I \rightarrow X$  so that  $g(x, 0) = f(x)$  and  $f(d_0, t) = \alpha(1 - t)$ . The *path-conjugate of  $f$  by  $\alpha$*  is the map  $\alpha * f : (S^n, d_0) \rightarrow (X, x_0)$  given by  $\alpha * f(x) = F(R(x, 1))$ .

Path-conjugation defines the basepoint-change isomorphism  $\varphi_\alpha : \pi_n(X, y_0) \rightarrow \pi_n(X, x_0)$ ,  $\varphi_\alpha([f]) = [\alpha * f]$ . In particular,  $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$ . Additionally, if  $[\alpha] = [\beta]$ , which we write to mean that the paths  $\alpha$  and  $\beta$  are homotopic relative to their endpoints, then  $[\alpha * f] = [\beta * f]$ . Note that when  $n = 1$ ,  $f : S^1 \rightarrow X$  is a loop and  $\alpha * f \simeq \alpha \cdot f \cdot \alpha^-$ .

**Definition 3.1.** Let  $n \geq 1$  and  $\alpha : (I, 0) \rightarrow (X, x_0)$  be a path and  $U$  be an open neighborhood of  $\alpha(1)$  in  $X$ . Define

$$[\alpha] * \pi_n(U) = \{[\alpha * f] \in \pi_n(X, x_0) \mid f(S^n) \subseteq U, f(d_0) = \alpha(1)\}.$$

Since  $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$ , the set  $[\alpha] * \pi_n(U)$  is a subgroup of  $\pi_n(X, x_0)$ .

**Definition 3.2.** Let  $n \geq 1$ ,  $\mathcal{U}$  be an open cover of  $X$ , and  $x_0 \in X$ . The  *$n$ -th Spanier group of  $(X, x_0)$  with respect to  $\mathcal{U}$*  is the subgroup  $\pi_n^{Sp}(\mathcal{U}, x_0)$  of  $\pi_n(X, x_0)$  generated by the subgroups  $[\alpha] * \pi_n(U)$  for all pairs  $(\alpha, U)$  with  $\alpha(1) \in U$  and  $U \in \mathcal{U}$ . In short

$$\pi_n^{Sp}(\mathcal{U}, x_0) = \langle [\alpha] * \pi_n(U) \mid U \in \mathcal{U}, \alpha(1) \in U \rangle.$$

The  *$n$ -th Spanier group of  $(X, x_0)$*  is the intersection

$$\pi_n^{Sp}(X, x_0) = \bigcap_{\mathcal{U} \in \mathcal{O}(X)} \pi_n^{Sp}(\mathcal{U}, x_0).$$

We may refer to subgroups of the form  $\pi_n^{Sp}(\mathcal{U}, x_0)$  as *relative Spanier groups* and to  $\pi_n^{Sp}(X, x_0)$  as the *absolute Spanier group*.

**Remark 3.3.** We note that our definition of  $n$ -th Spanier group is the “unbased” definition from [Bahredar et al. 2021]; see also [Fischer et al. 2011] for more on “based” Spanier groups, which is defined using covers of  $X$  by *pointed* open sets. The two notions agree for locally path connected spaces. When  $n = 1$ , Spanier groups (absolute and relative to a cover) are normal subgroups of  $\pi_1(X, x_0)$ . In the case  $n = 1$ , Spanier groups have been studied heavily due to their relationship to covering space theory [Spanier 1966].

**Remark 3.4** (functoriality). Let  $\text{Top}_*$  denote the category of based topological spaces and based continuous functions and  $\text{Grp}$  and  $\text{Ab}$  denote the usual categories



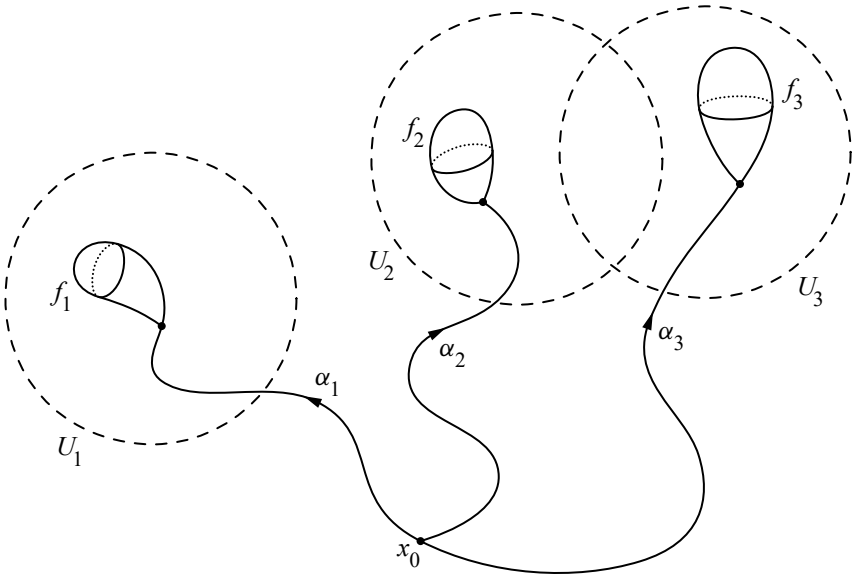
of groups and abelian groups respectively. If  $f : (X, x_0) \rightarrow (Y, y_0)$  is a map and  $\mathcal{V}$  is an open cover of  $Y$ , then  $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$  is an open cover of  $X$  such that  $f_{\#}(\pi_n(\mathcal{U}, x_0)) \subseteq \pi_n(\mathcal{V}, y_0)$ . It follows that  $f_{\#}(\pi_n^{Sp}(X, x_0)) \subseteq \pi_n^{Sp}(Y, y_0)$ . Thus  $(f_{\#})|_{\pi_n^{Sp}(X, x_0)} : \pi_n^{Sp}(X, x_0) \rightarrow \pi_n^{Sp}(Y, y_0)$  is well-defined showing that  $\pi_1^{Sp} : \text{Top}_* \rightarrow \text{Grp}$  and  $\pi_n^{Sp} : \text{Top}_* \rightarrow \text{Ab}$ ,  $n \geq 2$ , are functors [Bahredar et al. 2021, Theorem 4.2]. Moreover, if  $g : (Y, y_0) \rightarrow (X, x_0)$  is a based homotopy inverse of  $f$ , then  $(f_{\#})|_{\pi_n^{Sp}(X, x_0)}$  and  $(g_{\#})|_{\pi_n^{Sp}(Y, y_0)}$  are inverse isomorphisms. Hence, these functors descend to functors  $\text{hTop}_* \rightarrow \text{Grp}$  and  $\text{hTop}_* \rightarrow \text{Ab}$  where  $\text{hTop}_*$  is the category of based spaces and basepoint-relative homotopy classes of based maps.

**Remark 3.5** (basepoint invariance). Suppose  $x_0, x_1 \in X$  and  $\beta : I \rightarrow X$  is a path from  $x_1$  to  $x_0$ , and  $\varphi_{\beta} : \pi_n(X, x_0) \rightarrow \pi_n(X, x_1)$ ,  $\varphi_{\beta}([g]) = [\beta * g]$  is the basepoint-change isomorphism. If  $[\alpha * f]$  is a generator of  $\pi_n^{Sp}(\mathcal{U}, x_0)$ , then  $\varphi_{\beta}([\alpha * f]) = [(\beta \cdot \alpha) * f]$  is a generator of  $\pi_n^{Sp}(\mathcal{U}, x_1)$ . It follows that  $\varphi_{\beta}(\pi_n^{Sp}(\mathcal{U}, x_0)) = \pi_n^{Sp}(\mathcal{U}, x_1)$ . Moreover, in the absolute case, we have  $\varphi_{\beta}(\pi_n^{Sp}(X, x_0)) = \pi_n^{Sp}(X, x_1)$ . In particular, changing the basepoint of  $X$  does not change the isomorphism type of the  $n$ -th Spanier group, particularly its triviality.

In terms of our choice of generators, a generic element of  $\pi_n^{Sp}(\mathcal{U}, x_0)$  is a product  $\prod_{i=1}^m [\alpha_i * f_i]$  where each map  $f_i : S^n \rightarrow X$  has an image in some open set  $U_i \in \mathcal{U}$  (see Figure 1). The next lemma identifies how such products might actually appear in practice and motivates the proof of our key technical lemma, Lemma 5.1. Recall that  $(\text{sd}^m \Delta_{n+1})_n$  is the union of the boundaries of the  $(n+1)$ -simplices in the  $m$ -th barycentric subdivision  $\text{sd}^m \Delta_{n+1}$ .

**Lemma 3.6.** *For  $m, n \in \mathbb{N}$ , let  $\mathcal{U}$  be an open cover of  $X$  and  $f : ((\text{sd}^m \Delta_{n+1})_n, \mathbf{o}) \rightarrow (X, x_0)$  be a map such that for every  $(n+1)$ -simplex  $\sigma$  of  $\text{sd}^m \Delta_{n+1}$ , we have  $f(\partial\sigma) \subseteq U$  for some  $U \in \mathcal{U}$ . Then  $f_{\#}(\pi_n((\text{sd}^m \Delta_{n+1})_n, \mathbf{o})) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$ .*

*Proof.* The case  $n = 1$  is proved in [Brazas and Fabel 2014]. Suppose  $n \geq 2$  and set  $K = \text{sd}^m \Delta_{n+1}$ . The set  $\mathcal{W} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$  is an open cover of  $K_n = (\text{sd}^m \Delta_{n+1})_n$  such that  $f_{\#}(\pi_n^{Sp}(\mathcal{W}, \mathbf{o})) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$  and for every  $(n+1)$ -simplex  $\sigma$  in  $K$ , we have  $\partial\sigma \subseteq f^{-1}(U)$  for some  $U \in \mathcal{U}$ . Thus it suffices to prove  $\pi_n(K_n, \mathbf{o}) \subseteq \pi_n^{Sp}(\mathcal{W}, \mathbf{o})$ . Let  $S$  be the set of  $(n+1)$ -simplices of  $K$ . Since  $n \geq 2$ ,  $K_n$  is simply connected. Standard simplicial homology arguments give that the reduced singular homology groups of  $K_n$  are trivial in dimension  $< n$  and  $H_n(K_n)$  is a finitely generated free abelian group. A set of free generators for  $H_n(K_n)$  can be chosen by fixing the homology class of a simplicial map  $g_{\sigma} : \partial\Delta_{n+1} \rightarrow K_n$  that sends  $\partial\Delta_{n+1}$  homeomorphically onto the boundary of an  $(n+1)$ -simplex  $\sigma \in S$ . Thus  $K_n$  is  $(n-1)$ -connected and the Hurewicz homomorphism  $h : \pi_k(K_n, \mathbf{o}) \rightarrow H_k(K_n)$  is an isomorphism for all  $1 \leq k \leq n$ . In particular, let  $p_{\sigma} : I \rightarrow K_n$  be any path from  $\mathbf{o}$  to  $g_{\sigma}(\mathbf{o})$ . Then  $\pi_n(K_n, \mathbf{o})$  is freely generated by the path-conjugates  $[p_{\sigma} * g_{\sigma}]$ ,  $\sigma \in S$ . By assumption, for every  $\sigma \in S$ ,  $[p_{\sigma} * g_{\sigma}]$  is a generator of



**Figure 1.** An element of  $\pi_2^{Sp}(\mathcal{U}, x_0)$ , which is a product of three path-conjugate generators  $[\alpha_i * f_i]$ .

$\pi_n^{Sp}(\mathcal{W}, \mathbf{o})$ . Since  $\pi_n^{Sp}(\mathcal{W}, \mathbf{o})$  contains all the generators of  $\pi_n(K_n, \mathbf{o})$ , the inclusion  $\pi_n(K_n, \mathbf{o}) \subset \pi_n^{Sp}(\mathcal{W}, \mathbf{o})$  follows.  $\square$

To characterize the triviality of relative Spanier groups, we establish the following terminology.

**Definition 3.7.** Let  $n \geq 0$  and  $x \in X$ . We say the space  $X$  is:

- (1) *Semilocally  $\pi_n$ -trivial at  $x$*  if there exists an open neighborhood  $U$  of  $x$  in  $X$  such that every map  $S^n \rightarrow U$  is null-homotopic in  $X$ .
- (2) *Semilocally  $n$ -connected at  $x$*  if there exists an open neighborhood  $U$  of  $x$  in  $X$  such that every map  $S^k \rightarrow U$ ,  $0 \leq k \leq n$  is null-homotopic in  $X$ .

We say  $X$  is *semilocally  $\pi_n$ -trivial* (resp. *semilocally  $n$ -connected*) if it has this property at all of its points.

It is straightforward to see that  $X$  is semilocally  $n$ -connected at  $x$  if and only if  $X$  is semilocally  $\pi_k$ -trivial at  $x$  for all  $0 \leq k \leq n$ .

**Remark 3.8.** A space  $X$  is semilocally  $\pi_n$ -trivial if and only if  $X$  admits an open cover  $\mathcal{U}$  such that  $\pi_n^{Sp}(\mathcal{U}, x_0)$  is trivial [Bahredar et al. 2021, Theorem 3.7]. Moreover,  $X$  is semilocally  $n$ -connected if and only if  $X$  admits an open cover  $\mathcal{U}$  such that  $\pi_k^{Sp}(\mathcal{U}, x_0)$  is trivial for all  $1 \leq k \leq n$ . Note that local path connectivity is independent of the properties given in Definition 3.7.

Attempting a proof of [Theorem 1.1](#), one should not expect the groups  $\pi_n^{Sp}(\mathcal{U}, x_0)$  and  $\ker(p_{\mathcal{U}\#})$  to agree “on the nose.” Indeed, the following example shows that we should not expect the equality  $\pi_n^{Sp}(\mathcal{U}, x_0) = \ker(p_{\mathcal{U}\#})$  to hold even in the “nicest” local circumstances.

**Example 3.9.** Let  $X = S^2 \vee S^2$  and  $W$  be a contractible neighborhood of  $d_0$  in  $S^2$ . Set  $U_1 = S^2 \vee W$  and  $U_2 = W \vee S^2$  and consider the open cover  $\mathcal{U} = \{U_1, U_2\}$  of  $X$ . Then  $\pi_3^{Sp}(\mathcal{U}, x_0) \cong \mathbb{Z}^2$  is freely generated by the homotopy classes of the two inclusions  $i_1, i_2 : S^2 \rightarrow X$ . However,  $\pi_3(X) \cong \mathbb{Z}^3$  is freely generated by  $[i_1]$ ,  $[i_2]$ , and the Whitehead product  $[[i_1, i_2]]$ . However  $|N(\mathcal{U})|$  is a 1-simplex and is therefore contractible. Thus  $\ker(p_{\mathcal{U}\#})$  is equal to  $\pi_3(X)$  and contains  $[[i_1, i_2]]$ . Even though the spaces  $X, U_1, U_2$  are locally contractible and the elements of  $\mathcal{U}$  are 1-connected,  $\pi_n^{Sp}(\mathcal{U}, x_0)$  is a proper subgroup of  $\ker(p_{\mathcal{U}\#})$ . One can view this failure as the result of two facts: (1) The sets  $U_i$  are not 2-connected and (2) the definition of Spanier group does not allow one to generate homotopy classes by taking Whitehead products of maps  $S^2 \rightarrow U_i$  in the neighboring elements of  $\mathcal{U}$ .

First, we show the inclusion  $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$  holds in full generality. Recall that the intersections  $\pi_n^{Sp}(X, x_0) = \bigcap_{\mathcal{U} \in O(X)} \pi_n^{Sp}(\mathcal{U}, x_0)$  and  $\ker(\Psi_n) = \bigcap_{(\mathcal{U}, U_0) \in \Lambda} \ker(p_{\mathcal{U}\#})$  are formally indexed by different sets.

**Lemma 3.10.** *For every open cover  $\mathcal{U}$  of  $X$  and canonical map  $p_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$ , there exists a refinement  $\mathcal{V} \succeq \mathcal{U}$  such that  $\pi_n^{Sp}(\mathcal{V}, x_0) \subseteq \ker(p_{\mathcal{U}\#})$  in  $\pi_n(X, x_0)$ .*

*Proof.* Let  $\mathcal{U} \in O(X)$ . The stars  $\text{st}(U, |N(\mathcal{U})|)$ ,  $U \in \mathcal{U}$  form an open cover of  $|N(\mathcal{U})|$  by contractible sets and therefore  $\mathcal{V} = \{p_{\mathcal{U}}^{-1}(\text{st}(U, |N(\mathcal{U})|)) \mid U \in \mathcal{U}\}$  is an open cover of  $X$ . Since  $p_{\mathcal{U}}$  is a canonical map, we have  $p_{\mathcal{U}}^{-1}(\text{st}(U, |N(\mathcal{U})|)) \subseteq U$  for all  $U \in \mathcal{U}$ . Thus  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ . A generator of  $\pi_n^{Sp}(\mathcal{V}, x_0)$  is of the form  $[\alpha * f]$  for a map  $f : S^n \rightarrow p_{\mathcal{U}}^{-1}(\text{st}(U, |N(\mathcal{U})|))$ . However,  $p_{\mathcal{U}} \circ f$  has image in the contractible open set  $\text{st}(U, |N(\mathcal{U})|)$  and is therefore null-homotopic. Thus  $p_{\mathcal{U}\#}([\alpha * f]) = 0$ . We conclude that  $p_{\mathcal{U}\#}(\pi_n^{Sp}(\mathcal{V}, x_0)) = 0$ .  $\square$

**Corollary 3.11** [[Bahredar et al. 2021](#), Theorem 4.14]. *Let  $n \geq 1$ . For any based space  $(X, x_0)$ , we have  $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ .*

*Proof.* Suppose  $[f] \in \pi_n^{Sp}(X, x_0)$ . Given a normal, based open cover  $(\mathcal{U}, U_0) \in \Lambda$  and any canonical map  $p_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$ , [Lemma 3.10](#) ensures we can find a refinement  $\mathcal{V} \succeq \mathcal{U}$  such that  $\pi_n^{Sp}(\mathcal{V}, x_0) \subseteq \ker(p_{\mathcal{U}\#})$ . Thus  $[f] \in \pi_n^{Sp}(\mathcal{V}, x_0) \subseteq \ker(p_{\mathcal{U}\#})$ . Since  $(\mathcal{U}, U_0)$  is arbitrary, we conclude that  $[f] \in \ker(\Psi_n)$ .  $\square$

**Example 3.12** (higher earring spaces). An important space, which we will call upon repeatedly for examples, is the  $n$ -dimensional earring space

$$\mathbb{E}_n = \bigcup_{j \in \mathbb{N}} \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x} - (1/j, 0, 0, \dots, 0)\| = 1/j\},$$

which is a shrinking wedge (one-point union) of  $n$ -spheres with basepoint the origin  $\mathbf{o}$ . It is known that  $\mathbb{E}_n$  is  $(n - 1)$ -connected, locally  $(n - 1)$ -connected, and  $\pi_n$ -shape injective for all  $n \geq 1$  [Eda and Kawamura 2000a; Morgan and Morrison 1986]. However,  $\mathbb{E}_n$  is not semilocally  $\pi_n$ -trivial. Thus  $\pi_n^{Sp}(\mathcal{U}, \mathbf{o}) \neq 0$  for any open cover  $\mathcal{U}$  of  $\mathbb{E}_n$  even though in the absolute case  $\pi_n^{Sp}(\mathbb{E}_n, \mathbf{o})$  is trivial.

**Example 3.13.** Let  $n \geq 3$  and notice that  $\mathbb{E}_1 \vee \mathbb{E}_n$  is not semilocally  $\pi_1$ -trivial (since it has  $\mathbb{E}_1$  as a retract) and therefore fails to be semilocally  $(n - 1)$ -connected. However, it has recently been shown that  $\pi_k(\mathbb{E}_1 \vee \mathbb{E}_n) = 0$  for  $2 \leq k \leq n - 1$  and that  $\mathbb{E}_1 \vee \mathbb{E}_n$  is  $\pi_n$ -shape injective [Brazas 2021]. Thus  $\mathbb{E}_1 \vee \mathbb{E}_n$  is semilocally  $\pi_k$ -trivial for all  $k \leq n - 1$  except  $k = 1$  and  $\pi_n^{Sp}(\mathbb{E}_1 \vee \mathbb{E}_n, \mathbf{o}) = 0$ . Thus the failure to be semilocally  $n$ -connected can occur at single dimension less than  $n$ .

### 4. Recursive extension lemmas

Toward a proof of the inclusion  $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$  for  $LC^{n-1}$  space  $X$ , we introduce some convenient notation and definitions. If  $\mathcal{U}$  is an open cover and  $A \subseteq X$ , then  $\text{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid A \cap U \neq \emptyset\}$ . Note that if  $A \subseteq B$ , then  $\text{St}(A, \mathcal{U}) \subseteq \text{St}(B, \mathcal{U})$ . Also if  $\mathcal{V} \succeq \mathcal{U}$ , then  $\text{St}(A, \mathcal{V}) \subseteq \text{St}(A, \mathcal{U})$ . We take the following terminology from [Willard 1970].

**Definition 4.1.** Let  $\mathcal{U}, \mathcal{V} \in O(X)$ :

- (1) We say  $\mathcal{V}$  is a *barycentric-star refinement* of  $\mathcal{U}$  if for every  $x \in X$ , we have  $\text{St}(x, \mathcal{V}) \subseteq U$  for some  $U \in \mathcal{U}$ . We write  $\mathcal{V} \succeq_* \mathcal{U}$ .
- (2) We say  $\mathcal{V}$  is a *star refinement* of  $\mathcal{U}$  if for every  $V \in \mathcal{V}$ , we have  $\text{St}(V, \mathcal{V}) \subseteq U$  for some  $U \in \mathcal{U}$ . We write  $\mathcal{V} \succeq_{**} \mathcal{U}$ .

Note that if  $\mathcal{U} \preceq_* \mathcal{V} \preceq_* \mathcal{W}$ , then  $\mathcal{U} \preceq_{**} \mathcal{W}$ .

**Lemma 4.2** [Stone 1948]. *A  $T_1$  space  $X$  is paracompact if and only if for every  $\mathcal{U} \in O(X)$  there exists  $\mathcal{V} \in O(X)$  such that  $\mathcal{V} \succeq_* \mathcal{U}$ .*

**Definition 4.3.** [Mardešić and Segal 1982, Chapter I, Section 3.2.5] Let  $n \in \{0, 1, 2, 3, \dots, \infty\}$ . A space  $X$  is  $LC^n$  at  $x \in X$  if for every neighborhood  $U$  of  $x$ , there exists a neighborhood  $V$  of  $x$  such that  $V \subseteq U$  and such that for all  $0 \leq k \leq n$  ( $k < \infty$  if  $n = \infty$ ), every map  $f : \partial\Delta_{k+1} \rightarrow V$  extends to a map  $g : \Delta_{k+1} \rightarrow U$ . We say  $X$  is  $LC^n$  if  $X$  is  $LC^n$  at all of its points.

We have the following evident implications for both the point-wise and global properties:

$$X \text{ is locally } n\text{-connected} \Rightarrow X \text{ is } LC^n \Rightarrow X \text{ is semilocally } n\text{-connected.}$$

For first countable spaces, the  $LC^n$  property is equivalent to the “ $n$ -tame” property in [Brazas 2021] defined in terms of shrinking sequences of maps.

**Definition 4.4.** Suppose  $\mathcal{V} \succeq \mathcal{U}$  in  $O(X)$ :

- (1) We say  $\mathcal{V}$  is an *n-refinement* of  $\mathcal{U}$ , and write  $\mathcal{V} \succeq^n \mathcal{U}$ , if for all  $0 \leq k \leq n$ ,  $V \in \mathcal{V}$ , and maps  $f : \partial\Delta_{k+1} \rightarrow V$ , there exists  $U \in \mathcal{U}$  with  $V \subseteq U$  and a continuous extension  $g : \Delta_{k+1} \rightarrow U$  of  $f$ .
- (2) We say  $\mathcal{V}$  is an *n-barycentric-star refinement* of  $\mathcal{U}$ , and write  $\mathcal{V} \succeq_*^n \mathcal{U}$ , if for every  $0 \leq k \leq n$ , for every  $x \in X$ , and every map  $f : \partial\Delta_{k+1} \rightarrow \text{St}(x, \mathcal{V})$ , there exists  $U \in \mathcal{U}$  with  $\text{St}(x, \mathcal{V}) \subseteq U$  and a continuous extension  $g : \Delta_{k+1} \rightarrow U$  of  $f$ .

Note that if  $\mathcal{V} \succeq^n \mathcal{U}$  (resp.  $\mathcal{V} \succeq_*^n \mathcal{U}$ ), then  $\mathcal{V} \succeq^k \mathcal{U}$  (resp.  $\mathcal{V} \succeq_*^k \mathcal{U}$ ) for all  $0 \leq k \leq n$ .

**Lemma 4.5.** *Suppose  $X$  is paracompact, Hausdorff, and  $LC^n$ . For every  $\mathcal{U} \in O(X)$ , there exists  $\mathcal{V} \in O(X)$  such that  $\mathcal{V} \succeq_*^n \mathcal{U}$ .*

*Proof.* Let  $\mathcal{U} \in O(X)$ . Since  $X$  is  $LC^n$ , for every  $U \in \mathcal{U}$  and  $x \in U$ , there exists an open neighborhood  $W(U, x)$  such that  $W(U, x) \subseteq U$  and such that for all  $0 \leq k \leq n$ , each map  $f : \partial\Delta_{k+1} \rightarrow W(U, x)$  extends to a map  $g : \Delta_{k+1} \rightarrow U$ . Let  $\mathcal{W} = \{W(U, x) \mid U \in \mathcal{U}, x \in U\}$  and note  $\mathcal{W} \succeq^n \mathcal{U}$ . Since  $X$  is paracompact Hausdorff, by [Lemma 4.2](#), there exists  $\mathcal{V} \in O(X)$  such that  $\mathcal{V} \succeq_* \mathcal{W}$ .

Fix  $x' \in X$ . Then  $\text{St}(x', \mathcal{V}) \subseteq W(U, x)$  for some  $x \in U \in \mathcal{U}$ . Then  $\text{St}(x', \mathcal{V}) \subseteq U$ . Moreover, if  $0 \leq k \leq n$  and  $f : \partial\Delta_{k+1} \rightarrow \text{St}(x', \mathcal{V})$  is a map, then since  $f$  has image in  $W(U, x)$ , there is an extension  $g : \Delta_{k+1} \rightarrow U$ . This verifies that  $\mathcal{V} \succeq_*^n \mathcal{U}$ .  $\square$

For the next two lemmas, we fix  $n \in \mathbb{N}$ , a geometric simplicial complex  $K$  with  $\dim K = n + 1$ , and a subcomplex  $L \subseteq K$  with  $\dim L \leq n$ . Let  $M[k] = L \cup K_k$  denote the union of  $L$  and the  $k$ -skeleton of  $K$ . Since  $L \subseteq K_n$ ,  $M[n] = K_n$  is the union of the boundaries of the  $(n + 1)$ -simplices of  $K$ . Later we will consider the cases where (1)  $K = \text{sd}^m \Delta_{n+1}$  and  $L = \text{sd}^m \partial\Delta_{n+1}$  and (2)  $K = \text{sd}^m \partial\Delta_{n+2}$  and  $L = \{\mathfrak{o}\}$ .

**Lemma 4.6** (recursive extensions). *Suppose  $1 \leq k \leq n$ ,  $\mathcal{U} \preceq_* \mathcal{V} \preceq_*^{k-1} \mathcal{W}$ ,  $m \in \mathbb{N}$ , and  $f : M[k - 1] \rightarrow X$  is a map such that for every  $(n + 1)$ -simplex  $\sigma$  of  $K$ , we have  $f(\sigma \cap M[k - 1]) \subseteq W_\sigma$  for some  $W_\sigma \in \mathcal{W}$ . Then there exists a continuous extension  $g : M[k] \rightarrow X$  of  $f$  such that for every  $(n + 1)$ -simplex  $\sigma$  of  $K$ , we have  $g(\sigma \cap M[k]) \subseteq U_\sigma$  for some  $U_\sigma \in \mathcal{U}$ .*

*Proof.* Supposing the hypothesis, we must extend  $f$  to the  $k$ -simplices of  $M[k]$  that do not lie in  $L$ . Let  $\tau$  be a  $k$ -simplex of  $M[k]$  that does not lie in  $L$  and let  $S_\tau$  be the set of  $(n + 1)$ -simplices in  $K$  that contain  $\tau$ . By assumption,  $S_\tau$  is nonempty. We make some general observations first. Since  $f$  maps the  $(k - 1)$ -skeleton of each  $(n + 1)$ -simplex  $\sigma \in S_\tau$  into  $W_\sigma$  and  $\partial\tau$  lies in this  $(k - 1)$ -skeleton, we have

$f(\partial\tau) \subseteq \bigcap_{\sigma \in S_\tau} W_\sigma$ . Thus, for all  $\tau$ , we have

$$f(\partial\tau) \subseteq \bigcap_{\sigma \in S_\tau} \text{St}(W_\sigma, \mathcal{V}).$$

Fix a vertex  $v_\tau$  of  $\tau$  and let  $x_\tau = f(v_\tau)$ . Then  $x_\tau \in W_\sigma \subseteq \text{St}(x_\tau, \mathcal{W})$  whenever  $\sigma \in S_\tau$ . Since  $\mathcal{W} \succeq_*^{k-1} \mathcal{V}$ , we may find  $V_\tau \in \mathcal{V}$  such that  $\text{St}(x_\tau, \mathcal{W}) \subseteq V_\tau$  and such that every map  $\partial\Delta_k \rightarrow \text{St}(x_\tau, \mathcal{W})$  extends to a map  $\Delta_k \rightarrow V_\tau$ . In particular,  $f|_{\partial\tau} : \partial\tau \rightarrow W_\sigma$  extends to a map  $\tau \rightarrow V_\tau$ . We define  $g : M[k] \rightarrow X$  so that it agrees with  $f$  on  $M[k-1]$  and so that the restriction of  $g$  to  $\tau$  is a choice of continuous extension  $\tau \rightarrow V_\tau$  of  $f|_{\partial\tau}$ .

We now choose the sets  $U_\sigma$ . Fix an  $(n+1)$ -simplex  $\sigma$  of  $K$ . If the  $k$ -skeleton of  $\sigma$  lies entirely in  $L$ , we choose any  $U_\sigma \in \mathcal{U}$  satisfying  $W_\sigma \subseteq U_\sigma$ . Suppose there exists at least one  $k$ -simplex in  $\sigma$  not in  $L$ . Then whenever  $\tau$  is a  $k$ -simplex of  $\sigma$  not in  $L$ , we have  $W_\sigma \subseteq \text{St}(x_\tau, \mathcal{W}) \subseteq V_\tau$ . Fix a point  $y_\sigma \in W_\sigma$ . The assumption that  $\mathcal{V} \succeq_* \mathcal{U}$  implies that there exists  $U_\sigma \in \mathcal{U}$  such that  $\text{St}(y_\sigma, \mathcal{V}) \subseteq U_\sigma$ . In this case, we have  $W_\sigma \subseteq V_\tau \subseteq U_\sigma$  whenever  $\tau$  is a  $k$ -simplex of  $\sigma$  not in  $L$ .

Finally, we check that  $g$  satisfies the desired property. Again, fix an  $(n+1)$ -simplex  $\sigma$  of  $K$ . If  $\tau$  is a  $k$ -simplex of  $\sigma$  not in  $L$ , our definition of  $g$  gives  $g(\tau) \subseteq V_\tau \subseteq U_\sigma$ . If  $\tau'$  is a  $k$ -simplex in  $\sigma \cap L$ , then  $g(\tau') = f(\tau') \subseteq W_\sigma \subseteq U_\sigma$ . Overall, this shows that  $g(\sigma \cap M[k]) \subseteq U_\sigma$  for each  $(n+1)$ -simplex  $\sigma$  of  $K$ .  $\square$

A direct, recursive application of the previous lemma is given in the following statement.

**Lemma 4.7.** *Suppose there is a sequence of open covers*

$$\mathcal{W}_n \preceq_* \mathcal{V}_n \preceq_*^{n-1} \mathcal{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathcal{W}_2 \preceq_* \mathcal{V}_2 \preceq_*^1 \mathcal{W}_1 \preceq_* \mathcal{V}_1 \preceq_*^0 \mathcal{W}_0$$

and a map  $f_0 : M[0] \rightarrow X$  such that for every  $(n+1)$ -simplex  $\sigma$  of  $K$ , we have  $f_0(\sigma \cap M[0]) \subseteq W$  for some  $W \in \mathcal{W}_0$ . Then there exists an extension  $f_n : M[n] \rightarrow X$  of  $f_0$  such that for every  $(n+1)$ -simplex  $\sigma$  of  $K$ , we have  $f_n(\partial\sigma) \subseteq U$  for some  $U \in \mathcal{W}_n$ .

### 5. A proof of Theorem 1.1

We apply the extension results of the previous section in the case where  $K = \text{sd}^m \Delta_{n+1}$  for some  $m \in \mathbb{N}$  and  $L = \text{sd}^m \partial\Delta_{n+1}$  so that  $M[k] = L \cup K_k$  consists of the  $n$ -simplices of the boundary of  $\Delta_{n+1}$  and the  $k$ -simplices of  $\text{sd}^m \Delta_{n+1}$  not in the boundary. Note that  $M[n]$  is the union of the boundaries of the  $(n+1)$ -simplices of  $\text{sd}^m \Delta_{n+1}$ .

**Lemma 5.1.** *Let  $n \geq 1$ . Suppose  $X$  is paracompact, Hausdorff, and  $LC^{n-1}$ . Then for every open cover  $\mathcal{U}$  of  $X$ , there exists  $(\mathcal{V}, V_0) \in \Lambda$  such that  $\ker(p_{\mathcal{V}\#}) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$ .*

*Proof.* Suppose  $\mathcal{U} \in O(X)$ . Since  $X$  is paracompact, Hausdorff, and  $LC^{n-1}$ , we may apply Lemmas 4.2 and 4.5 to first find a sequence of refinements

$$\mathcal{U} = \mathcal{U}_n \preceq_* \mathcal{V}_n \preceq_*^{n-1} \mathcal{U}_{n-1} \preceq_* \cdots \preceq_*^2 \mathcal{U}_2 \preceq_* \mathcal{V}_2 \preceq_*^1 \mathcal{U}_1 \preceq_* \mathcal{V}_1 \preceq_*^0 \mathcal{U}_0$$

and then one last refinement  $\mathcal{U}_0 \preceq_* \mathcal{V}_0 = \mathcal{V}$ . Let  $V_0 \in \mathcal{V}$  be any set containing  $x_0$  and recall that since  $X$  is paracompact Hausdorff  $(\mathcal{V}, V_0) \in \Lambda$ . We will show that  $\ker(p_{\mathcal{V}\#}) \subseteq \pi_n^{Sp}(\mathcal{U}, x_0)$ . Note that  $p_{\mathcal{V}}^{-1}(\text{st}(V, N(\mathcal{V}))) \subseteq V$  by the definition of canonical map  $p_{\mathcal{V}}$ .

Suppose  $[f] \in \ker(p_{\mathcal{V}\#})$  is represented by a map  $f : (|\partial \Delta_{n+1}|, \mathbf{o}) \rightarrow (X, x_0)$ . We will show that  $[f] \in \pi_n^{Sp}(\mathcal{U}, x_0)$ . Then  $p_{\mathcal{V}} \circ f : |\partial \Delta_{n+1}| \rightarrow |N(\mathcal{V})|$  is null-homotopic and extends to a map  $h : |\Delta_{n+1}| \rightarrow |N(\mathcal{V})|$ . Set  $Y_V = h^{-1}(\text{st}(V, N(\mathcal{V})))$  so that  $\mathcal{Y} = \{Y_V \mid V \in \mathcal{V}\}$  is an open cover of  $|\Delta_{n+1}|$ .

We find a particular simplicial approximation for  $h$  using the cover  $\mathcal{Y}$  [Munkres 1984, Theorem 16.1]: let  $\lambda$  be a Lebesgue number for  $\mathcal{Y}$  so that any subset of  $\Delta_{n+1}$  of diameter less than  $\lambda$  lies in some element of  $\mathcal{Y}$ . Find  $m \in \mathbb{N}$  such that each simplex in  $\text{sd}^m \Delta_{n+1}$  has diameter less than  $\lambda/2$ . Thus the star  $\text{st}(a, \text{sd}^m \Delta_{n+1})$  of each vertex  $a$  in  $\text{sd}^m \Delta_{n+1}$  lies in a set  $Y_{V_a} \in \mathcal{Y}$  for some  $V_a \in \mathcal{V}$ . The assignment  $a \mapsto V_a$  on vertices extends to a simplicial approximation  $h' : \text{sd}^m \Delta_{n+1} \rightarrow N(\mathcal{V})$  of  $h$ , i.e., a simplicial map  $h'$  such that

$$h(\text{st}(a, \text{sd}^m \Delta_{n+1})) \subseteq \text{st}(h'(a), N(\mathcal{V})) = \text{st}(V_a, N(\mathcal{V}))$$

for each vertex  $a$  [Munkres 1984, Lemma 14.1].

Let  $K = \text{sd}^m \Delta_{n+1}$  and  $L = \text{sd}^m \partial \Delta_{n+1}$  so that  $M[k] = L \cup K_k$ . First, we extend  $f : L \rightarrow X$  to a map  $f_0 : M[0] \rightarrow X$ . For each vertex  $a$  in  $K$ , pick a point  $f_0(a) \in V_a$ . In particular, if  $a \in L$ , take  $f_0(a) = f(a)$ . This choice is well defined since, for a boundary vertex  $a \in L$ , we have  $p_{\mathcal{V}} \circ f(a) = h(a) \in \text{st}(V_a, |N(\mathcal{V})|)$  and thus  $f(a) \in p_{\mathcal{V}}^{-1}(\text{st}(V_a, |N(\mathcal{V})|)) \subseteq V_a$ .

Note that  $h'$  maps every simplex  $\sigma = [a_0, a_1, \dots, a_k]$  of  $K$  to the simplex of  $N(\mathcal{V})$  spanned by  $\{h'(a_i) \mid 0 \leq i \leq k\} = \{V_{a_i} \mid 0 \leq i \leq k\}$ . By definition of the nerve, we have  $\bigcap \{V_{a_i} \mid 0 \leq i \leq k\} \neq \emptyset$ . Pick a point  $x_\sigma \in \bigcap \{V_{a_i} \mid 0 \leq i \leq k\}$ .

By our initial choice of refinements, we have  $\mathcal{U}_0 \preceq_* \mathcal{V}$ . If  $\sigma = [a_0, a_1, \dots, a_{n+1}]$  is an  $(n+1)$ -simplex of  $K$ , then  $\text{St}(x_\sigma, \mathcal{V}) \subseteq U_\sigma$  for some  $U_\sigma \in \mathcal{U}$ . In particular  $\{f_0(a_i) \mid 0 \leq i \leq n+1\} \subseteq \bigcup \{V_{a_i} \mid 0 \leq i \leq n+1\} \subseteq U_\sigma$ . Thus  $f_0$  maps the 0-skeleton of  $\sigma$  into  $U_\sigma$ . If  $1 \leq k \leq n$ ,  $\tau$  is a  $k$ -face of  $\sigma \cap L$  with  $a_i \in \tau$ , then  $p_{\mathcal{V}} \circ f_0(\text{int}(\tau)) = p_{\mathcal{V}} \circ f(\text{int}(\tau)) = h(\text{int}(\tau)) \subseteq h(\text{st}(a_i, K)) \subseteq \text{st}(V_{a_i}, |N(\mathcal{V})|)$ . It follows that

$$f_0(\tau) \subseteq p_{\mathcal{V}}^{-1}(\text{st}(V_{a_i}, |N(\mathcal{V})|)) \subseteq V_{a_i} \subseteq U_\sigma.$$

Thus for every  $n$ -simplex in  $\sigma \cap L$ , we have  $f_0(\tau) \subseteq U_\sigma$ . We conclude that for every  $(n+1)$ -simplex  $\sigma$  of  $K$ , we have  $f_0(\sigma \cap M[0]) \subseteq U_\sigma$ .



By our choice of sequence of refinements, we are precisely in the situation to apply [Lemma 4.7](#). Doing so, we obtain an extension  $f_n : M[n] \rightarrow X$  of  $f$  such that for every  $(n + 1)$ -simplex  $\sigma$  of  $K$ , we have  $f_n(\partial\sigma) \subseteq U_\sigma$  for some  $U_\sigma \in \mathcal{U}_n = \mathcal{U}$ . By [Lemma 3.6](#), we have  $[f] = [f_n|_{\partial\Delta_{n+1}}] \in \pi_n^{Sp}(\mathcal{U}, x_0)$ .  $\square$

Finally, both inclusions have been established and provide a proof of our main result.

*Proof of Theorem 1.1.* The inclusion  $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$  holds in general by [Corollary 3.11](#). Under the given hypotheses, the inclusion  $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$  follows from [Lemma 5.1](#).  $\square$

When considering examples relevant to [Theorem 1.1](#), it is helpful to compare  $\pi_n$ -shape injectivity with the following weaker property from [[Ghane and Hamed 2009](#)].

**Definition 5.2.** We say a space  $X$  is *n-homotopically Hausdorff* at  $x \in X$  if no nontrivial element of  $\pi_n(X, x)$  has a representing map in every neighborhood of  $x$ . We say  $X$  is *n-homotopically Hausdorff* if it is *n-homotopically Hausdorff* at each of its points.

Clearly,  $\pi_n$ -shape injectivity  $\Rightarrow$  *n-homotopically Hausdorff*. The next example, which highlights the effectiveness of [Theorem 1.1](#), shows the converse is not true even for  $LC^{n-1}$  Peano continua.

**Example 5.3.** Fix  $n \geq 2$  and let  $\ell_j : S^n \rightarrow \mathbb{E}_n$  be the inclusion of the  $j$ -th sphere and define  $f : \mathbb{E}_n \rightarrow \mathbb{E}_n$  to be the shift map given by  $f \circ \ell_j = \ell_{j+1}$ . Let  $M_f = \mathbb{E}_n \times [0, 1] / \sim$ ,  $(x, 0) \sim (f(x), 1)$  be the mapping torus of  $f$ . We identify  $\mathbb{E}_n$  with the image of  $\mathbb{E}_n \times \{0\}$  in  $M_f$  and take  $\mathbf{o}$  to be the basepoint of  $M_f$ . Let  $\alpha : I \rightarrow M_f$  be the loop where  $\alpha(t)$  is the image of  $(\mathbf{o}, t)$ . Then  $M_f$  is locally contractible at all points other than those in the image of  $\alpha$ . Also, every point  $\alpha(t)$  has a neighborhood that deformation retracts onto a homeomorphic copy of  $\mathbb{E}_n$ . Thus, since  $\mathbb{E}_n$  is  $LC^{n-1}$ , so is  $X$ . It follows from [Theorem 1.1](#) that  $\pi_n^{Sp}(M_f, \mathbf{o}) = \ker(\pi_n(M_f, \mathbf{o}) \rightarrow \check{\pi}_n(M_f, \mathbf{o}))$ . In particular, the Spanier group of  $M_f$  contains all elements  $[\alpha^k * g]$  where  $g : S^n \rightarrow \mathbb{E}_n$  is a based map and  $k \in \mathbb{Z}$ . Using the universal covering map  $E \rightarrow M_f$  that “unwinds”  $\alpha$  and the relation  $[g] = [\alpha * (f \circ g)]$  in  $\pi_n(M_f, \mathbf{o})$ , it is not hard to show that these are, in fact, the only elements of the  $n$ -th Spanier group. Hence,

$$\ker(\pi_n(M_f, \mathbf{o}) \rightarrow \check{\pi}_n(M_f, \mathbf{o})) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, \mathbf{o}), k \in \mathbb{Z}\},$$

which is an uncountable subgroup. Moreover, since  $M_f$  is shape equivalent to the aspherical space  $S^1$ , we have  $\check{\pi}_n(M_f, \mathbf{o}) = 0$  and thus  $\pi_n(M_f, \mathbf{o}) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, \mathbf{o}), k \in \mathbb{Z}\}$ .



It follows from this description that, even though  $M_f$  is not  $\pi_n$ -shape injective,  $M_f$  is  $n$ -homotopically Hausdorff. Indeed, it suffices to check this at the points  $\alpha(t)$ ,  $t \in I$ . We give the argument for  $\alpha(0) = \mathbf{o}$ , the other points are similar. If  $0 \neq h \in \pi_n(M_f, \mathbf{o})$  has a representative in every neighborhood of  $\mathbf{o}$  in  $M_f$ , then clearly  $h \in \ker(\Psi_n)$ . Hence,  $h = [\alpha^k * g]$  for  $[g] \in \pi_n(\mathbb{E}_n, \mathbf{o})$  and  $k \in \mathbb{Z}$ . Since  $M_f$  retracts onto the circle parametrized by  $\alpha$ , the hypothesis on  $h$  can only hold if  $k = 0$ . However, there is a basis of neighborhoods of  $\mathbf{o}$  in  $M_f$  that deformation retract onto an open neighborhood of  $\mathbf{o}$  in  $\mathbb{E}_n$ . Thus  $[g]$  has a representative in every neighborhood of  $\mathbf{o}$  in  $\pi_n(\mathbb{E}_n, \mathbf{o})$ , giving  $h = [g] \in \ker(\pi_n(\mathbb{E}_n, \mathbf{o}) \rightarrow \check{\pi}_n(\mathbb{E}_n, \mathbf{o})) = 0$ .

It is an important feature of [Example 5.3](#) that  $M_f$  is not simply connected and has multiple points at which it is not semilocally  $\pi_n$ -trivial. This motivates the following application of [Theorem 1.1](#), which identifies a partial converse of the implication  $\pi_n$ -shape injective  $\Rightarrow$   $n$ -homotopically Hausdorff.

**Corollary 5.4.** *Let  $n \geq 2$  and  $X$  be a simply connected,  $LC^{n-1}$ , compact Hausdorff space such that  $X$  fails to be semilocally  $\pi_n$ -trivial only at a single point  $x \in X$ . Then for every element  $g \in \ker(\Psi_n) \subseteq \pi_n(X, x)$  and neighborhood  $V$  of  $x$ ,  $g$  is represented by a map with image in  $V$ . In particular, if  $X$  is  $n$ -homotopically Hausdorff at  $x$ , then  $X$  is  $\pi_n$ -shape injective.*

*Proof.* Let  $0 \neq g \in \ker(\Psi_n) \subseteq \pi_n(X, x)$ . By [Theorem 1.1](#),  $g \in \pi_n^{Sp}(X, x)$ . Since  $X$  is compact Hausdorff, we may replace  $O(X)$  by the cofinal subdirected order  $O_F(X)$  consisting of finite open covers  $\mathcal{U}$  of  $X$  with the property that there is a unique  $A_{\mathcal{U}} \in \mathcal{U}$  with  $x \in A_{\mathcal{U}}$ . For each  $\mathcal{U} \in O_F(X)$ , we can write  $g = \prod_{i=1}^{m_{\mathcal{U}}} [\alpha_{\mathcal{U},i} * f_{\mathcal{U},i}]$  where  $f_{\mathcal{U},i} : S^n \rightarrow U_{\mathcal{U},i}$  is a non-nullhomotopic map for some  $U_{\mathcal{U},i} \in \mathcal{U}$  and  $\alpha_{\mathcal{U},i}$  is a path from  $x$  to  $f_{\mathcal{U},i}(d_0)$ .

Let  $V$  be an open neighborhood of  $x$ . We check that  $g$  is represented by a map with image in  $V$ . Since  $X$  is  $LC^0$  at  $x$ , there exists an open neighborhood  $V'$  of  $x$  such that any two points of  $V'$  may be connected by a path in  $V$ . Fix  $\mathcal{U}_0 \in O_F(X)$  such that  $A_{\mathcal{U}_0} \subseteq V'$ . Then  $A_{\mathcal{V}} \subseteq V'$  whenever  $\mathcal{V} \in O_F(X)$  refines  $\mathcal{U}_0$ .

We claim that for sufficiently refined  $\mathcal{V}$ , all of the maps  $f_{\mathcal{V},i}$  have image in  $V'$ . Suppose, to obtain a contradiction, there is a subset  $T \subseteq \{\mathcal{V} \in O_F(X) \mid \mathcal{V} \succeq \mathcal{U}_0\}$ , which is cofinal in  $O_F(X)$  and such that for every  $\mathcal{V} \in T$  there exists  $i_{\mathcal{V}} \in \{1, 2, \dots, m_{\mathcal{V}}\}$  and  $d_{\mathcal{V}} \in S^n$  such that  $f_{\mathcal{V},i_{\mathcal{V}}}(d_{\mathcal{V}}) \in U_{\mathcal{V},i_{\mathcal{V}}} \setminus V' \subseteq U_{\mathcal{V},i_{\mathcal{V}}} \setminus A_{\mathcal{U}_0}$ . Since  $X$  is compact, we may replace  $\{f_{\mathcal{V},i_{\mathcal{V}}}(d_{\mathcal{V}})\}$  with a subnet  $\{x_j\}_{j \in J}$  that converges to a point  $y \in X$ . Here,  $x_j = f_{\mathcal{V}_j,i_{\mathcal{V}_j}}(d_{\mathcal{V}_j})$  for some directed set  $J$  and monotone, final function  $J \rightarrow T$  given by  $j \mapsto \mathcal{V}_j$ . Let  $Y$  be an open neighborhood of  $y$  in  $X$ . Find  $\mathcal{W} \in O_F(X)$  such that there exists  $W_0 \in \mathcal{W}$  such that  $y \in W_0$  and  $\text{St}(W_0, \mathcal{W}) \subseteq Y$ . Since  $\{x_j\}$  is subnet that converges to  $y$ , there exists  $k \in J$  such that  $\mathcal{V}_k \succeq \mathcal{W}$  and  $x_k \in W_0$ . We have  $x_k \in \text{Im}(f_{\mathcal{V}_k,i_{\mathcal{V}_k}}) \subseteq U_{\mathcal{V}_k,i_{\mathcal{V}_k}} \subseteq W$  for some  $W \in \mathcal{W}$  and thus  $\text{Im}(f_{\mathcal{V}_k,i_{\mathcal{V}_k}}) \subseteq U_{\mathcal{V}_k,i_{\mathcal{V}_k}} \subseteq \text{St}(W_0, \mathcal{W}) \subseteq Y$ . However, for every  $\mathcal{V} \in O_F(X)$ ,  $f_{\mathcal{V},i_{\mathcal{V}}}$  is

not null-homotopic in  $X$ . Thus, since  $Y$  represents an arbitrary neighborhood of  $y$ ,  $X$  is not semilocally  $\pi_n$ -trivial at  $y$ . By assumption, we must have  $x = y$ . Since  $\{x_j\} \rightarrow x$ , the same argument, but where  $Y$  is replaced by  $V'$ , shows that there exists sufficiently refined  $\mathcal{V}$  for which  $\text{Im}(f_{\mathcal{V},i}) \subseteq V'$ ; a contradiction. Since the claim is proved, there exists  $\mathcal{U}_1 \geq \mathcal{U}_0$  in  $O_F(X)$  such that whenever  $\mathcal{V} \geq \mathcal{U}_1$ , we have  $\text{Im}(f_{\mathcal{V},i}) \subseteq V'$  for all  $i \in \{1, 2, \dots, m_{\mathcal{V}}\}$ .

Fix  $\mathcal{V} \geq \mathcal{U}_1$  in  $O_F(X)$ . For all  $i \in \{1, 2, \dots, m_{\mathcal{V}}\}$ , we may find a path  $\beta_{\mathcal{V},i} : I \rightarrow V$  from  $x$  to  $f_{\mathcal{V},i}(d_0)$ . Since  $X$  is simply connected, we have  $[\alpha_{\mathcal{V},i} * f_{\mathcal{U},i}] = [\beta_{\mathcal{V},i} * f_{\mathcal{U},i}]$  for all  $i$ . Thus  $g$  is represented by  $\prod_{i=1}^{m_{\mathcal{V}}} \beta_{\mathcal{V},i} * f_{\mathcal{V},i}$ , which has image in  $V$ .  $\square$

**Remark 5.5** (topologies on homotopy groups). Given a group  $G$  and a collection of subgroups  $\{N_j \mid j \in J\}$  of  $G$  such that for all  $j, j' \in J$ , there exists  $k \in J$  such that  $N_k \subseteq N_j \cap N_{j'}$ , we can generate a topology on  $G$  by taking the set  $\{gN_j \mid j \in J, g \in G\}$  of left cosets as a basis. We can apply this to both the collection of Spanier subgroups  $\pi_n^{Sp}(\mathcal{U}, x_0)$  and the collection of kernels  $\ker(p_{\mathcal{U}\#})$  to define two natural topologies on  $\pi_n(X, x_0)$ :

- (1) The *Spanier topology* on  $\pi_n(X, x_0)$  is generated by the left cosets of Spanier groups  $\pi_n(\mathcal{U}, x_0)$  for  $\mathcal{U} \in O(X)$ .
- (2) The *shape topology* on  $\pi_n(X, x_0)$  is generated by left cosets of the kernels  $\ker(p_{\mathcal{U}\#})$  where  $(\mathcal{U}, U_0) \in \Lambda$ . Equivalently, the shape topology is the initial topology with respect to the map  $\Psi_n$  where the groups  $\pi_n(|N(\mathcal{U})|, U_0)$  are given the discrete topology and  $\check{\pi}_n(X, x_0)$  is given the inverse limit topology.

**Lemma 3.10** ensures the Spanier topology is always finer than the shape topology. **Lemma 5.1** then implies that, whenever  $X$  is paracompact, Hausdorff, and  $LC^{n-1}$ , the two topologies agree. Moreover,  $\pi_n(X, x_0)$  is Hausdorff in the shape topology if and only if  $X$  is  $\pi_n$ -shape injective.

### 6. When is $\Psi_n$ an isomorphism?

It is a result of Kozłowski and Segal [1978] that if  $X$  is paracompact Hausdorff and  $LC^n$ , then  $\Psi_n : \pi_n(X, x) \rightarrow \check{\pi}_n(X, x)$  is an isomorphism. This result was first proved for compact metric spaces in [Kuperberg 1975]. The assumption that  $X$  is  $LC^n$  assumes that small maps  $S^n \rightarrow X$  may be contracted by small null-homotopies. However, if  $\mathbb{E}_n$  is the  $n$ -dimensional earring space, then the cone  $C\mathbb{E}_n$  is  $LC^{n-1}$  but not  $LC^n$ . However,  $C\mathbb{E}_n$  is contractible and so  $\Psi_n$  is an isomorphism of trivial groups. Certainly, many other examples in this range exist. Our Spanier group-based approach allows us to generalize Kozłowski–Segal’s theorem in a way that includes this example by removing the need for “small” homotopies in dimension  $n$ . In this section, when  $\mathcal{U}$  is an open cover of a space  $X$  and a distinguished element

$U_0 \in \mathcal{U}$  containing the basepoint  $x_0$  has been established or is clear from context, we will often write  $\mathcal{U}$  to represent the pair  $(\mathcal{U}, U_0) \in \Lambda$ .

**Lemma 6.1.** *Let  $n \geq 1$ . Suppose that  $X$  is paracompact, Hausdorff, and  $LC^{n-1}$ . If  $([f_{\mathcal{U}}])_{\mathcal{U} \in \Lambda} \in \check{\pi}_1(X, x_0)$ , then for every  $\mathcal{U} \in \Lambda$ , there exists  $[g] \in \pi_n(X, x)$  such that  $(p_{\mathcal{U}})_{\#}([g]) = [f_{\mathcal{U}}]$ .*

*Proof.* With  $(\mathcal{U}, U_0) \in \Lambda$  and  $p_{\mathcal{U}}$  fixed, consider a representing map

$$f_{\mathcal{U}} : (|\partial \Delta_{n+1}|, \mathbf{o}) \rightarrow (|N(\mathcal{U})|, U_0).$$

Let  $\mathcal{U}' = \{p_{\mathcal{U}'}^{-1}(\text{st}(U, |N(\mathcal{U}')|)) \mid U \in \mathcal{U}\}$ . Since  $p_{\mathcal{U}'}^{-1}(\text{st}(U, |N(\mathcal{U}')|)) \subseteq U$  for all  $U \in \mathcal{U}$ , we have  $\mathcal{U} \preceq \mathcal{U}'$ . Applying Lemmas 4.2 and 4.5 we can choose the following sequence of refinements of  $\mathcal{U}'$ . First, we choose a star refinement  $\mathcal{U}' \preceq_{**} \mathcal{W}$  so that for every  $W \in \mathcal{W}$ , there exists  $U' \in \mathcal{U}'$  such that  $\text{St}(W, \mathcal{W}) \subseteq U'$ . In this case, we can choose the projection map  $p_{\mathcal{W}'\mathcal{W}} : |N(\mathcal{W}')| \rightarrow |N(\mathcal{U}')|$  so that if  $p_{\mathcal{W}'\mathcal{W}}(W) = U'$  on vertices, then  $\text{St}(W, \mathcal{W}') \subseteq U'$  in  $X$ . This choice will be important near the end of the proof.

To construct  $g$ , we must take further refinements. First, we choose a sequence of refinements

$$\mathcal{W} = \mathcal{W}_n \preceq_* \mathcal{V}_n \preceq_*^{n-1} \mathcal{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathcal{W}_2 \preceq_* \mathcal{V}_2 \preceq_*^1 \mathcal{W}_1 \preceq_* \mathcal{V}_1 \preceq_*^0 \mathcal{W}_0$$

followed by one last refinement  $\mathcal{W}_0 \preceq_* \mathcal{V}_0 = \mathcal{V}$ . Let  $V_0 \in \mathcal{V}$  be any set containing  $x_0$  and recall that since  $X$  is paracompact Hausdorff  $(\mathcal{V}, V_0) \in \Lambda$ . For some choice of canonical map  $p_{\mathcal{V}}$ , we have  $p_{\mathcal{V}}^{-1}(\text{st}(V, N(\mathcal{V}))) \subseteq V$  for all  $V \in \mathcal{V}$ .

Recall that we have assumed the existence of a map

$$f_{\mathcal{V}} : (\partial \Delta_{n+1}, \mathbf{o}) \rightarrow (|N(\mathcal{V})|, V_0)$$

such that  $p_{\mathcal{W}'\mathcal{V}\#}([f_{\mathcal{V}}]) = [f_{\mathcal{U}}]$ . Set  $Y_V = f_{\mathcal{V}}^{-1}(\text{st}(V, N(\mathcal{V})))$  so that  $\mathcal{Y} = \{Y_V \mid V \in \mathcal{V}\}$  is an open cover of  $\partial \Delta_{n+1}$ . As before, we find a simplicial approximation for  $f_{\mathcal{V}}$ . Find  $m \in \mathbb{N}$  such that the star  $\text{st}(a, \text{sd}^m \partial \Delta_{n+1})$  of each vertex  $a$  in  $\text{sd}^m \partial \Delta_{n+1}$  lies in a set  $Y_{V_a} \in \mathcal{Y}$  for some  $V_a \in \mathcal{V}$ . Since  $f_{\mathcal{V}}(\mathbf{o}) = V_0$ , we may take  $V_{\mathbf{o}} = V_0$ . The assignment  $a \mapsto V_a$  on vertices extends to a simplicial approximation  $f' : \text{sd}^m \partial \Delta_{n+1} \rightarrow |N(\mathcal{V})|$  of  $f_{\mathcal{V}}$ , i.e., a simplicial map  $f'$  such that

$$f_{\mathcal{V}}(\text{st}(a, \text{sd}^m \partial \Delta_{n+1})) \subseteq \text{st}(f'(a), |N(\mathcal{V})|) = \text{st}(V_a, |N(\mathcal{V})|)$$

for each vertex  $a$ .

We begin to define  $g$  with the constant map  $\{\mathbf{o}\} \rightarrow X$  sending  $\mathbf{o}$  to  $x_0$ . In preparation for applications of Lemma 4.6, set  $K = \text{sd}^m \partial \Delta_{n+1}$  and  $L = \{\mathbf{o}\}$  so that  $K[k] = K_k$ . First, we define a map  $g_0 : M[0] \rightarrow X$  by picking, for each vertex  $a \in K_0$ , a point  $g_0(a) \in V_a$ . In particular, set  $g_0(\mathbf{o}) = x_0$ . This choice is well defined since we have  $p_{\mathcal{V}}(x_0) = V_0 \in \text{st}(V_{\mathbf{o}}, N(\mathcal{V}))$  and thus  $g_0(\mathbf{o}) = x_0 \in p_{\mathcal{V}}^{-1}(\text{st}(V_{\mathbf{o}}, N(\mathcal{V}))) \subseteq V_{\mathbf{o}}$ . Note that  $f'$  maps every simplex  $\sigma = [a_0, a_1, \dots, a_k]$  of  $K$  to the simplex of  $|N(\mathcal{V})|$  spanned by  $\{V_{a_i} \mid 0 \leq i \leq k\}$ . By definition of the

nerve, we have  $\bigcap\{V_{a_i} \mid 0 \leq i \leq k\} \neq \emptyset$ . Pick a point  $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \leq i \leq k\}$ . By our initial choice of refinements, we have  $\mathcal{U}_0 \preceq_* \mathcal{V}$ . If  $\sigma = [a_0, a_1, \dots, a_n]$  is a  $n$ -simplex of  $K$ , then  $\text{St}(x_\sigma, \mathcal{V}) \subseteq U_{0,\sigma}$  for some  $U_{0,\sigma} \in \mathcal{U}_0$ . In particular  $\{g_0(a_i) \mid 0 \leq i \leq n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \leq i \leq n\} \subseteq U_{0,\sigma}$ . Thus  $g_0$  maps the 0-skeleton of  $\sigma$  into  $U_{0,\sigma}$ . If  $\mathbf{o} \in \sigma$ , then  $g_0(\mathbf{o}) \in p_{\mathcal{V}}^{-1}(\text{st}(V_{\mathbf{o}}, N(\mathcal{V}))) \subseteq V_{\mathbf{o}} \subseteq U_{0,\sigma}$ . Hence, for every  $n$ -simplex  $\sigma$  of  $K$ , we have  $g_0(\sigma \cap M[0]) \subseteq U_{0,\sigma}$ .

We are now in the situation to recursively apply [Lemma 4.6](#). This is similar to the application in the proof of [Lemma 5.1](#) with the dimension  $n+1$  shifted down by one so we omit the details. Recalling that  $M[n] = \text{sd}^m \partial \Delta_{n+1}$ , we obtain an extension  $g : K = M[n] \rightarrow X$  of  $g_0$  such that for every  $n$ -simplex  $\sigma$  of  $K$ , we have  $g(\sigma) \subseteq W_\sigma$  for some  $W_\sigma \in \mathcal{W} = \mathcal{U}_n$ .

With  $g$  being defined, we seek show that  $f_{\mathcal{U}} \simeq p_{\mathcal{U}} \circ g$ . Since  $f' \simeq f_{\mathcal{V}}$  (by simplicial approximation),  $p_{\mathcal{U}\mathcal{V}} \simeq p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'\mathcal{W}} \circ p_{\mathcal{W}\mathcal{V}}$  (for any choice of projection maps), and  $p_{\mathcal{U}\mathcal{V}} \circ f_{\mathcal{V}} \simeq f_{\mathcal{U}}$  (for any choice of projection  $p_{\mathcal{U}\mathcal{V}}$ ), it suffices to show that  $p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'\mathcal{W}} \circ p_{\mathcal{W}\mathcal{V}} \circ f' \simeq p_{\mathcal{U}} \circ g$ . We do this by proving that the simplicial map  $F = p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'\mathcal{W}} \circ p_{\mathcal{W}\mathcal{V}} \circ f' : K \rightarrow |N(\mathcal{U})|$  is a simplicial approximation for  $p_{\mathcal{U}} \circ g$ . Recall that this can be done by verifying the “star-condition”  $p_{\mathcal{U}} \circ g(\text{st}(a, K)) \subseteq \text{st}(F(a), |N(\mathcal{U})|)$  for any vertex  $a \in K$  [[Munkres 1984](#), Chapter 2, Section 14]. Since  $n \geq 1$ , we have  $\mathcal{W} \preceq_{**} \mathcal{V}$ . Hence, just like our choice of  $p_{\mathcal{U}'\mathcal{W}}$ , we may choose  $p_{\mathcal{W}\mathcal{V}}$  so that whenever  $p_{\mathcal{W}\mathcal{V}}(V) = W$ , then  $\text{St}(V, \mathcal{V}) \subseteq W$ . Also, we choose  $p_{\mathcal{U}\mathcal{U}'}$  to map  $p_{\mathcal{U}}^{-1}(\text{st}(U, |N(\mathcal{U})|)) \mapsto U$  on vertices.

Fix a vertex  $a_0 \in K$ . To check the star-condition, we’ll check that  $p_{\mathcal{U}} \circ g(\sigma) \subseteq \text{st}(F(a_0), |N(\mathcal{U})|)$  for each  $n$ -simplex  $\sigma$  having  $a_0$  as a vertex. Pick an  $n$ -simplex  $\sigma = [a_0, a_1, \dots, a_n] \subseteq K$  having  $a_0$  as a vertex. Recall that  $f'(a_i) = V_{a_i}$  for each  $i$ . Set  $p_{\mathcal{W}\mathcal{V}}(V_{a_i}) = W_i$  and  $p_{\mathcal{U}'\mathcal{W}}(W_i) = p_{\mathcal{U}'}^{-1}(\text{st}(U_i, |N(\mathcal{U}')|)) \in \mathcal{U}'$  for some  $U_i \in \mathcal{U}'$ . Then  $F(a_i) = U_i$  for all  $i$ . It now suffices to check that  $p_{\mathcal{U}} \circ g(\sigma) \subseteq \text{st}(U_0, |N(\mathcal{U})|)$ . Recall that by our choice of  $p_{\mathcal{U}'\mathcal{W}}$ , we have  $\text{St}(W_0, \mathcal{W}) \subseteq p_{\mathcal{U}'}^{-1}(\text{st}(U_0, |N(\mathcal{U}')|))$ . Thus it is enough to check that  $g(\sigma) \subseteq \text{St}(W_0, \mathcal{W})$ . By construction of  $g$ , we have  $g(\sigma) \subseteq W_\sigma$  for some  $W_\sigma \in \mathcal{W}$ . Since  $g(a_0) \in W_0 \cap W_\sigma$ , we have  $g(\sigma) \subseteq W_\sigma \subseteq \text{St}(W_0, \mathcal{W})$ , completing the proof.  $\square$

Finally, we prove our second result, [Theorem 1.2](#).

*Proof of Theorem 1.2.* Since  $X$  is paracompact, Hausdorff,  $LC^{n-1}$ , we have  $\pi_n^{SP}(X, x_0) = \ker(\Psi_n)$  by [Theorem 1.1](#). Since  $X$  is semilocally  $\pi_n$ -trivial, we have  $\pi_n^{SP}(\mathcal{U}, x_0) = 1$  for some  $\mathcal{U} \in \Lambda$ . It follows that  $\Psi_n$  is injective. Moreover, by [Lemma 5.1](#), we may find  $\mathcal{V} \in \Lambda$  with  $\ker(p_{\mathcal{V}\#}) \subseteq \pi_n^{SP}(\mathcal{U}, x_0)$ . Thus  $p_{\mathcal{V}\#} : \pi_n(X, x_0) \rightarrow \pi_n(|N(\mathcal{V})|, V_0)$  is injective. Let  $([f_{\mathcal{U}}])_{\mathcal{U} \in \Lambda} \in \check{\pi}_n(X, x_0)$ . By [Lemma 6.1](#), for each  $\mathcal{U} \in \Lambda$ , there exists  $[g_{\mathcal{U}}] \in \pi_n(X, x_0)$  such that  $p_{\mathcal{U}}([g_{\mathcal{U}}]) = [f_{\mathcal{U}}]$ . If  $\mathcal{V} \preceq \mathcal{W}$ , then we have

$$p_{\mathcal{V}\#}([g_{\mathcal{V}}]) = [f_{\mathcal{V}}] = p_{\mathcal{V}\mathcal{W}\#}([f_{\mathcal{W}}]) = p_{\mathcal{V}\mathcal{W}\#} \circ p_{\mathcal{W}\#}([g_{\mathcal{W}}]) = p_{\mathcal{V}\#}([g_{\mathcal{W}}]).$$

Since  $p_{\mathcal{V}\#}$  is injective, it follows that  $[g_{\mathcal{W}}] = [g_{\mathcal{V}}]$  whenever  $\mathcal{V} \preceq \mathcal{W}$ . Setting  $[g] = [g_{\mathcal{V}}]$  gives  $\Psi_n([g]) = ([f_{\mathcal{W}}])_{\mathcal{W} \in \Lambda}$ . Hence,  $\Psi_n$  is surjective.  $\square$

## 7. Examples

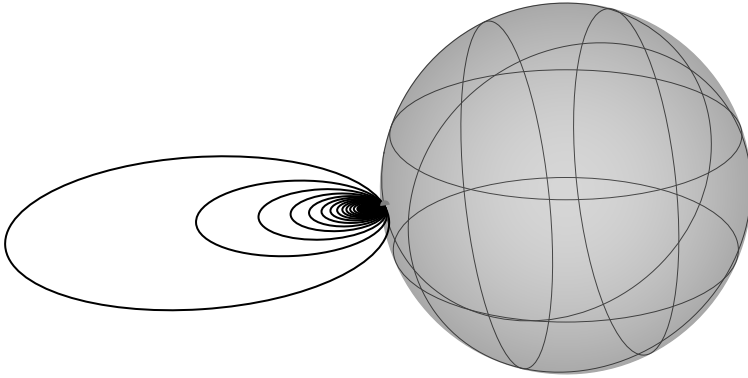
**Example 7.1.** Fix  $n \geq 2$ . When  $X$  is a metrizable  $LC^{n-1}$  space, the cone  $CX$  and unreduced suspension  $SX$  are  $LC^{n-1}$  and semilocally  $\pi_n$ -trivial but need not be  $LC^n$ . This occurs in the case  $X = \mathbb{E}_n$  or if  $X = Y \vee \mathbb{E}_n$  where  $Y$  is a CW-complex. In such cases,  $\Psi_n : \pi_n(SX) \rightarrow \check{\pi}_n(SX)$  is an isomorphism. One point unions of such cones and suspensions, e.g.,  $CX \vee CY$  or  $CX \vee SY$  also meet the hypotheses of [Theorem 1.2](#) (checking this is fairly technical [[Brazas 2021](#)]) but need not be  $LC^n$ .

**Example 7.2.** The converse of [Theorem 1.2](#) does not hold. For  $n \geq 2$ ,  $\mathbb{E}_n$  is  $LC^{n-1}$  but is not semilocally  $\pi_n$ -trivial at the wedgepoint  $x_0$ . However,  $\Psi_n : \pi_n(\mathbb{E}_n, x_0) \rightarrow \check{\pi}_n(\mathbb{E}_n, x_0)$  is an isomorphism where both groups are canonically isomorphic to  $\mathbb{Z}^{\mathbb{N}}$  [[Eda and Kawamura 2000a](#)]. Additionally, for the infinite direct product  $\prod_{\mathbb{N}} S^n$ ,  $\Psi_k : \pi_k(\prod_{\mathbb{N}} S^n, x_0) \rightarrow \check{\pi}_k(\prod_{\mathbb{N}} S^n, x_0)$  is an isomorphism for all  $k \geq 1$  even though  $\prod_{\mathbb{N}} S^n$  is not  $LC^{k-1}$  when  $k-1 \geq n$ .

**Example 7.3.** We can also modify the mapping torus  $M_f$  from [Example 5.3](#) so that  $\Psi_n$  becomes an isomorphism (recall that  $n \geq 2$  is fixed). Let  $X = M_f \cup C\mathbb{E}_n$  be the mapping cone of the inclusion  $\mathbb{E}_n \rightarrow M_f$ . For the same reason  $M_f$  is  $LC^{n-1}$ , the space  $X$  is  $LC^{n-1}$ . Moreover, if  $U$  is a neighborhood of  $\alpha(t)$  that deformation retracts onto a homeomorphic copy of  $\mathbb{E}_n$ , then any map  $S^n \rightarrow U$  may be freely homotoped “around” the torus and into the cone. It follows that  $X$  is semilocally  $\pi_n$ -trivial. We conclude from [Theorem 1.2](#) that  $\Psi_n : \pi_n(X) \rightarrow \check{\pi}_n(X)$  is an isomorphism. Since sufficiently fine covers of  $X$  always give nerves homotopy equivalent to  $S^1 \vee S^{n+1}$ , we have  $\check{\pi}_n(X) = 0$ . Thus  $\pi_n(X) = 0$ .

**Example 7.4.** Let  $n \geq 2$  and  $X = \mathbb{E}_1 \vee S^n$  (see [Figure 2](#)). Note that because  $\mathbb{E}_1$  is aspherical [[Cannon et al. 2002](#); [Curtis and Fort 1957](#)],  $X$  is semilocally  $\pi_n$ -trivial. However,  $X$  is not  $LC^1$  because it has  $\mathbb{E}_1$  as a retract. It is shown in [[Brazas 2021](#)] that  $\pi_n(X) \cong \bigoplus_{\pi_1(\mathbb{E}_1)} \pi_n(S^n) \cong \bigoplus_{\pi_1(\mathbb{E}_1)} \mathbb{Z}$  and that  $\Psi_n : \pi_n(X) \rightarrow \check{\pi}_n(X)$  is injective. In particular, we may represent elements of  $\pi_n(X)$  as finite-support sums  $\sum_{\beta \in \pi_1(\mathbb{E}_1)} m_{\beta}$  where  $m_{\beta} \in \mathbb{Z}$ . We show that  $\Psi_n$  is not surjective.

Identify  $\pi_1(X)$  with  $\pi_1(\mathbb{E}_1)$  and recall from [[Eda 1992](#)] that we can represent the elements of  $\pi_1(\mathbb{E}_1)$  as countably infinite reduced words indexed by a countable linear order (with a countable alphabet  $\beta_1, \beta_2, \beta_3, \dots$ ). Here  $\beta_j$  is represented by a loop  $S^1 \rightarrow \mathbb{E}_1$  going once around the  $j$ -th circle. Let  $X_j$  be the union of  $S^n$  and the largest  $j$  circles of  $\mathbb{E}_1$  so that  $X = \varprojlim_j X_j$ . Identify  $\pi_1(X_j)$  with the free group  $F_j$  on generators  $\beta_1, \beta_2, \dots, \beta_j$  and note that  $\pi_n(X_j) \cong \bigoplus_{F_j} \mathbb{Z}$ . Thus we may view an element of  $\pi_n(X_j)$  as a finite-support sums  $\sum_{w \in F_j} m_w$  of integers indexed over reduced words in  $F_j$ . Let  $d_{j+1,j} : F_{j+1} \rightarrow F_j$  be the homomorphism that deletes the



**Figure 2.** The one point union  $\mathbb{E}_1 \vee S^2$ .

letter  $\beta_{j+1}$ . Consider the inverse limit  $\check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$ . The map  $X \rightarrow X_j$  that collapses all but the first  $j$ -circles of  $\mathbb{E}_1$  induces a homomorphism  $d_j : \pi_1(X) \rightarrow F_j$ . There is a canonical homomorphism  $\phi : \pi_1(X) \rightarrow \check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$  given by  $\phi(\beta) = (d_1(\beta), d_2(\beta), \dots)$ , which is known to be injective [Morgan and Morrison 1986] but not surjective. For example, if  $x_k = \prod_{j=1}^k [\beta_1, \beta_j]$ , then  $(x_1, x_2, x_3, x_4, \dots)$  is an element of  $\check{\pi}_1(X)$  not in the image of  $\phi$ .

The bonding map  $b_{j+1,j} : \pi_n(X_{j+1}) \rightarrow \pi_n(X_j)$  sends a sum  $\sum_{w \in F_{j+1}} m_w$  to  $\sum_{v \in F_j} p_v$  where  $p_v = \sum_{d_{j+1,j}(w)=v} m_w$ . Similarly, projection map  $b_j : \pi_n(X) \rightarrow \pi_n(X_j)$  sends the sum  $\sum_{\beta \in \pi_1(X)} n_\beta$  to  $\sum_{v \in F_j} m_v$  where  $m_v = \sum_{d_j(\beta)=v} m_\beta$ . Let  $y_j \in \pi_n(X)$  be the sum whose only nonzero coefficient is the  $x_j$ -coefficient, which is 1. Since  $d_{j+1,j}(x_{j+1}) = x_j$ , it's clear that  $(y_1, y_2, y_3, \dots) \in \check{\pi}_n(X)$ . Suppose  $\Psi_n(\sum_\beta m_\beta) = (y_1, y_2, y_3, \dots)$ . Writing  $\sum_\beta m_\beta$  as a finite sum  $\sum_{i=1}^r m_{\beta_i}$  for nonzero  $m_{\beta_i}$ , we must have  $\sum_{d_j(\beta_i)=x_j} m_{\beta_i} = 1$  for all  $j \in \mathbb{N}$ . Since there are only finitely many  $\beta_i$  involved, there must exist at least one  $i$  for which  $d_j(\beta_i) = x_j$  for infinitely many  $j$ . For such  $i$ , we have  $\phi(\beta_i) = (x_1, x_2, x_3, \dots)$ , which, as mentioned above, is impossible. Hence  $\Psi_n$  is not surjective.

The previous example shows why we cannot remove the  $LC^{n-1}$  hypothesis in Theorem 1.2. Since we weakened the hypothesis from [Kozłowski and Segal 1978] in dimension  $n$  and no hypothesis in dimension  $n$  is required for Theorem 1.1, one might suspect that we might be able to remove the dimension  $n$  hypothesis completely. The next example, which is a higher analogue of the harmonic archipelago [Bogley and Sieradski 1998; Conner et al. 2015; Karimov and Repovš 2012] shows why this is not possible.

**Example 7.5.** Let  $n \geq 2$  and  $\ell_j : S^n \rightarrow \mathbb{E}_n$  be the inclusion of the  $j$ -th  $n$ -sphere in  $\mathbb{E}_n$ . Let  $X$  be the space obtained by attaching  $(n+1)$ -cells to  $\mathbb{E}_n$  using the attaching maps  $\ell_j$ . Since  $\mathbb{E}_n$  is  $LC^{n-1}$ , it follows that  $X$  is  $LC^{n-1}$ . However,  $X$  is not

semilocally  $\pi_n$ -trivial at the wedgepoint  $\mathbf{o}$  of  $\mathbb{E}_n$ . Indeed, the infinite concatenation maps  $\prod_{j \geq k} \ell_j = \ell_k \cdot \ell_{k+1} \cdots$  are not null-homotopic (using a standard argument that works for the harmonic archipelago) but are all homotopic to each other. Thus,  $\pi_n(X, \mathbf{o}) \neq 0$ . However, for sufficiently fine open covers  $\mathcal{U} \in \mathcal{O}(X)$ ,  $|N(\mathcal{U})|$  is homotopy equivalent to a wedge of  $(n + 1)$ -spheres and thus  $\check{\pi}_n(X, \mathbf{o}) = 0$ . Therefore, despite  $X$  being  $LC^{n-1}$ ,  $\Psi_n$  is not an isomorphism. In fact,  $\pi_n(X, \mathbf{o}) = \pi_n^{Sp}(X, \mathbf{o}) = \ker(\Psi_n)$ . The reader might also note that since  $\mathbb{E}_{n-1}$  is  $(n - 1)$ -connected and  $\pi_n(\mathbb{E}_n, \mathbf{o}) \cong H_n(\mathbb{E}_n) \cong \mathbb{Z}^{\mathbb{N}}$ ,  $X$  will also be  $(n - 1)$ -connected. A Meyer–Vietoris sequence argument similar to that in [Karimov and Repovš 2012] can then be used to show  $\pi_n(X, \mathbf{o}) \cong H_n(X) \cong \mathbb{Z}^{\mathbb{N}} / \bigoplus_{\mathbb{N}} \mathbb{Z}$ .

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Received August 14, 2022. Revised February 17, 2023.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PJM peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 322    No. 2    February 2023

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Elements of higher homotopy groups undetectable by polyhedral approximation	221
JOHN K. ACETI and JEREMY BRAZAS	
Regularity for free multiplicative convolution on the unit circle	243
SERBAN T. BELINSCHI, HARI BERCOVICI and CHING-WEI HO	
Invariant theory for the free left-regular band and a $q$ -analogue	251
SARAH BRAUNER, PATRICIA COMMINS and VICTOR REINER	
Irredundant bases for finite groups of Lie type	281
NICK GILL and MARTIN W. LIEBECK	
Local exterior square and Asai $L$ -functions for $GL(n)$ in odd characteristic	301
YEONGSEONG JO	
On weak convergence of quasi-infinitely divisible laws	341
ALEXEY KHARTOV	
$C^*$ -irreducibility of commensurated subgroups	369
KANG LI and EDUARDO SCARPARO	
Local Maass forms and Eichler–Selberg relations for negative-weight vector-valued mock modular forms	381
JOSHUA MALES and ANDREAS MONO	
Representations of orientifold Khovanov–Lauda–Rouquier algebras and the Enomoto–Kashiwara algebra	407
TOMASZ PRZEŹDZIECKI	