Topololological Methods in Group Theory:
The Adjunction Problem.

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Also I would like to thank Dr. Steve Pride for his many helpful comments on previous versions of this thesis, and I thank him and Dr. Holt for the time they have taken over examining it; I also apologise for the troubles I have caused them. I would also like to thank Peta McAllister for typing one of the early versions (and correcting my spelling).

My family have given me many years of patience and understanding for which I am extremely grateful; I only wish that my father could have lived to see the day.

While originally carrying out this research, I was in receipt of a Science Research Council Grant.

Declaration

I declare that none of the work presented here has been published elsewhere, or been submitted for another degree.

The results of Chapter 1 are well-known (creditted to their original sources where possible) except for the new results quoted from Chapter 4.

Chapter 2 grew out of work by Dr. Rourke on presentations of the trivial group (see [Rou]) and suggestion that this approach should yield new results on the Adjunction Problem. Chapters 3 and 4 are original except where otherwise is indicated.

SUMMARY

The work presented here is a new method of attack on an old group theory problem known as the Adjunction Problem, defined by B H Neumann in 1943 (see [N]). The problem is the following : given a group, form a new group by adding one new generator and one new relation ; determine the conditions under which the natural map from the original to the modified group is an injection. (For instance the new relator must not be conjugate to a word in the original group.) The main result obtained using the new methods is that the map is indeed an injection when the original group is locally indicable - a new result independently obtained by Howie [H] and Brodskii [Br] .

Chapter 1 consists of some basic definitions and some of the known results, together with statements of the new results and some instances of where the problem arises in low-dimensional topology.

In Chapter 2 we introduce the new methods - showing that a non-trivial element in the kernal of the natural map for a given group and added relator (a "counter-example") gives us a labelled, planar graph with certain properties (a "special diagram") and that this special diagram in its turn defines a counter-example (these results are summed up in 2.22). These topologically obtained diagrams turn out (2.10) to be dual to the "Dehn diagrams" of Small Cancellation Theory (see for instance [LS] or [L2]) .

In Chapter 3 a class of such diagrams is constructdd and it is shown that none of these corresponds to a counter-example. This class contains the only diagrams known to the author which give potential counter-examples ("triples") such that the new generator appears with exponent-sum non-zero in the added relator.

Chapter 4 begins with the construction of a potential function on a diagram, based on work by Lyndon [L2]. This is then used to prove the main result of the thesis, the Freiheitssatz for locally indicable groups, a new proof of the result which (as noted above) has been independently obtained by Howie and by Brodskii. Finally it is show that the existence of a counter-example for a given group G and a given relator r depends upon the existence of a counter-example for $G * \langle s \rangle$ and added relator r'' , where r'' is one of two words obtained from r using a homomorphism from $G * \langle t \rangle$ to $\mathbb{Z}$ which takes r to zero.

Historical Note

In 1928, Dehn suggested the Freiheitsatz as a research problem to his student Wilhelm Magnus, thinking that Magnus would be able to provide a proof using his (Dehn's) diagrams. Magnus was eventually able to prove the Freiheitssatz and told Dehn so.

Dehn asked him if he has used diagrams in the proof, and on being told that the method was purely algebraic, Dehn said "Da sind Sie also blind gegangen": "So you proceeded blindfolded".

(Reported by Magnus in his article on Dehn in 'The Mathematical Intelligencer', Volume 3, 1978).

Here we shall be proceeding without blindfolds.

Terminology

$\mathbb{Z}$ - the integers

I - the unit interval $[0,1]$

S^1 - the circle

∂X - the boundary of X

$F(Y)$ - the free group on the set Y

$N_G(R)$ - the normal closure of the set R in the group G.

$\langle X;R\rangle$ - the group $\frac{F(X)}{N_F(R)}$

$\mathrm{sbgp}_G(g_1,g_2,\dots,g_n)$ - the subgroup of G generated by the set $g_1,\dots,g_n$

$G_1 * G_2$ - the free product of the groups G_1 and G_2

$\langle G,t;r\rangle$ - the group $\frac{G * \langle t\rangle}{N(r)}$ where r is a cyclically reduced word in $G * \langle t\rangle$.

A word $w = b_1b_2\dots b_n$ in $\langle X;R\rangle$ where each $b_i = c_{j_i}^{\pm 1}$ for some c_{j_i} in X is *reduced* if $b_i^{-1} \neq b_{i+1}$ in $F(X)$ for each $i < n$; w is *cyclically reduced* if in addition $b_1^{-1} \neq b_n$.

$\sigma_t(r)$ - the exponent sum of t in r ; if $r = a_1 t^{\alpha_1} a_2^{\alpha_2} \dots a_n t^{\alpha_n}$ then $\sigma_t(r) = \sum_{i=1}^{n} \alpha_i$.

r *has a solution over* G if G naturally injects into $\langle G,t;r\rangle$.

$\{G,r,w\}$ is a *triple* if G is a group , $r \in G * \langle t\rangle$,and w is an element in the kernal of the natural map G to $\langle G,t;r\rangle$.

A *counter-example* is a triple where r is a cyclically reduced word containing occurences of t ,and w is non-trivial in G.

CHAPTER 1

We open with a description of the problems with which we are concerned in this dissertation. The second section reviews the literature in this and related fields, and in section 3 we give a summary of the principal results of this thesis. Section 4 is a brief collection of some elementary results in the area, some of which we shall require later on, and we conclude with a description of some topological problems which reduce to the group-theoretic problems of section 1.

Section 1. The Problems

We are concerned with the following problems from combinatorial group theory:

The Adjunction Problem

Let G be a group, and let r be a cyclically reduced word in $G * \langle t \rangle$ which contains t non-trivially.

Under what conditions is the natural map G to $\langle G,t;r \rangle$ an injection?

We can think of $\langle G,t;r \rangle$ as G together with one new generator and one new relation, and the problem is to determine whether any elements of G have been killed. We shall see in section 5 of this chapter that this question arises in low-dimensional topology.

A more general form of the question is:

The Generalised Freiheitssatz

Let $\{H_i\}_{i \in I}$ be a collection of non-trivial groups, and let r be a cyclically reduced word in $H = \underset{i\in I}{*} H_i$, and let I' be a proper subset of I such that r contains occurences from H_j , for some $j \in I-I'$.

Under what conditions is the natural map $\underset{i\in I'}{*} H_i$ to $\frac{H}{N(r)}$ an injection?

The classical Freiheitssatz, due to Magnus (see e.g. [MKS]) states that the above map is an injection when the H_i are free groups. We say therefore that the Freiheitssatz holds for free groups.

Related to the Adjunction Problem we have two conjectures ; the first is usually attributed to Kervaire (see [Kv] , pages 116-117 , though the conjecture is not explicitly made there) and we shall refer to it by his name. (In [LS] it is also attributed to Laudenbach via Serre : the conjecture was made by Laudenbach during a course on low-dimensional topology, in connection with problem 1.9 .)

The Kervaire Conjecture

If $\langle G,t;r\rangle$ is trivial then G is trivial.

Some partial results on this are given in Section 2 and in Section 4.

All known examples of cases where G does not inject into $\langle G,t;r\rangle$ have the following two properties : the exponent sum of t in r is zero , and G has elements of finite order ; e.g. in the group $\langle\langle a,b;b^2\rangle,t;\ btat^{-1}\rangle$ a has order two, whereas a has infinite order in $\langle a,b;b^2\rangle$.

In [Lev 1] Levin conjectures that G injects into $\langle G,t;r\rangle$ if G is torsion-free. In view of the above remarks, the obvious conjectures to make are :

Conjecture A : G naturally injects into $\langle G,t;r\rangle$ when $\sigma_t(r) \neq 0$.

Conjecture B : G naturally injects into $\langle G,t;r\rangle$ except when $\sigma_t(r) = 0$ and G has elements of finite order.

Section 2. The Literature

The Adjunction Problem was first raised in 1943 by B.H. Neumann, [N] , in the following form:

Given a set of m equations in n unknowns , i.e. a set of m words $f_i(x_1,x_2,..x_n)$ in $G * F(x_1,x_2,..x_n)$, does there exist an 'overgroup' G' containing G and elements $a_1,..a_n$ in G' such that $f_i(a_1,a_2,..a_n) = 1$ in G' ? If so we say that the equations are soluble over G . Neumann shows that the equation $x^n = g$ is soluble over any group G with $g \in G$. In [B] , Baumslag produces a new proof of this result using wreath products, and Levin uses a similar construction in [Lev 1] to show that any equation of the form $a_1t^{\alpha_1}a_2t^{\alpha_2}...a_nt^{\alpha_n}$ where $a_i \in G$ and each α_i is a positive integer, is soluble over any group G . In a second paper , [Lev 2] , Levin looks at the general case of m equations in n unknowns, and then there is a group generated by G and a single new generator containing solutions to the equations.

Continuing on the general case, in [GR] it is shown that if G is a finite group, and we have n equations f_j in n unknowns x_i , then the equations have a solution over G if the determinant of the matrix $(\sigma_{i,j})$ is non-zero; $\sigma_{i,j}$ is the exponent sum of x_i in f_j . We shall give the proof of this for the case $n = 1$ in 1.4 .

In [R] , Rothaus looks at the question of proper injectivity; that is, assuming that the equations have a solution over G, when is the natural map G to $\dfrac{G * F(x_1,x_2,..x_n)}{N(f_1,f_2,...f_m)}$ not a surjection ?

Conditions on a certain matrix in the Whithead group are obtained to ensure that the injection is proper, thus providing a partial solution to a conjecture of M.Cohen (Conjecture A of M.Cohen, 'Whitehead Torsion Group Extensions and Zeeman's Conjecture in high dimensions', Topology 16, 1977, p 79-97). Also in this paper Rothaus extends the result of [GR] to locally residually finite groups.

As we said in Section 1, we are here interested in the case of just one equation in one unknown, the Adjunction Problem; Magnus' Freiheitssatz gives the result for free products of free groups, and further results in this come under the heading of 'one-relator groups' (see e.g. [LS] chapter 11, page 111 for a comprehensive survey) where there are many results on the form of the relator and whether there is torsion etc. .(See also [S] where the methods of [L2], used in Chapter 4 Section 1, are used to obtain a 'spelling theorem' for one-relator groups.)

In [L2] Lyndon proved that the Generalised Freiheitssatz holds for sugroups of the group of real numbers under addition, and this result was extended by Gildenhuys in [G] to torsion-free abelian groups. S.J. Pride has proved in [P] that the Generalised Freiheitssatz holds for locally residually free groups, and in Chapter 4 we extend this result, using an adaptation of the techniques of [L2], to locally indicable groups (4.8), a result independently obtained by J.Howie [H] in Edinburgh and Brodskii in Moscow using completely different methods.

On the Kervaire Conjecture, the principal results known are all derived from the above results ; e.g. if G is residually finite then <G,t;r> is non-trivial when G is non-trivial by [R] .

Finally we note that Gutierrez has published a proof that if G is residually nilpotent, then <G,t;r> is non-trivial [Gu] . (He has also produced several close but unsuccessful attempts to prove the Kervaire Conjecture using crossed modules.)

Section 3. Summary of Results

The main contribution of this thesis is the method introduced in Chapter 2 to attack the Adjunction Problem. We first obtain a picture, a "special diagram", representing an element in the kernal of the natural map from G to $\langle G,t;r\rangle$ by omitting some edges and discs from a diagram which is essentially dual to a Dehn diagram of Small Cancellation theory. A brief description of Dehn diagrams is given in 2.1 - 2.3 , and the duality is described in 2.10 . A topological derivation of the diagrams, due to Rourke in [Rou] , is given in 2.4 - 2.9, but this can be omitted by the reader who is familiar with Dehn diagrams; in this case the duality description of 2.10 can be taken as a definition. We prefer to work from the topological description as it seems to us to be the more natural method of obtaining the diagrams.

We show that the existence of a non-trivial element w in the kernal of the map from G to $\langle G,t;r\rangle$ (a "counter-example" $\{G,r,w\}$) implies the existence of a special diagram which in turn defines a counter-example $\{G',r',w'\}$. There is a homomorphism h from $\langle G',t;r'\rangle$ to $\langle G,t;r\rangle$ such that $h(r') = r$, and $\{G,r,h(w)\}$ is a counter-example. (Note also that G' is finitely generated.) This means that the search for a counter-example to Conjecture A is equivalent to the search for a special diagram which defines a counter-example $\{G',r',w'\}$ such that $\sigma_t(r') \neq 0$. We close the chapter with the construction of some counter-examples with $\sigma_t(r') = 0$ using the special diagrams we have introduced.

It turns out to be rather difficult to construct reduced special diagrams, and the only class which we have been able to construct where the added relator in the triple defined by the diagram has non-zero exponent sum (in t), is described in Chapter 3, where it is shown that none of these diagrams ever defines a counter-example. The proof is principally group theoretic, reducing a graph theoretic problem to a problem in triangle groups; originally a purely graph theoretic proof was planned.

The main results of this thesis are contained in Chapter 4. In the first section we define a "potential function" on a special diagram, and we construct several such functions. In 4.4 we prove the Lyndon Lemma, adapted from [L2], showing that if a diagram represents a counter-example which is in some sense minimal for a certain class of groups, then any potential function on it satisfies an extra condition. In the proof we take a diagram representing a failure of the theorem; we then add a subscript, whose value is the potential, to the label on some of the edges to give us a new diagram, representing a "smaller" failure of the theorem for the same class of groups.

We now use this lemma to prove the Freiheitssatz for locally indicable groups (a group is locally indicable if any finitely generated subgroup has $\mathbb{Z}$ as a homomorphic image). The class of locally indicable groups contains the locally residually free groups (previously the largest class of groups for which the Freiheitssatz had been established) and the fundamental groups of irreducible 3-manifolds with boundary.

In the final section we use a homomorphism from $\langle G,t;r\rangle$ to $\mathbb{Z}$ to define two words $r_{\min}$ and $r_{\max}$ in $G*\langle s\rangle*\langle t\rangle$ and show that if both these words have solutions in $G*\langle s\rangle*\langle t\rangle$ then r has a solution in $G*\langle t\rangle$. It was originally hoped (and claimed) that we could prove a strengthened form of a theorem of Schiek [Sch 1] but this is no longer possible.

Section 4. Some Initial Results

We here give a brief collection of some of the elementary results known concerning the Kervaire Conjecture and the Adjunction Problem.

Proposition 1.1

Let r be a cyclically reduced element of $G * \langle t \rangle$, and let r_0 be r with the t-occurrences omitted. Then :

i) $r_0 = 1$ in G implies that r has a solution in G .

ii) $\langle G,t;r \rangle$ is trivial implies that $\sigma_t(r) = \pm 1$ and G is perfect.

Proof

i) If $r_0 = 1$ in G , then the composite map $G \to \langle G,t;r \rangle \to \dfrac{G}{N(r_0)} \cong G$ is an isomorphism.

ii) $\langle G,t;r \rangle$ is trivial means that $G * \langle t \rangle = N(r)$.

Abelianising both sides we have that $\dfrac{G}{[G,G]} \oplus \langle t \rangle = \langle s \rangle$

where $s = r_0 . t^{\sigma_t(r)}$. Hence killing t we see that G is perfect, and killing G we see that $\sigma_t(r) = \pm 1$.

■

The next proposition is basically the Higman, Neumann, Neumann theorem, which provides the only fully understood case where $\sigma_t(r) = 0$.

Proposition 1.2

Let $r = atbt^{-1}$ be a cyclically reduced word in $G * \langle t \rangle$.

Then G injects into $\langle G,t;r \rangle$ if and only if a and b have the same order in G .

Proof

In $\langle G,t;r \rangle$, a and b have the same orders, as $a^{-1} = tbt^{-1}$, and hence G does not inject into $\langle G,t;r \rangle$ if they have different orders in G .

Conversely, if a and b have the same order in G , then $\langle G,t;r \rangle$ is an HNN-extension of G with stable letter t , and hence G injects into $\langle G,t;r \rangle$. (See e.g. [LS] or [MKS]; and 2.25.)

■

The next two results are proved in a similar manner : we try to embed G in another (well-understood) group H in such a way that we can extend this to a map from $G * \langle t \rangle$ to H so that r maps to the identity. In this way we have, in a sense, 'found a solution for $r(t) = 0$ ' regarding r as an equation in t . The first of these results is very easily established.

Note 1.3

If G is a subgroup of the group of real numbers under addition, and $\sigma_t(r) \neq 0$, then G injects into $\langle G,t;r\rangle$.

Proof

Let ϕ be a homomorphism from G into R^+, the group of reals under addition, and extend ϕ to ϕ_* on $G * \langle t\rangle$ by

$\phi_*(t) = -\dfrac{\phi(r_0)}{\sigma_t(r)}$; note that $\phi(r_0) \neq 0$ as r_0 is non-trivial by 1.1 .

Then $\phi_* : G * \langle t\rangle \rightarrow R^+$ is a homomorphism and $\phi_*(r) = 0$; thus we have extended ϕ as required and so the injection $G \longrightarrow R^+$ factors through $\langle G,t;r\rangle$.

■

We now prove the particular case of Gerstenhaber and Rothaus' result [GR] :

Proposition 1.4

If G is finite and $\sigma_t(r) \neq 0$ then G injects into $\langle G,t;r\rangle$.

Proof

We can embed G in a compact connected Lie group U. We now extend this to a map $h : G * \langle t\rangle \longrightarrow U$ such that $h(r) = id$. To do this we must define $h(t)$ so that the equation $h(r) = id$ is satisfied.

As U is arcwise connected, we can define paths $p_i : [0,1] \longrightarrow U$ such that $p_i(0) = id$, $p_i(1) = a_i$, where $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} \ldots a_n t^{\alpha_n}$, $a_i \in G-\{1\}$ $\alpha_i \in \mathbb{Z}-\{0\}$. We now define a homotopy:

$P : U \times [0,1] \rightarrow U$, by $P(u,s) = p_1(s)u^{\alpha_1} p_2(s)u^{\alpha_2} \ldots p_n(s)u^{\alpha_n}$,

and $\sigma_t(r) \neq 0$ means that $P(u,0) = u^{\sigma_t(r)} \neq 1$.

Hence the map $P(\cdot,1)$ is homotopic to the map $u \rightarrow u^{\sigma_t(r)}$, which is a map of non-zero degree on a compact Lie group, i.e. on a compact manifold, and is therefore onto. (e.g. for a proper map, $\deg f = \deg_Q f$ for any point Q, using the notation of Dold 'Lectures on Algebraic Topology' pages 267,8. $\deg_Q f$ is the degree of the map $H_n(U,U-f^{-1}(Q)) \longrightarrow H_n(U,U-Q)$ and hence $f^{-1}(Q)$ is non-empty if $\deg f$ is non-zero.)

That $P(\ ,1)$ is onto means that there is a $u_0 \in U$ such that $P(u_0,1) = 1$, and hence we can define $h(t) = u_0$, giving the required map $G * \langle t\rangle \longrightarrow U$.

■

The above proposition can be extended to cover the case of groups which are residually finite. In fact we can prove a slightly more general result concerning residual properties, but first we need some definitions :

Definition 1.5

Let X be a property of groups, and G a (not necessarily finitely generated) group.

G is fully residually X if for any finite set $\{w_1, w_2, .. w_n\}$ of non-trivial words in G there is a normal subgroup $N \lhd G$ such that for each i, $w_i \notin N$, and G/N has the property X. G is residually X iff for any word w in G there is a normal subgroup N such that $w \notin N$ and G/N has property X.

If $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} .. a_n t^{\alpha_n}$ with $a_i \in G - \{1\}$, $\mathbb{Z} - \{0\}$ we say r has t-shape $(\alpha_1, \alpha_2, .. \alpha_n)$.

Proposition 1.6

Let R be a set of t-shapes such that if r has t-shape in R and G has property X, then r has a solution over G

Then if r' has t-shape in R, and H is fully residually X, then r' has a solution over H.

Proof

Let $r' = a_1 t^{\alpha_1} .. a_n t^{\alpha_n}$, with $a_i \in H$; assume that $w \in \ker H \to \langle H, t; r' \rangle$ and let N be a normal subgroup such that w and $a_i \notin N$, and let ϕ be the natural map $H * \langle t \rangle \to H/N * \langle t \rangle$

Then $\phi(r') = \phi(a_1) t^{\alpha_1} \phi(a_2) ... \phi(a_n) t^{\alpha_n}$, and the t-shape of $\phi(r')$ is the same as the t-shape of r', as each $\phi(a_i) \neq 1$, and so as H/N has property X, r has a solution over H/N, and hence $\phi(w) = 1$ in H/N, contradicting the choice of N.

■

Notice that if G is residually finite then it is fully residually finite; for if $w_1, .. w_n$ are non-trivial elements of G, then for each i there is a homomorphism ϕ_i from G onto a finite group H_i such that $\phi_i(w_i) \neq 1$. Taking cartesian products we have the required map taking the set $\{w_i\}$ to non-trivial elements of a finite group, and hence G is also fully residually finite.

By putting $R = \{$words whose t-shape is $(\alpha_1, .., \alpha_n)$ with $\sum_{i=1}^{n} \alpha_i \neq 0\}$ and X the property of being finite in 1.6 we get :

Corollary 1.7

Let G be a residually finite group and r a cyclically reduced word in $G * \langle t \rangle$ with $\sigma_t(r) \neq 0$.

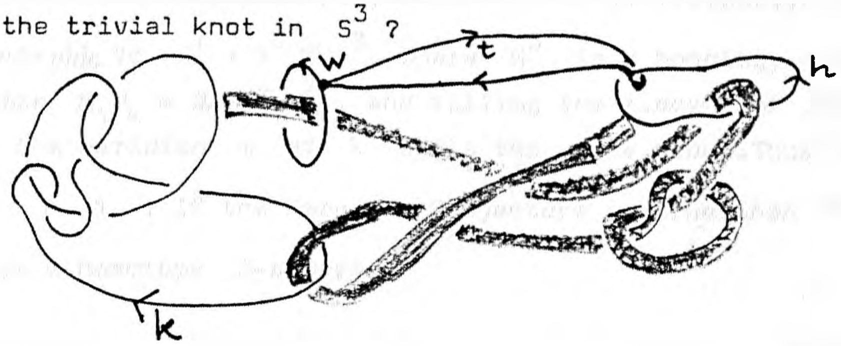
Then r has a solution over G.

Section 5

Having given some (most) of the known results for the Adjunction Problem proved via group-theoretic methods, we shall finish this chapter with some problems in 3 and 4 dimensional topology which raise problems involving the adjoining of generators and relations to groups.

1.8 Band Sum Problem (1.1 in [K])

Given a knot k in S^3 and an unlinked, unknotted loop h in $S^3 - k$, is it possible to band together k and h to obtain $k \#_b h$ which is the trivial knot in S^3 ?



$k \#_b h$ is obtained by : embed a rectangle $b = I \times I$ in S^3 such $b \cap k = \{0\} \times I$, $b \cap h = \{1\} \times I$.

$k \#_b h = \{k - \{0\} \times I\} \cup \{h - \{1\} \times I\} \cup \{I \times \{0,1\}\}$.

Let w be the element of $\pi_1(S^3 - k\#_b h)$ corresponding to the loop around the band as shown, and let t be the element of $\pi_1(S^3 - k\#_b h)$ corresponding to the loop around h . Killing w corresponds to putting a disc across the band b and so we can consider that the band has no knots and does not link k or h , that is, that there is a natural map

$\phi : \pi_1(S^3 - k\#_b h)) \longrightarrow \dfrac{\pi_1(S^3 - k) * <t>}{N(w)}$ which is a surjection.

But $k \#_b h$ is the trivial knot if and only if $\pi_1(S^3 - k\#_b h) = \mathbb{Z}$.

Hence if we know that if w has a solution over $\pi_1(S^3 - k)$ for all choices of w , then we know that the answer to the question is 'no', it is not possible to obtain the unknot as the band sum of the unknot with a non-trivial knot.

In fact, by Thurston's recent work [T], it is known that knot groups are residually finite, and we can show that in the above construction the word w has $\sigma_t(w) \neq 0$ (e.g. by looking at homology) and hence by 1.7 w always has a solution over $\pi_1(S^3 - k)$. In fact it can be shown that knot groups are locally indicable (see [Sh]) and hence using 4.8 we also obtain the answer 'no'.

1.9 Property R for Knots (5.7 in [K])

Is it possible to obtain $S^1 \times S^2$ by surgery on a knot in S^3 ? Let M_k denote the manifold obtained by doing 0-surgery on the knot k in S^3 .(It is known that if M_k is $S^1 \times S^2$ we must do 0-surgery.)

If M_k is NOT homeomorphic to $S^1 \times S^2$, k has property R .

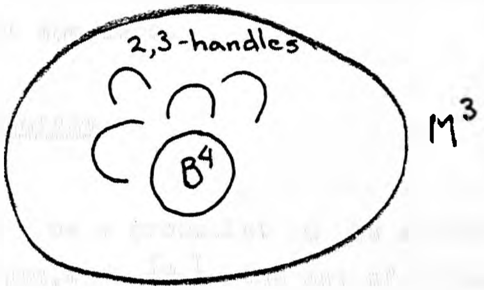
There is a natural map $f_k : M_k \rightarrow S^1 \times S^2$, which induces an isomorphism on integral homology and induces a $\mathbb{Z}[\mathbb{Z}]$ -homology isomorphism if and only if the Alexander polynomial of k is trivial.In this case M_k is diffeomorphic to $S^1 \times S^2 \# H^3$, where H^3 is a homology 3-shere.

Hence $\pi_1 M_k = \mathbb{Z} * \pi_1 H^3$, and killing the element of $\pi_1 M_k$ corresponding to the meridian m of k kills the whole group.Thus we have that $\frac{\mathbb{Z} * \pi_1 H^3}{N(m)} = 1$. If the Kervaire Conjecture is true,then $\pi_1 H^3 = 1$, i.e. H^3 is a homotopy 3-sphere.

1.10 Contractible 4-Manifolds with Boundary (4.18 in [K])

Let M be a 3-manifold bounding a contractible 4-manifold H^4 ; can we choose H^4 to have no 1-handles ?

If this were possible,then H^4 is obtained by adding an equal number of 2- and 3-handles to a 4-ball B^4 .(We must add an equal number else we get non-trivial higher homology.)



Turning this construction upside down,we find that we get a manifold with two boundary components, M and S^3 with an equal number of 1- and 2-handles,and this corresponds to killing $\pi_1 M^3$ by adding an equal number of generators and relations.

This is a partial generalisation of the Adjunction Problem,and it is shown in [GR] that there are groups which cannot be killed in this manner.Casson has shown that there are therefore 3-manifolds bounding contractible 4-manifolds which must have 1-handles.

CHAPTER 2

We start with a brief description of 'Small Cancellation Diagrams' which we shall call Dehn-diagrams.These pictures were first used at the beginning of this century by Dehn and Van Kampen to attack the word problem for certain groups.A full description of Dehn-diagrams is available in Chapter V of Lyndon and Schupp 's book [LS] where they are called R-diagrams.We go on to give a topological derivation of a diagram (2.7,2.8) which is essentially dual to a Dehn-diagram (2.9) .Those who are used to working with Dehn-diagrams may prefer to think of our diagrams as dual Dehn-diagrams and omit the topological section 2.3-2.9 and use the construction of 2.10 to define the relevant diagram.

We proceed to define the group defined by a diagram,and then,concentrating on the Adjunction Problem,in Section 3 we define special diagrams which are the main tool used in the rest of the thesis.These are essentially diagrams with some edges and discs omitted.It is in this context that it seems easier to think of diagrams rather than Dehn-diagrams as in the dual Dehn-diagram,omitting an edge corresponds to identifying certain edges to points.

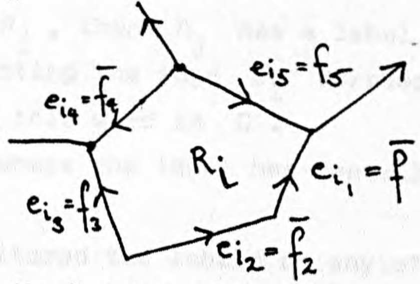
We develop some of the properties of special diagrams ,the principal results being the construction of 2.18 , and the summing-up of 2.22 .

In Section 4 we use special diagrams to construct some counter-examples with exponent sum zero.

Section 1. Dehn - Diagrams

Definition 2.1

Let $G = \langle X;R \rangle$ be a group.Let D be a finite planar graph,with $E = \{e_i\}$ the set of edges,and $\{R_j\}$ the set of regions.Suppose that each edge is oriented,and has a label in G , i.e. there is a map $\phi : E \rightarrow G$ such that $\phi(\bar{e}_i) = (\phi(e_i))^{-1}$ where $\bar{e}_i$ is e_i with the opposite orientation.Let R_i be a region of D, and let the oriented boundary of R_i ,read clockwise,be $f_1, f_2, \ldots f_n$, where for each i , $f_i \in E$ or $\bar{f}_i \in E$.



Define $\phi(R_i) = \phi(f_1)\phi(f_2)\ldots\phi(f_n)$.

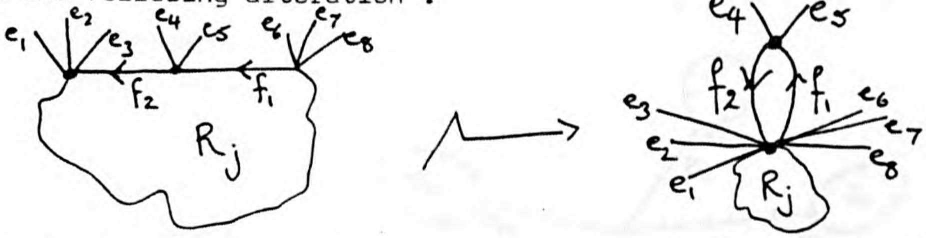
We say that D is a Dehn-diagram if $\phi(R_i)$ is trivial in G for each i , i.e. if $\phi(R_i) \in N_{F(X)}(R)$ where $G = \langle X;R \rangle$ .The Dehn-diagram then illustrates that $\phi(\partial D)$ is trivial in G ,where ∂D is the boundary of the region containing the point at infinity.

2.2 Reduction of Dehn - Diagrams

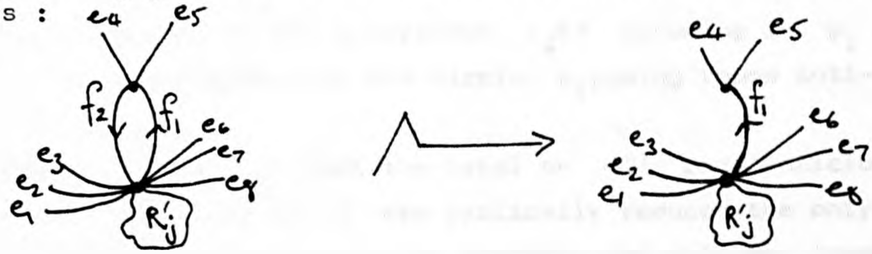
We can define reductions of Dehn-diagrams as follows :

i) If there is an edge e_i in D with only one end attatched to the rest of D (i.e. D - Int(e_i) has two components,one of which is a point) then omit edge e_i from D .

ii) If there is a region R_j of D which has two consecutive edges f_1,f_2 on its oriented boundary such that $\phi(f_1) = \phi(f_2)^{-1}$, then perform the following alteration :



iii) If there is a region R_j with just two edges,such that the label on R_j is $a_j a_j^{-1}$,then we discard one of the edges;e.g. the above becomes :



Note that in the above,a 'region' may be the outside region,i.e. the region containing the point at infinity

The changes in the Dehn-diagram D described above are allowed as they correspond to reduction of the labels on regions,or cyclic reduction of the labels.

In i) , if e_i lies in the region R_j , then R_j has a label with $\phi(e_i).\phi(e_i)^{-1}$ as a subword,and so omitting the edge e_i corresponds to cancellation in this word,or reduction of this word in G .

In ii) we again have a region R_j where the label has cancellation as above.

After performing iii) we have not altered the labels on any of the regions,and merely omitted a region with trivial label $a_j a_j^{-1}$, $a_j \in G$.

We now show that we have a Dehn-diagram for any element of $F(X)$ which is trivial in $G = \langle X;R \rangle$. This is what is called an R-diagram in [LS], chapter V ; the following theorem is given in [LS] as a note on p.240.

Proposition 2.3

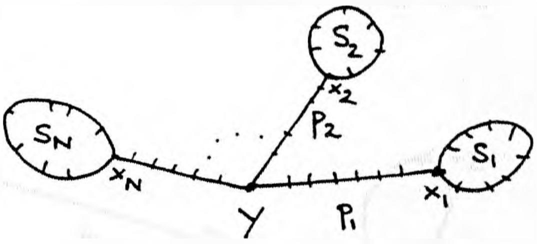
Let w be a cyclically reduced word in $F(X)$ such that $w = 1$ in $G = \langle X;R \rangle$.

Then there is a reduced Dehn-diagram D such that $\phi(\partial D) = w$, and the label on each interior region R_i is a cyclic conjugate of an element of R.

Proof

$w = 1$ in G means that $w = \prod_{j=1}^{N} p_j s_j p_j^{-1}$ where $s_j^{\pm 1} \in R$ and $p_j \in F(X)$

Let Y be a base point in the plane with N radiating finite arcs, labelled p_1 to p_N in anticlockwise order. At the end X_i of the arc p_i place a circle labelled s_i.



We now split the arc p_i into segments corresponding to the generators $x_j \in X$ occuring in p_i. Similarly split the circumferance of the circle s_i, going round anti-clockwise from X_i.

This diagram D' is such that the label on $\partial D'$ read anticlockwise is w. Assuming that the words in R are cyclically reduced, the only place that reduction can occur is on the outside boundary, and this may involve cyclic reduction of w.

■

We call the above a Dehn-diagram representing $w = 1$ in G.

Readers who are not interested in a topological derivation and are familiar with Dehn diagrams may move directly to 2.10 to define a diagram.

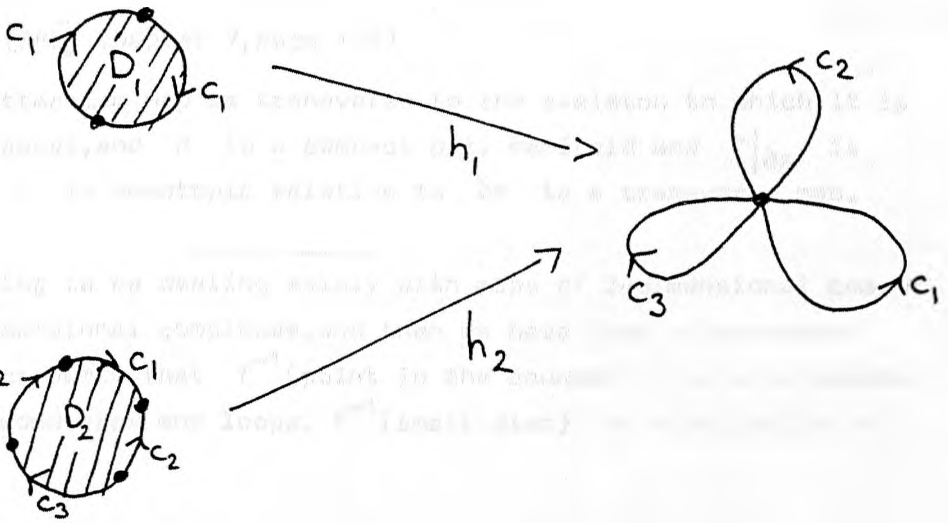
Section 2. Topology : Diagrams

Given a presentation of a group $G = \langle X_n;R \rangle$, there is a standard method of forming a 2-dimensional CW-complex L such that $\pi_1 L = G$, as follows :

Take one o-cell, which will be the base point x of L ;
take n oriented 1-cells, which will form a bouquet K of oriented loops $c_1, c_2 \dots c_n$, and $\pi_1 K = F_n$, the free group on n generators $\{a_1, \dots a_n\} = X_n$;
take m 2-cells $\{D_1, D_2 \dots D_m\}$ corresponding to the m elements $\{r_1, r_2, \dots, r_m\} = R$ of $F(X_n)$, and glue these to K as follows :
if $r_i = a_{i,1}^{\varepsilon_{i,1}} \dots\dots a_{i,k}^{\varepsilon_{i,k}}$, where $\varepsilon_{i,j} = \pm 1$, define $h_i : \partial D_i \rightarrow K$ such that $h_i(\partial D_i) = c_{i,1}^{\varepsilon_{i,1}} c_{i,2}^{\varepsilon_{i,2}} \dots c_{i,k}^{\varepsilon_{i,k}}$.

then define $L = \dfrac{K \cup \bigcup_{i=1}^{m} D_i}{\{\text{identifications } h_i\}}$; $\pi_1 L = G$.

e.g. we illustrate the example $G = \langle a_1, a_2, a_3 ; a_1^{\,2}, a_1 a_2^{-1} a_3 a_2 \rangle$:



Definition 2.4

(L,K) is called the complex associated with the presentation of the group $G = \langle X_n ; R \rangle$, or a complex for G .

Note that a group has many different presentations and there are correspondingly many different complexes associated with a given group.

Using complexes associated with group presentations, topological solutions of group-theoretic problems are possible; for instance the proofs of the Nelson-Schreier, Kuros and Grusko subgroup theorems. (See e.g [R] pages 258 and 267 for the first two, and for the third, see Stallings "A Topological Proof of Grusko's theorem on Free Products", Math. Zeit. 90 (1965), pages 1-8) . These proofs use covering-space theory and allied topological techniques, as well as some elements of transversality, which latter we now use as our main tool, using the definition from [BRS] chapter 7 .

Definition 2.5 Transversality

Let L be a CW-complex, M a closed p.l. manifold.We say that a map $f : M \longrightarrow L$ is transverse on the i-cells of L ,if for each i-cell D^i_j either $f^{-1}(D^i_j) = \emptyset$

or there is a commuting diagram :

$$\begin{tikzcd}
cl(f^{-1}(D^i_j)) \arrow[rr, "t_{i,j}"] \arrow[dr, "f|"'] & & D^i \arrow[dl, "e_{i,j}"] \\
& L &
\end{tikzcd}$$

where $e_{i,j}$ is the characteristic map for D^i_j , and $t_{i,j}$ is the projection of a trivial p.l.bundle,and $cl(f^{-1}(D^i_j))$ has codimension zero in M.

If $\partial M \neq \emptyset$, then $f : M \longrightarrow L$ is transverse if in addition $t^{-1}_{i,j}(D^i) \cap \partial M = T$, where $T \subset \partial(t^{-1}_{i,j}(D^i))$ has codimension zero.

This ensures that $f|_{\partial M}$ is transverse.

Theorem 2.6 ([BRS] Chapter 7,page 135)

If each attaching map is transverse to the skeleton to which it is attached in the above,and M is a compact p.l. manifold and $f|_{\partial M}$ is transverse,then f is homotopic relative to ∂M to a transverse map.

We are going to be dealing solely with maps of 2-dimensional complexes into 2-dimensional complexes,and then we have that a transverse map f has the property that f^{-1}(point in the boundary) is a collection of disjoint embedded arcs and loops, f^{-1}(small disc) is a collection of disjoint discs.

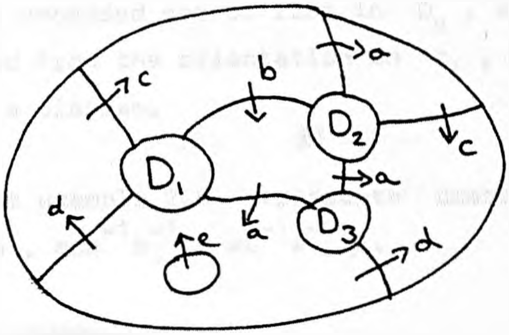
Definition 2.7

A diagram is a 2-complex in the plane consisting of a finite number of disjoint 2-cells $D_1, D_2, .. D_m$ in the interior of the disc D, together with a finite number of disjoint,transverse-oriented 1-cells which are loops or properly embedded arcs in $cl(D - \bigcup_{i=1}^{m} D_i)$.

The 2-cells are the discs of the diagram,the 1-cells are the edges, the endpoints of the embedded edges are the vertices.

The diagram is labelled if each edge is labelled by some letter a_i.

Example 2.8



We define a word r_i corresponding to a disc D_i , or the label r_i on the disc D_i as follows :

Choose any point on D_i – (vertices) .Read clockwise round D_i writing down,in order,the labels of the edges traversed,with exponent +1 or –1 accordingly as the edge is oriented clockwise or anti-clockwise near D_i.

Note that r_i is only defined up to cyclic conjugation.In example 2.8 associated words are $ad^{-1}a^{-1}$, adcb , $aca^{-1}b^{-1}$.

Proposition 2.9

Corresponding to a word w in $F(X_n)$ which is trivial in $G = \langle X_n ; R \rangle$, there is a diagram,which we call a diagram representing w = 1 in G .

Proof

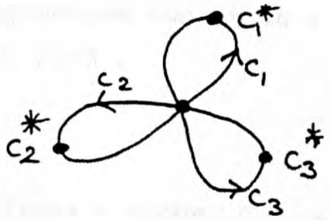
Let (L,K) be the CW-complex associated with the presentation of $G = \langle X_n ; R \rangle$, chosen such that the 2-cells are attached to K by transverse maps.Then the word $w \in F(X_n)$ is represented by a loop c in K which bounds a singular disc in L .

Thus we have $f : (D^2, \partial D^2) \longrightarrow (L,K)$, $f(\partial D^2) = c$, and we can choose f to be transverse on c .

Now the conditions of theorem 2.6 are fulfilled with M = 2-ball and so we can homotop f rel∂D^2 to get a map,also called f , which is transverse on the 2-cells of L , and thus,for each 2-cell D_i of L , $f^{-1}(D_i) = \bigcup_j D_{i,j}$ where each $D_{i,j} \subset \mathrm{Int}D^2$, $D_{i,j} \cap D_{k,l} = \emptyset$ unless $(i,j) = (k,l)$.

Let $D_0 = cl(D - \bigcup_{i,j} D_{i,j})$; then $f|_{D_0}$ is transverse on D_0 , so we can homotop $f|_{D_0}$ (rel ∂D_0) such that $f|_{D_0}$ is transverse on K .

For each 1-cell c_i of K , let c_i^* be the midpoint of L;



Then $f^{-1}(c_i^*)$ is an embedded arc or loop in D_0 , with a transverse orientation inherited from the orientation on c_i , which we label a_i.

Thus we have a diagram.

■

So we see that example 2.8 represents $dcacd^{-1} = 1$ in the group $G = \langle a,b,c,d,e;\ adcb\ ,\ aca^{-1}b^{-1}\ ,\ ad^{-1}a^{-1} \rangle$.

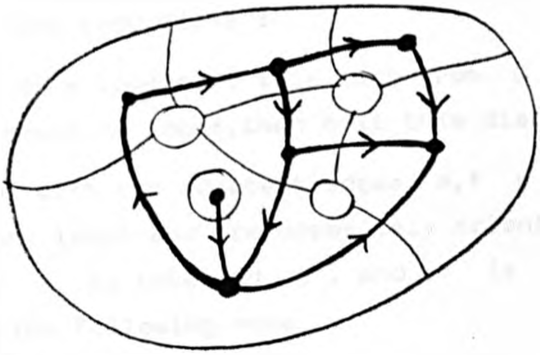
Henceforth we shall assume a diagram is labelled unless otherwise is stated.

2.10 Diagram / Dehn - Diagram Duality

We can now obtain a new proof of a weak form of 2.3 which will illustrate the duality existing between diagrams and Dehn-diagrams :

If $w \in F(X)$ and $w = 1$ in $G = \langle X;R \rangle$, then there is a (reduced) Dehn-diagram representing $w = 1$ in G .

Let D be a diagram representing $w = 1$ in G obtained by 2.8 . Regarding the discs of D as points, and ignoring ∂D and the outside region, let D' be the dual of D . Orient each edge of D' by the transverse orientation on the dual edge of D, and label the edge with the label on the dual edge in D. What we now have is a Dehn-diagram representing $w = 1$ in G , which we can then reduce.



■

The inverse proceedure to the above provides the justification for our diagrams for those who are familiar with Dehn-diagrams :

Given a Dehn-diagram D' representing $w = 1$ in G we form a diagram representing $w = 1$ in G as follows :

In each interior region of D' draw a small circle, and enclose D' inside a large circle. Now whenever two regions of D' have a common boundary edge, join the corresponding circles by an edge with a transverse orientation and a label inherited from D' . The result is a diagram representing $w = 1$ in G . This may in fact be taken to be a definition of a diagram representing $w = 1$ in G .

Returning to diagrams, we now state a converse to 2.8 which will be slightly improved in 2.13 .

Proposition 2.11

A diagram D defines a group $G = \langle X;R \rangle$, and D represents $w = 1$ in G for some word w in F(X) .

Proof

Let X be the set of labels on the edges of D , and let R be the set of labels on the discs of D , we can reverse the construction of

2.7 , to build a complex K for the group $\langle X;R\rangle$, and we can then see that D represents $w = 1$ in $G = \langle X;R\rangle$ where w is the label on the outside boundary of D .

2.12 Reduction of Diagrams

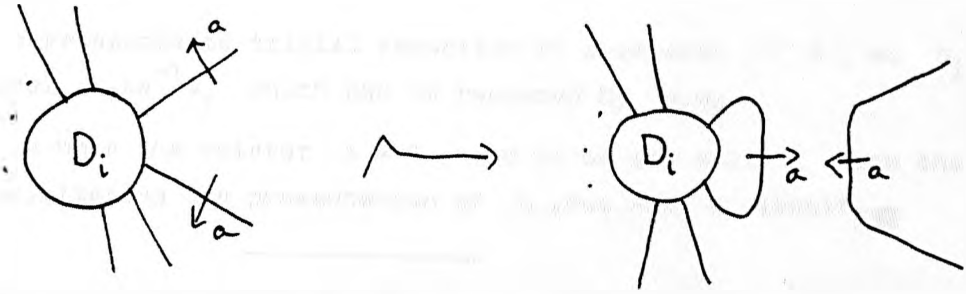
As with Dehn-diagrams,we now give a set of moves,or alterations, which we can perform on a diagram D ,which we shall subsequently show do not alter the properties of D in which we are interested.These changes are analagous to the reduction operations of 2.2 .

We call the following alterations reductions :

i) a) If D contains an edge which is a loop,omit this loop from D .

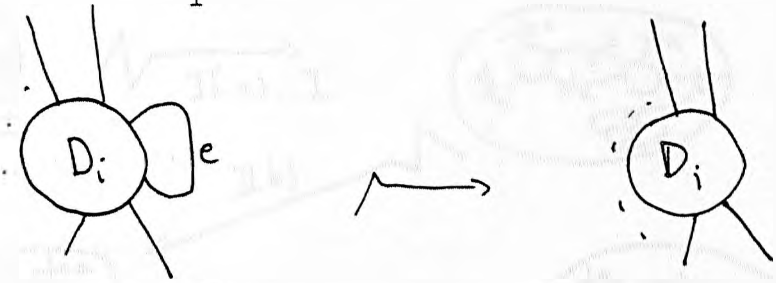
b) If D contains a disc which meets no edges,then omit this disc.

ii)a) If there is a disc D_i in D with two adjacent edges e,f $e \neq f$ where e and f bear the same label and are oppositely oriented (or are similarly oriented,but e is labelled a , and f is labelled a^{-1}) then perform the following move



(We see here that move i)a) is necessary,as free loops may be generated here.)

b) If there is an edge e in D which has as endpoints two adjacent points on a disc D_i , then omit e .



iii) If there is a disc D_i in D such that D_i meets only one edge e , in just one point,and e is labelled by the letter a, then omit all edges labelled a .

(Here we use i)b) to omit the disc D_i)

We say that a diagram is reduced if none of these moves can be performed.

Proposition 2.13

Let D be a diagram, and let G be the group defined by D (2.11). If D' is obtained from D by performing reduction operations, then D' defines the group G', where $G = G' * F_m$ for some m.

Proof

We need only show that performing the reduction operations is the same as performing Tietze moves on the presentation of G, and therefore alters the presentation but not the group, except i)a) which may forget generators which do not appear elsewhere, thus contributing to the free factor F_m.

i) a) -omits a loop which is not involved in showing that w, the label on D, is trivial in G. This alters the map from D to the complex associated with G by a homotopy rel∂D, and may alter G by omitting a generator which does not appear in any relation.

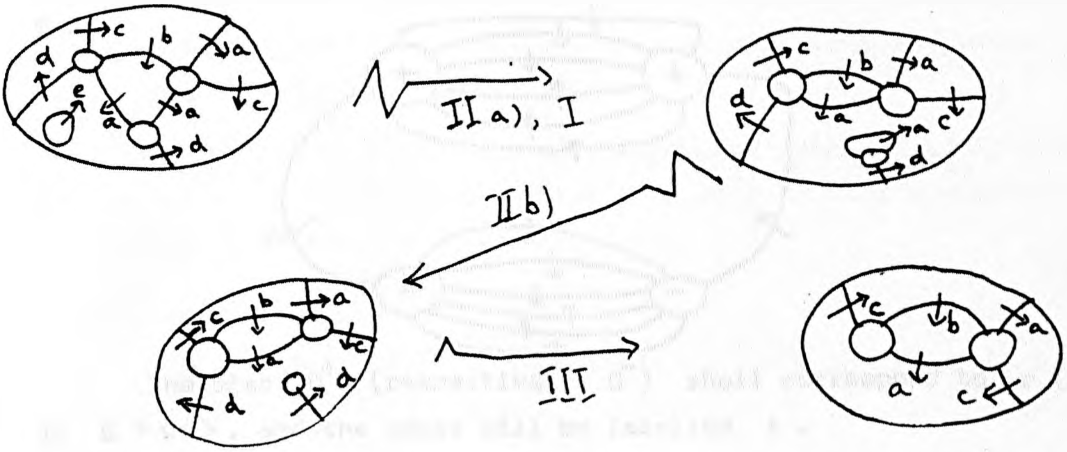
b) -a disc with no edges is the trivial relator 1, and so can be ignored.

ii)a,b)-this corresponds to trivial reduction of a relator in R, as D_i has label $w_1 a a^{-1} w_2$ which can be replaced by $w_1 w_2$.

iii) -here we have the relator $a = 1$, and so we can omit a from the diagram, altering the presentation of G, but not G itself. ■

Hence if we are interested in the group defined by the diagram, we may assume that the diagram is reduced.

Example - The group defined by the unreduced diagram has presentation $\langle a,b,c,d,e\ ;\ adcb\ ,\ aca^{-1}b^{-1}\ ,\ ad^{-1}a^{-1}\rangle$.



After reduction, the diagram represents $ac^2 = 1$ in the group $\langle a,b,c\ ;\ acb\ ,\ aca^{-1}b^{-1}\rangle$.

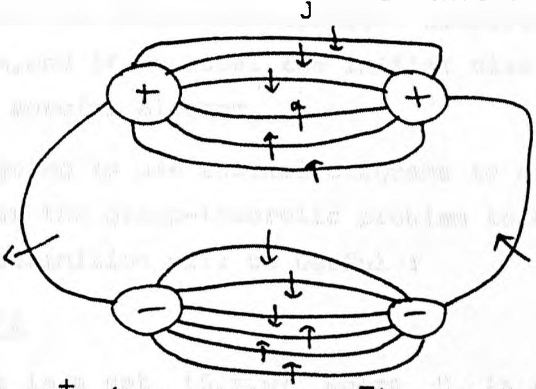
Section 3

2.14 Special Diagrams

From these (general) diagrams we are now going to define special diagrams, to enable us to look at the Adjunction Problem (of when does G inject into $\langle G,t;r \rangle$). In this case we are particularly interested in the new generator t , the relation r , and the words in G which are trivial in the group $\langle G,t;r \rangle$,and hence we are interested in diagrams for $\langle G,t;r \rangle$ in which no edge labelled t meet the boundary, and the t-edges only meet the discs labelled $r^{\pm 1}$. With this in mind, we make the following construction, before giving the definition.

Define a disc with spokes D^+ of order n to be a disc together with n disjoint, transverse-oriented arcs radiating outwards from D^+ , D^+ is labelled $+$, and there is a distinguished point p on $(D^+ - (\text{arcs}))$. Reflect D^+ in a planar axis, and call the result D^- ,labelled $-$.

Now form a planar complex by taking a finite number of copies of D^+ $D_1^+, D_2^+, \ldots D_k^+$, and a number of copies of D^- , $D_1^-, D_2^- \ldots D_{k'}^-$, lying in the plane such that the end point of an arc of $D_i^{\pm}$ is identified with an endpoint of just one arc from $D_j^{\pm}$ such that the orientations match. e.g.



The disc D^+ (respectively D^-) shall correspond to r (resp r^{-1}) in $G * \langle t \rangle$, and the edges will be labelled t .

We now give a formal definition of a special diagram, which is what we have just constructed.

Definition 2.15

An (unlabelled) special diagram is a planar 2-complex D in the interior of a 2-disc D^2 such that :

i) D contains a finite number of disjoint 2-discs $D_1, D_2 \ldots D_m$ in $\mathrm{Int}D^2$, each labelled + or −, and a finite number of transverse oriented arcs and loops embedded in $\mathrm{cl}(D^2 - \biguplus_{i=1}^{m} D_i) - \partial D^2$.

ii) For each i, $D_i \cap \{\text{edges}\} = \{n \text{ vertices}\}$. Call n the index of D.

iii) On each disc D_i we can choose a distinguished point p_i on $D_i - \{\text{edges}\}$ such that reading clockwise round ∂D_i from p_i, writing +1 (resp. −1) for each arc which we meet that is oriented clockwise (anti-clockwise) near D_i, we get the n-tuple $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ (resp. $(-\varepsilon_n, -\varepsilon_{n-1}, \ldots, -\varepsilon_1)$) where D_i is labelled '+' (resp. '−') .

As before, the D_i are called the <u>discs</u> of D, the arcs are called the <u>edges</u>, and the points where the edges meet the discs are the <u>vertices</u> of D. The n edges meeting the disc D_i divide ∂D_i into n <u>segments</u>. We call the region of the plane bounded by ∂D the <u>outside</u> region, and ∂D we call the <u>outside</u> edge of D ; the other regions of (plane) − D the <u>inner</u> regions of D.

If each segment of each ∂D_i is labelled by a letter (or possibly word) b_k, such that reading clockwise (anti-clockwise) round ∂D_i if D_i is labelled + (−) , gives the word $b_1 t^{\varepsilon_1} b_2 t^{\varepsilon_2} \ldots b_n t^{\varepsilon_n}$ we say that D is a labelled special diagram.

It is easy to see that an unlabelled special diagram can be made into a labelled special diagram because of condition iii) . It is also easy to see that the 'disc with spokes' construction gives an unlabelled special diagram, and if we label the initial disc in the construction we get a labelled special diagram.

We are going to use special diagrams to attack the Adjunction Problem, to alter the group-theoretic problem to a diagram-theoretic one. The following definition will be useful :

Definition 2.16

A triple is a set $\{G, r, w\}$ where G is a group, r is a cyclically reduced word in $G * \langle t \rangle$, and w an element of G which is trivial in $\langle G, t; r \rangle$.

A triple is a counter-example if r has length greater than 1, i.e. $r \notin G$, and w is non-trivial in G.

Proposition 2.17

Corresponding to a counter-example $\{G,r,w\}$ there is a (labelled) special diagram, which we call a special diagram representing $\{G,r,w\}$.

Proof

By Proposition 2.9 there is a diagram D representing $w = 1$ in $\langle G,t;r\rangle$. In fact by 2.11 the diagram may be assumed to be reduced.

Now omit from D

i) all edges labelled in G

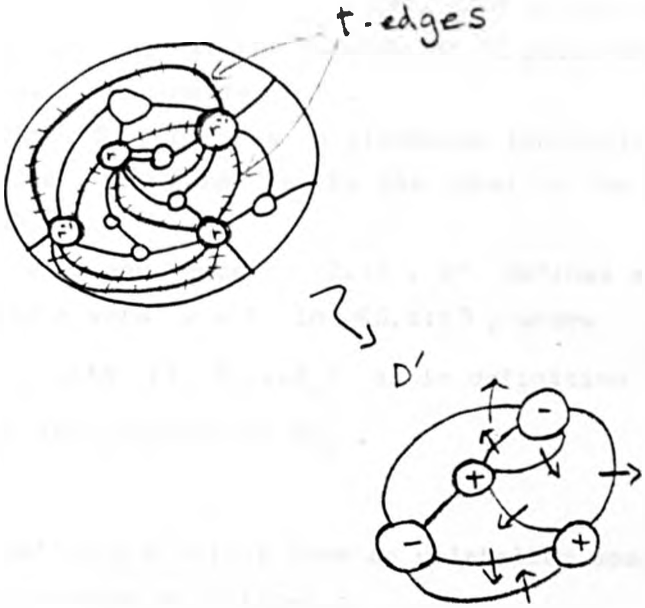
ii) all discs labelled in G

iii) ∂D

to give D'.

What remains are the discs labelled $r^{\pm 1}$ and edges labelled t. (If an edge is labelled t^{-1} we can reverse the orientation and label it t.

On a disc of D' labelled r^{+1} write '+', if labelled r^{-1} write '−'.

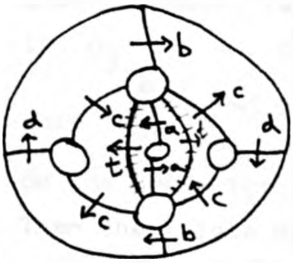


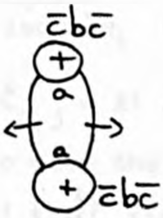
On the segments of each + disc D_i in D' where a G-edge e of D met D_i, write the label of e with exponent +1 if e was oriented clockwise, else with exponent −1. If D_i is a −disc write the label of e with exponent −1 if e was oriented clockwise, else +1.

What remains is a labelled special diagram.

■

e.g.





This starts with a triple $\{G,r,w\}$ where $r = atc^{-1}bc^{-1}t^{-1}$, $G = \langle a,b,c,d,X;\ a^2,\ dc^2,\ R\rangle$ where X is the set of generators of G which do not appear in D, R the set of relators for G which do not appear in D, and $w = (bd)^2$.

The special diagram shows that $\{G,(atc^{-1}bc^{-1}t^{-1}),(c^{-2}b)^2\}$ is a counter-example (remember $c^{-2} = d$).

Proposition 2.18

An (unlabelled) special diagram D defines a triple $\{G,r,w\}$, which is a counter-example if w is non-trivial in G .

Proof

Label each segment of each + (resp. −) disc D_i by proceeding clockwise (anti-clockwise) from p_i the distinguished point writing the letters $c_1,c_2,..c_n$ in each segment as in 2.15 .Label any free loops b.

In each region R_j of D draw a disc B_j , and draw an arc from B_j to each disc D_k on R_j (if R_j is the outside of D ,i.e. $\partial R_j = \partial D$ then draw B_j around D , i.e. D is inside B_j) .

If an edge e meets disc D_i , give e a clockwise (anti-clockwise) or near D_i if it is a + (−) disc , and label e by the label on the segment of ∂D_i where e meets ∂D_i .

We now have a diagram D' , and hence by 2.11 , D' defines a group G , and $\partial D'$ represents a word $w = 1$ in $\langle G,t;r\rangle$, where $r = c_1t^{\varepsilon_1}c_2t^{\varepsilon_2}c_3.......c_nt^{\varepsilon_n}$, with $(\varepsilon_1,\varepsilon_2,..\varepsilon_n)$ as in definition 2.15 and $c_1,c_2..,c_n$ the labels on the segments of ∂D_i .

In fact the method of defining a triple from an unlabelled special diagram can be described and presented as follows :

I label the diagram by $c_1,..c_n$ written on the segments of the discs going round clockwise (anti-clockwise) from the distinguished point on the + (−) discs, and any free loops label b .

II for each inner region of the diagram R_i write the word w_i by :
choose an edge on ∂R_i ,
read clockwise around ∂R_i ,
when a segment labelled c_k of a disc D_j is on ∂R_i write c_k^{-1} if D_j is + , c_k^{+1} if D_j is − , the loop ∂R_i then defines a word $c_{j_1}^{\varepsilon_{j,1}}c_{j_2}^{\varepsilon_{j,2}}...c_{j_m}^{\varepsilon_{j,m}}$, where each $\varepsilon_{i,j} = \pm 1$.

III do the above for the outer edge , ∂D , to give the word w .
Then the triple defined is $\{\langle c_1,..c_n,b;\ \{w_i\}\rangle\ ,r,w\}$ where $r = c_1t^{\varepsilon_1}c_2t^{\varepsilon_2}....c_nt^{\varepsilon_n}$, $(\varepsilon_1,\varepsilon_2,..,\varepsilon_n)$ is the n-tuple defined in 2.14 .

∎

Continuing as with general diagrams,we now define reduction operations which alter a diagram and the triple defined,but preserve the property of representing a counter-example(and more besides,as we shall see).

2.19 Reduction of (labelled) Special Diagrams

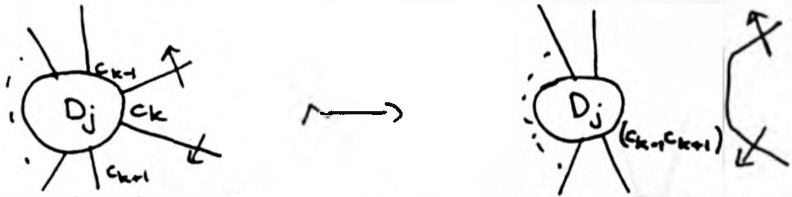
Let D be a labelled special diagram, where the segments of each disc are labelled $c_1, c_2, \ldots c_n$.

We call the following alterations of D reductions :

i) Discard loops.

ii) If there is an edge e which has as endpoints two adjacent vertices on some disc D_i , with the label a_k on the segment between the vertices, do the following operations : on D_i omit edge e and label c_k ,

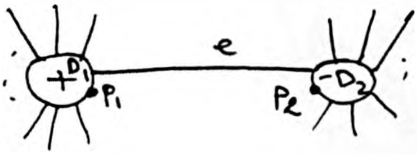
elsewhere do :

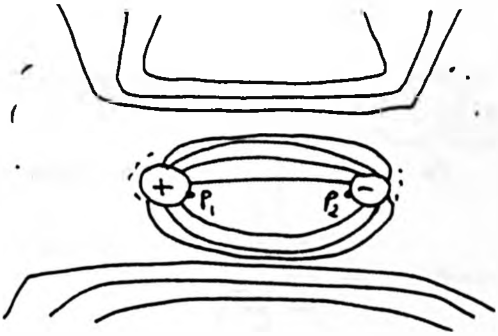


Hence one segment is now labelled $c_{k-1}c_{k+1}$.

iii) If two discs D_1 and D_2 are connected by an edge e , and D_1 and D_2 are oppositely labelled, and the distinguished points of D_1 and D_2 lie in the same region, on segments adjacent to e , then do the following opertations :

iii)a

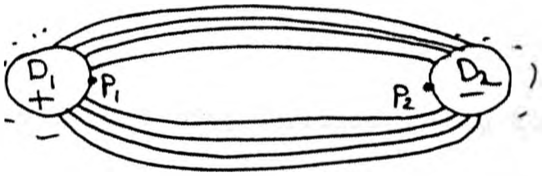




Notice that all the orientations match up by the properties of special diagrams, reading the n-tuple $(\mathcal{E}_1, \mathcal{E}_2, \ldots \mathcal{E}_n)$ clockwise from p_1 on D_1 , and the n-tuple $(-\mathcal{E}_1, -\mathcal{E}_2, \ldots -\mathcal{E}_n)$ reading anticlockwise from p_2 on D_2 .

iii)b If there is a connected component of D which consists of two discs

D_1 and D_2 with opposite sign, in which the distinguished points lie in the same region, discard this component ;
i.e. discard :

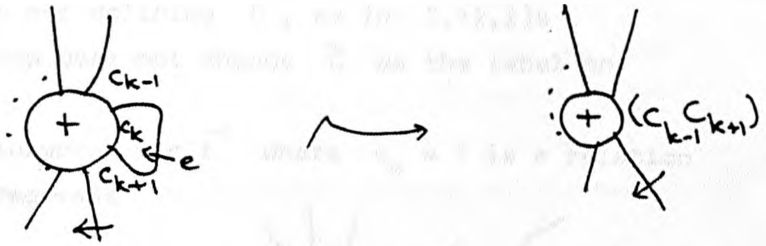


Operation III)a followed by III)b is called trivial reduction. We say that a special diagram is reduced if none of these operations can be performed.

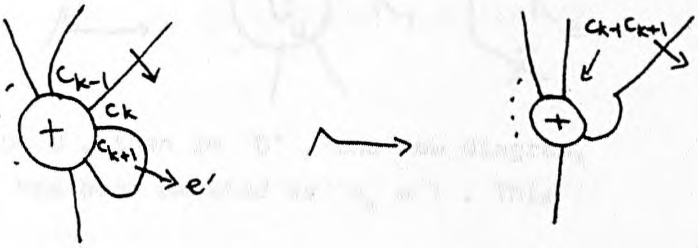
Note that operation II) may lead to the reduction process giving different reduced diagrams, e.g. suppose we have discs D_i , with edge e around label c_k , and D_j with edge e' around label c_{k+1}. Then doing operation II) to edge e , gives a different diagram from doing operation II) to edge e' :

A) do II) to edge e :

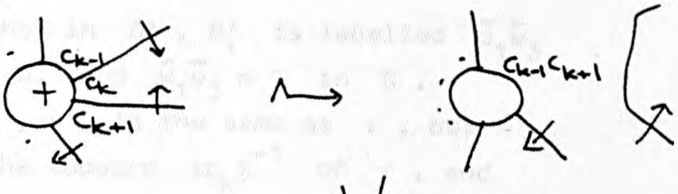
on D_i



on D_j

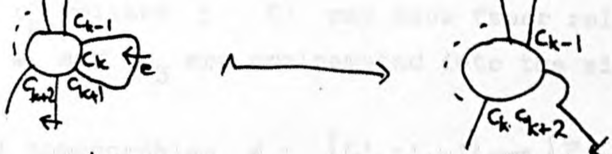


elsewhere

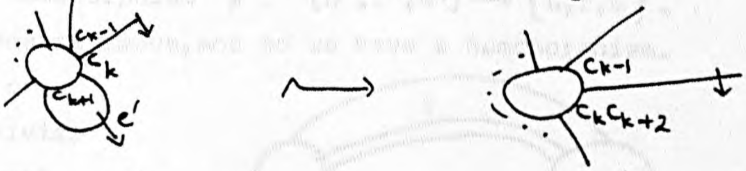


B) do II) to edge e':

on D_i



on D_j



elsewhere

We now strengthen Proposition 2.19 to show that a reduced diagram representing a counter-example defines a sort of 'canonical' counter-example.

Proposition 2.20

Let D be an unlabelled special diagram, representing the triple $\{G,r,w\}$.

i) Reduction of D gives a special diagram D' which defines a triple $\{G',r',w'\}$ such that there is a homomorphism ϕ such that:

$$\phi : G' * \langle t\rangle \longrightarrow G * \langle t\rangle , \quad \phi(r') = r , \quad \phi(w') = w$$

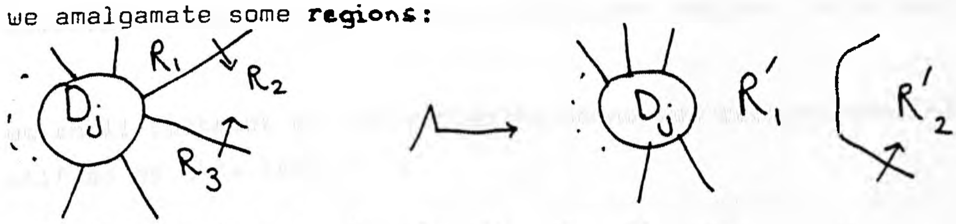
and $\phi(G') = \mathrm{Sbgp}_G\langle a_1,a_2,..a_n\rangle$, where $r = a_1t^{\alpha_1}a_2t^{\alpha_2}...a_nt^{\alpha_n}$.

ii) If $\{G,r,w\}$ is a counter-example then $\{G',r',w'\}$ is too.

Proof

I) We show that each reduction operation gives a diagram which defines a new group such that the homomorphism is defined obviously. Suppose that $\{\bar{G},\bar{r},\bar{w}\}$ is the triple defined by D as in 2.18. Operation I) is represented by a homotopy of the map defining D, as in 2.12,i)a. As in 2.9, omission of this loop does not change $\bar{G}$ as the label on this edge is t.

II) Here we omit from $\bar{r}$ a subword $t\,c_kt^{-1}$ where $c_k = 1$ is a relation in $\bar{G}$. Also we amalgamate some **regions**:



If R_2 is labelled w_2c_k in D, then in D', the new diagram, R_2' is labelled w_2; the letter c_k has been omitted as $c_k = 1$. This does not alter $\bar{G}$.

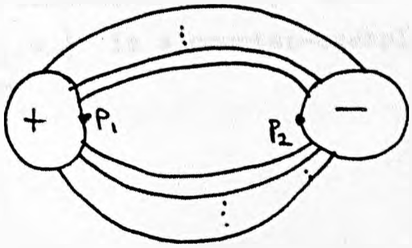
If R_1,R_3 are labelled w_1,w_3 then in D', R_1' is labelled $\bar{w}_1\bar{w}_3$ where $\bar{w}_i$ is some cyclic permutation of w_i, and $\bar{w}_1\bar{w}_3 = 1$ in G.

Hence if D' defines $\{G',r',w'\}$, r' is the same as r, but with trivial cancellation performed on the subword tc_kt^{-1} of r, and w' is $\bar{w}$ with occurrences of c_k omitted; G' may have fewer relators than $\bar{G}$, as the two relators w_1 and w_3 are amalgamated into the single relator $\bar{w}_1\bar{w}_3$ in G'.

Hence there is a natural homomorphism $\phi : \{G',r',w'\} \longrightarrow \{\bar{G},\bar{r},\bar{w}\}$.

III)a Again we amalgamate regions as above, and so we have a homomorphism.

III)b Here we omit a component of a type which only contributes trivial relations $c_ic_i^{-1}$, so this component may be omitted without altering the triple defined.



Hence reduction gives D' and there is a natural homomorphism $\{G',r',w'\} \longrightarrow \{\bar{G},\bar{r},\bar{w}\}$. The map $\phi : \{\bar{G},\bar{r},\bar{w}\} \longrightarrow \{G,r,w\}$ is defined as :

$\bar{r} = c_1 t^{\varepsilon_1} c_2 t^{\varepsilon_2} \ldots c_m t^{\varepsilon_m}$, $\varepsilon_i = \pm 1$, $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} \ldots . a_n t^{\alpha_n}$, $\alpha_i \in \mathbb{Z} - \{0\}$

Where $\varepsilon_i = \dfrac{\alpha_j}{|\alpha_j|}$ for $\sum_{k=1}^{j-1} |\alpha_k| < i \leqslant \sum_{k=1}^{j} |\alpha_k|$

and $\phi(c_i) = 1$ for $\left(\sum_{k=1}^{j-1} |\alpha_k|\right) + 1 < i \leqslant \sum_{k=1}^{j} |\alpha_k|$

$\phi(c_p) = a_j$ for $p = \left(\sum_{k=0}^{j-1} |\alpha_k|\right) + 1$.

Then $\phi(\bar{w}) = w$, and $\phi(\bar{r}) = r$.

This map is a homomorphism as each interior region of G defines a trivial word in G .And so there is a homomorphism

$\{G',r',w'\} \longrightarrow \{\bar{G},\bar{r},\bar{w}\} \longrightarrow \{G,r,w\}$.

ii) Follows immediately from the properties of ϕ .

■

Recall that reduction of a diagram is not unique,and hence the triple $\{G',r',w'\}$ is not uniquely defined by a diagram D .This is seen in the note after 2.19 ,where it is seen that operation II) may be possible at more than one place,at $c_k, c_{k'}$, and hence a different order of performing operation II) can give rise to different regions being amalgamated.

Now we shall restrict our attention to connected reduced special diagrams justified by this lemma :

Lemma 2.21

Let D be a reduced special diagram with more than one component representing a counter-example $\{G,r,w\}$. Then

i) Each component D_i of D defines a triple $\{G_i,r_i,w_i\}$ where there is a homomorphism $\phi_i : G_i * \langle t \rangle \longrightarrow G * \langle t \rangle$ such that $\phi_i(r_i) = r$.

ii) At least one triple $\{G_i,r_i,w_i\}$ is a counter-example .

Proof

Each component D_i defines a triple by 2.18 . The homomorphism follows as again the regions of $D - D_i$ are labelled by trivial words in G , and hence there is a natural homomorphism ϕ_i.

ii) Let D_i be a component inside another component,i.e. D_i is contained in an inner region of D_j.If $\{G_i,r_i,w_i\}$ is a counter-example then we are finished.Else $w_i = 1$ in G_i and so $\phi_i(w_i) = 1$ in G . Omit this D_i.Now assume that all components contribute to the boundary label.If no component gives a counter-example then the label on the boundary of D is trivial in G , i.e. $\{G,r,w\}$ is not a counter-example.Thus at least one component gives a counter-example.

■

We can sum up the results of this section on special diagrams in the following theorem :

Theorem 2.22

i) Suppose that r does not have a solution over G.
Then there is a connected,reduced,special diagram representing a counter-example $\{G,r,w\}$, and D defines a counter-example $\{G',r',w'\}$ such that $\exists \phi : G' * \langle t\rangle \longrightarrow G * \langle t\rangle$ with $\phi(r') = r$, and $\phi(w') = w^n$, a non-trivial element of G .

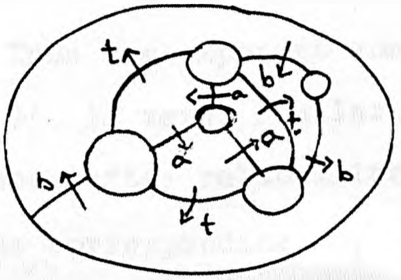
ii) Let D be a connected,reduced,special diagram.
Then D defines a triple $\{G,r,w\}$ which is a counter-example if w is non-trivial in G .

■

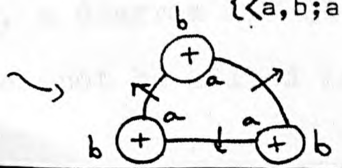
Thus we have,as promised,moved from the study of groups in general, to the study of groups defined by diagrams,and the study of diagrams in general.The following chart shows the progression we have passed through :

$$\begin{tikzcd}
\text{Counter-example } \{G,r,w\} \arrow[r, "\text{Prop.2.9}"] \arrow[dd, "\text{homomorphism}"', "\phi \text{ from } 2.20"'] & \text{Diagram} \arrow[d, "\text{Prop.2.17}"] \\
 & \text{Special Diagrams} \arrow[d, "2.20{,}2.21"] \\
\text{Counter-example } \{G',r',w'\} \arrow[r, shift left] & \text{Reduced,connected Special Diagram} \arrow[l, shift left]
\end{tikzcd}$$

The fact that the map ϕ is a homomorphism which is not necessarily injective or surjective means that G' may have 'fewer' relations than G; e.g. let $\{G,r,w\} = \{\langle a,b;\ a^3\ ,\ b^2\rangle,\ atbt^{-1}\ ,\ b\}$.
Then a diagram for this is :



which gives the special diagram defining the counter-example $\{\langle a,b;a^3\rangle, atbt^{-1}, b^3\}$.



As an example of the essential difference between a counter-example and the counter-example defined by a diagram, we have the following proposition which provides a class of groups and added relators which satisfy the Kervaire Conjecture.

Proposition 2.23

If $\{G,r,w\}$ is a triple defined by a special diagram, then $\langle G,t;r\rangle$ is non-trivial.

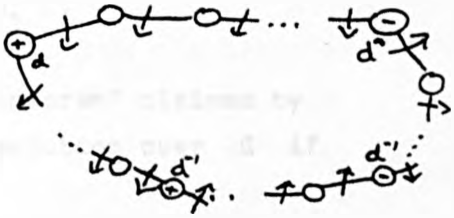
Proof

By proposition 1.6 we need only consider the case when $\sigma_t(r) = \pm 1$.

Let $r = c_1 t^{\varepsilon_1} c_2 t^{\varepsilon_2} \ldots c_n t^{\varepsilon_n}$ where $\varepsilon_i = \pm 1$; as $\sigma_t(r)$ is non-zero, we have that $\varepsilon_{i-1} = \varepsilon_i$ for some values of i . We shall show that for these values of i , c_i is non-trivial in $\langle G,t;r\rangle$.

Let D be the diagram which defines $\{G,r,w\}$; we now relabel some of the segments of D to obtain a diagram D' as follows : when $\varepsilon_{i-1} = \varepsilon_i = +1$, replace c_i by d , and when $\varepsilon_{i-1} = \varepsilon_i = -1$, replace c_i by d^{-1} . D' defines the triple $\{G',r',w'\}$ such that G' is a homomorphic image of G.

For any region of D' ,it is clear that its label has as many occurrences of d as it has of d^{-1}; reading anti-clockwise around the boundary of an interior region of D', a series of "inward" oriented t-edges ends when an r'-disc is reached where the label is d or d^{-1} , depending upon whether the r'-discs are labelled + or – .



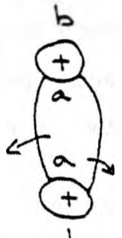
Thus the exponent sum of d in any of the defining relations for G' is zero. Similarly any special diagram representing $\{G,r,w\}$ becomes, after relabelling, a diagram for $\{G',r',w''\}$ where $\sigma_d(w'') = 0$ so the corresponding c_i cannot be killed in $\langle G,t;r\rangle$. ∎

Section 4

2.24 Examples

We can use special diagrams to construct counter-examples. As we noted in 1.1, the standard example of a word without a solution over a group has the form $r = atbt^{-1}$ where a and b are elements of G with different orders. We now show how this, and other examples can be drawn as diagrams.

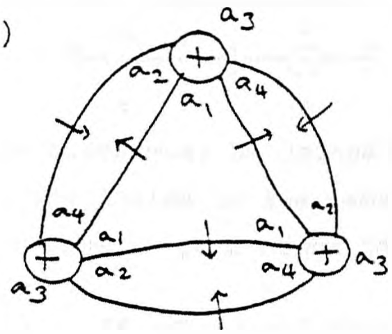
i)



This diagram defines the above counter-example : $G = \langle a,b;a^2\rangle$

$\langle G,t;r\rangle = \langle a,b,t;a^2,atbt^{-1}\rangle$, and the counter-example defined is $\{G,atbt^{-1},b^2\}$.

ii)



This defines $G = \langle a_1,a_2,a_3,a_4;a_1^3,a_2a_4\rangle = \langle a_1,a_2,a_3;a_1^3\rangle$.

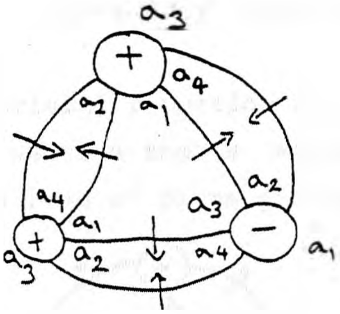
$\langle G,t;r\rangle = \langle a_1,a_2,a_3,t\ ;\ a_1^3,a_1ta_2t^{-1}a_3ta_2^{-1}t^{-1}\rangle$

and the counter-example defined is

$\{G,a_1ta_2t^{-1}a_3ta_2^{-1}t^{-1},a_3^3\}$.

Note that r has the form $a_3 = wa_1w^{-1}$.

iii)



Here $G = \langle a_1,a_2,a_3,a_4;\ a_1^2a_3^{-1},a_4a_2,a_4a_2^{-1}\rangle$

$= \langle a_1,a_2;\ a_2^2\rangle$.

$\langle G,t;r\rangle = \langle a_1,a_2,t;\ a_2^2,a_1ta_2t^{-1}a_1^2ta_2t^{-1}\rangle$

and the counter-example defined is

$\{G,a_1ta_2t^{-1}a_1^2ta_2t^{-1},a_1^3\}$.

This last example is a counter-example to a 'theorem' claimed by Schiek [Sch 2]. Schiek claims to show that r has a solution over G if r has the form $aEbE^{-1}$, where $a,b\in G$, and

$$E = (t^{-s_1}a_1t^{s_1})(t^{-s_2}a_2t^{s_2})\dots(t^{-s_n}a_nt^{s_n}) \quad , \quad n\geqslant 1 \ , \quad a_i\in G\ ;\ s_i\neq 0\ .$$

Levin points out in his review of Schiek's article (Maths. Reviews 55 number 3096) that the theorem is untrue if G contains elements of odd order, and this example shows that it also fails if G has elements of even order.

Note that in all these examples the exponent sum of t in r is zero.

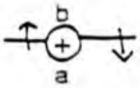
We now give a diagrammatic proof of 1.2.In fact we show something slightly more general :

Note 2.25

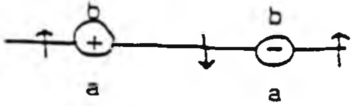
If r contains only two t-occurrences and r is cyclically reduced in $G * \langle t \rangle$, then r has a solution in G, unless $r = atbt^{-1}$ and a and b have different orders in G.

Proof

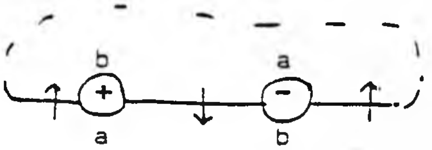
i) If $\sigma_t(r) = \pm 2$, then look at a reduced special diagram D representing some word $w = 1$ in $\langle G,t;r \rangle$.

Then a +disc in D looks like 

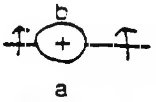
, and hence we can see that a +disc cannot be joined to another +disc .Therefore a +disc may only be connected to -discs,and one way of doing this will give an unreduced special diagram :

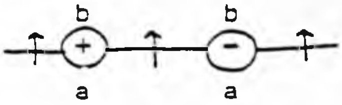


reduced:

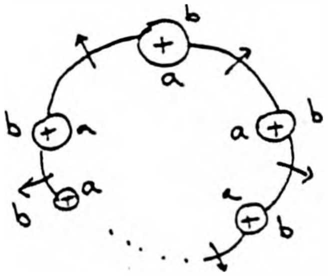


The discs must be joined together so as to form a loop,such that the label on the inside is the same as the label on the outside,which is $(ab^{-1})^k$ for some k ,and hence the diagram does not give a counter-example.

ii) If $\sigma_t(r) = 0$ then the +disc looks like 

and if we have a +disc joined to a -disc they must look like 

and so trivial reduction is possible,and so assuming that the diagram was reduced we have that a +disc can only be joined to a -disc,and again we get a loop of discs giving the relation $a^k = 1$ in G from the relation inside.Hence G injects into $\langle G,t;r \rangle$ if the order of a in G is the same as the order of b in G .If the orders of a and b are different in G , then the diagram shows that b^k is trivial in $\langle G,t;r \rangle$.



∎

Note that a new proof of the Higman,Neumann,Neumann theorem that the base group of an HNN extension injects into the HNN extension can be given on the above lines.

CHAPTER 3. A Class of Special Diagrams

Introduction

Attempts to draw special diagrams have not proved fruitful; the conditions on the transverse orientations of the edges make the step from regular planar graph to reduced special diagram difficult. In this section we give a description of a general class of reduced special diagrams which we shall then show (theorem 3.4) does not contain a diagram representing a counter-example. This class of diagrams is interesting because :

a) $\sigma_t(r) = \pm 1$ and so they are candidates for representing counter-examples to the Kervaire Conjecture.

b) The high degree of symmetry in the construction means that the number of relations in the group defined by such a diagram appears to be low (in fact the deficiency is $\geqslant (2n + 1) - (2n + 1) = 0$) which seems to be a good thing as the more relations we have, the more chance there is that the outside label w is trivial in G , the group defined by the diagram.

c) They are the only reduced diagrams with $\sigma_t(r) \neq 0$ which we know how to draw, and so for this reason alone we must show that they do not represent counter-examples.

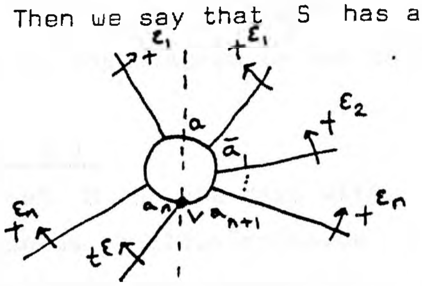
We start this short chapter with the construction in 3.1 ; the proof that these diagrams do not give counter-examples (3.4) is basically group-theoretic.

Definition 3.1

Let S be a disc with transverse oriented spokes (2.14) of index $2n + 1$, such that there is an axis X in the plane through a vertex v about which S – (edge at v) is symmetric .

Then we say that S has an axis of symmetry.

e.g.



Note that in a special diagram made from these discs, the triple $\{G, r, w\}$ has the property that $\sigma_t(r) = \pm 1$, and r has the form :

$$r = t^{\varepsilon} a_n t^{\varepsilon_n} a_{n-1} t^{\varepsilon_{n-1}} \ldots a_2 t^{\varepsilon_2} a_1 t^{\varepsilon_1} a t^{-\varepsilon_1} \bar{a}_1 t^{-\varepsilon_2} \bar{a}_2 \ldots \bar{a}_{n-1} t^{-\varepsilon_n} a_{n+1} \; ,$$

where $\pm 1 = \varepsilon = \sigma_t(r)$.

Definition 3.1 (cont.)

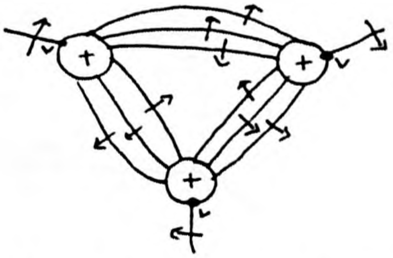
A +unit U in a special diagram D is a collection of k +discs (of index $2n + 1$) such that each disc has an axis of symmetry through v, such that :

i) U is connected to $D - U$ by the k edges through the k points corresponding to v .

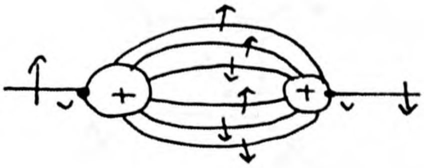
ii) if two discs D_1 and D_2 in U are connected by an edge, then then they are connected by $2n$ or n edges, depending on whether U has just two discs or more .

Define -units the same way using -discs.

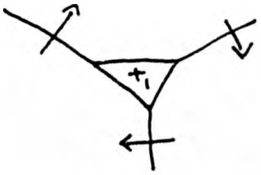
e.g.



or



If U is a +unit with k discs, built using axis of symmetry X_i then we draw it as a k-sided polygon with an edge radiating from each vertex, and label it $+_i$. If $k=2$ we draw a rectangle with an edge radiating from each of the shorter sides. Then the two examples above become :



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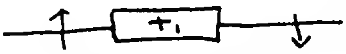
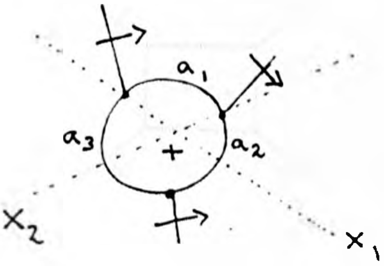


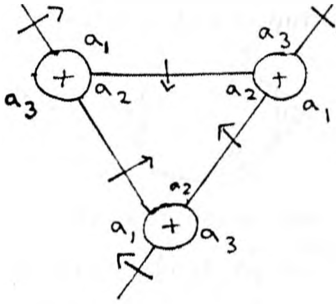
We say that a diagram is made up of units if each disc is in just one unit, each disc has just two axes of symmetry, and the outside word w defined by the diagram is not in just one unit.

Example 3.2

Let D be the disc with three spokes, as illustrated. We see that there are two axes of symmetry, X_1 and X_2



We see that we can build units using this disc with spokes :



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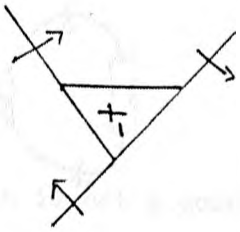

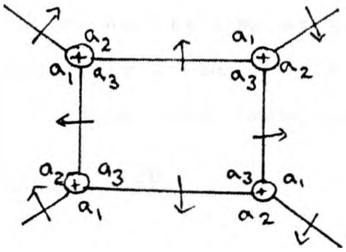

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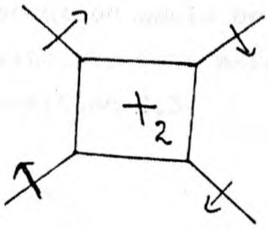
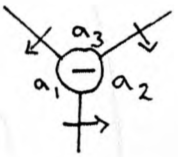
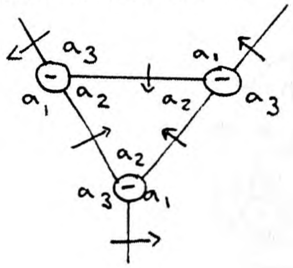


For this case the -disc looks like :



and as before we build units :



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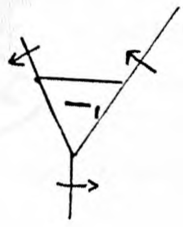

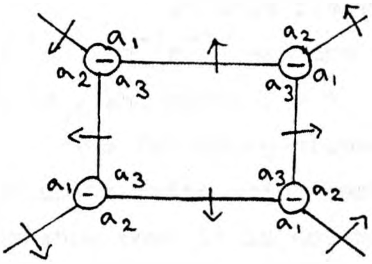

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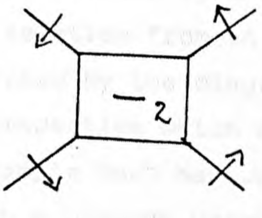
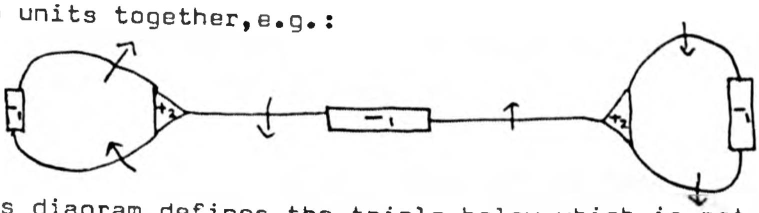


Using these units we may draw a special diagram made up of units, by joining units together, e.g.:

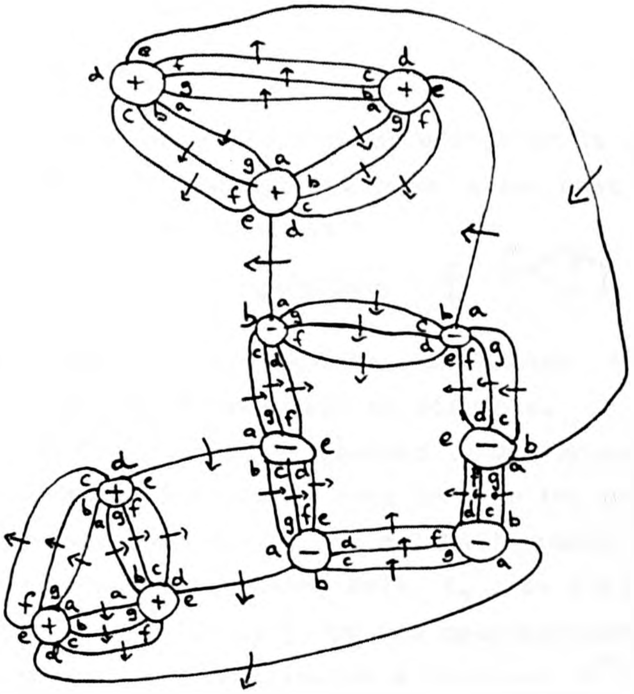


This diagram defines the triple below, which is not a counter-example, as $(a_2a_1a_3^{-1}a_1^{-1}) = 1$ in G ;

$$\{<a_1, a_2a_3;\ a_2^2, a_3^3, a_2a_1a_3^{-1}a_1^{-1}>,\ a_1ta_2t^{-1}a_3t\ ,\ (a_2a_1a_3^{-1}a_1^{-1})^4\}$$

Note that in this example, all the +discs use axis of symmetry X_1, all the -units use axis X_2. This is because reduction would be possible immediately if we have a +unit and a -unit with the same axis of symmetry which were joined, as we shall see in Proposition 3.3.

Example 3.2b



In this diagram we see that the word defined by the outside is $eda^{-1}b^{-1}eda^{-1}b^{-1}$, and the word $eda^{-1}b^{-1}$ is a relation from an inner region e.g. R , and hence w = 1 in the group G defined by the diagram.

The following proposition sums up the properties which a diagram made up of units which represents a counter-example must have. We shall then show that it is not possible to build such a diagram using discs with just two axes of symmetry.

Proposition 3.3

Let D be a reduced, connected non-empty diagram made up of units, which represents a counter-example.

Then :

i) there are no edges joining units of the same sign.

ii) the number of +discs is equal to the number of -discs.

iii) a +disc built using axis of symmetry X_i is not connected to a -disc built using axis of symmetry X_i.

iv) the orders of the +(resp.-) units have a highest common factor d (resp. d') > 1 .

v) the numbers of edges bounding interior regions have a highest common factor 4n , where $n > 1$.

vi) if 4n' is the number of edges on the outside of D , then n does not divide n' .

Proof

i) If an edge joins a +unit to another unit, then it arrives at the other unit with an anti-clockwise orientation near that unit, and hence that unit must be a -unit :

ii) This follows from i) , as each disc contributes 1 edge to the diagram where the units are drawn as polygons.

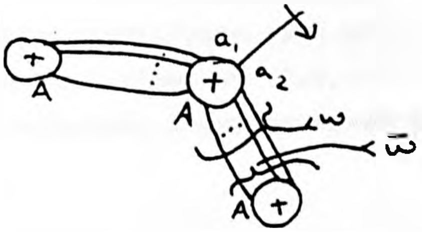
iii) This follows as the diagram is assumed to be reduced.

Assuming that the two axes of symmetry used to make the units of D are X_1 and X_2 , we have that the +units are all built using the axis X_1 (say) , and the -units are built using axis X_2 , by iii) and the fact that D is connected. Let $\{G', r', w'\}$ be the counter-example defined by D.

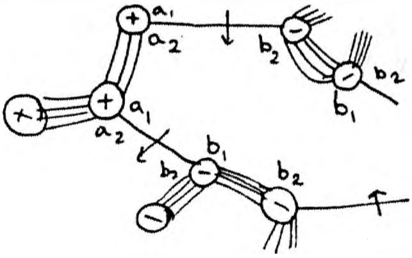
iv) A +unit of degree k_i contributes a relation $A^{k_i} = 1$ and so if d is the highest common factor of the k_i, we have that $A^d = 1$ in G' , and so $(\phi(A))^d = 1$ in G .

We see that r' has the form $a_2 wA\bar{w}a_1 t$, and so if $d = 1$, $r' = a_1 a_2 t$ and so $\langle G', t; r' \rangle =$ G' , and hence D does not define a counter-example.

Similarly we have the same result for -units.



v)vi) An interior region R_i of D outside of the units contributes a relation $(a_1a_2b_2^{-1}b_1^{-1})^{n_i}$. So,as above,if n is the highest common factor of the n_i ,we have the relation $(a_1a_2b_2^{-1}b_1^{-1})^n$ in G'.But w is the word $(a_1a_2b_2^{-1}b_1^{-1})^{n'}$ and so if $n = 1$, D does not define a counter-example.Note that the number of edges of R_i is $4n_i$, and that in fact D represents a counter-example only if n does not divide n' .



The point of this chapter is to prove :

Theorem 3.4

A diagram made up of units does not represent a counter-example.

From 3.3 we see that there is a diagram made up of units which represents a counter-example only if there exists a planar graph with certain properties,and so the following graph-theoretic theorem implies 3.4:

Theorem 3.5

Let G be a planar graph with the following properties :

1) G is bipartite with corresponding sets of vertices A and B;

2) The orders of the vertices of A (resp. B) have a highest common factor d (resp. d') > 1 .

3) The numbers of edges bounding interior regions of G have a highest common factor 2n , $n > 1$.

If 2n' is the number of edges bounding the outside of G , then n divides n' .

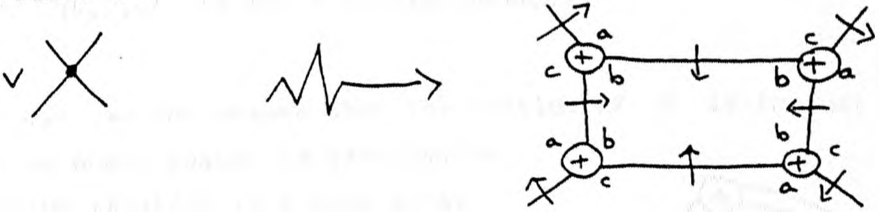
<u>3.5 Implies 3.4</u> : Let D be a diagram made up of units representing a counter-example.Let P be the graph obtained by identifying each unit of D to a point.Then by 3.3 ,P fulfills the conditions of 3.5,and n divides n' and so by 3.3vi) , D does not represent a counter-example.

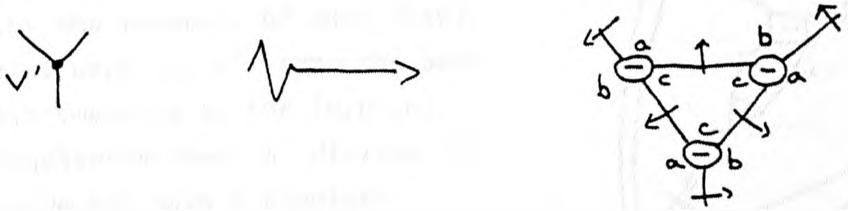
Proof of theorem 3.5

The method of proof is,strangely enough,group-theoretic ; having moved from group theory problems to graph-theory constructions,we now move full-circle, and prove this result by converting a graph which fulfills the conditions into a diagram defining a triple which we shall show is not a counter-example, in fact which has the property that n divides n'.

Let G be a planar graph fulfilling 1), 2), and 3).

Replace a vertex v in A (respectively v' in B) of order k_i (resp k_i') by a +unit (resp −unit) with k_i (resp k_i') discs with 3 spokes as below :

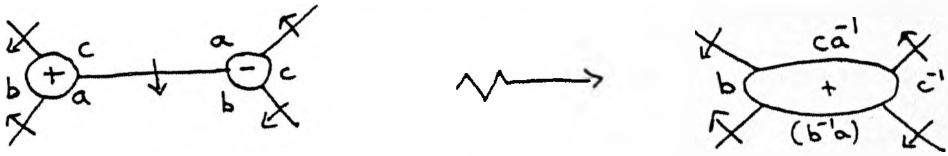




Then the diagram D obtained defines the triple :

$$\left\{ \langle a,b,c \; ; \; b^{d},\ c^{d'},\ (aca^{-1}b^{-1})^{n} \rangle ,\ atbt^{-1}ct \ ,\ (aca^{-1}b^{-1})^{n'} \right\} .$$

As we noted in 3.3 there is a pairing of the discs of D induced by the edges joining the units, and so we can replace each (+,−) pair of discs by a single disc labelled $bt^{-1}ca^{-1}t^{-1}c^{-1}tb^{-1}at$:



This new diagram D' represents the triple

$$\left\{ \langle a,b,c \; ; \; b^{d},\ c^{d'},\ (aca^{-1}b^{-1})^{n} \rangle ,\ atbt^{-1}ca^{-1}t^{-1}c^{-1}tb^{-1},\ (aca^{-1}b^{-1})^{n'} \right\}$$

that is $w = (aca^{-1}b^{-1})^{n'}$ is trivial in $\langle G,t;r \rangle$, using the usual notation.

Adding the relation $a = 1 = t$, $\langle G,t;r \rangle \longrightarrow H = \langle b,c; b^{d}, c^{d'}, (cb^{-1})^{n} \rangle$ and $w = (aca^{-1}b^{-1})^{n'}$ is mapped to $(cb^{-1})^{n'}$.

But H is the (d,d',n) triangle group (see e.g. Coxeter and Moser "Generators and Relations for Discrete Groups", page 67) and hence (cb^{-1}) has order n , and so n divides n' as required.

■

We can now generalise theorem 3.4 slightly; we note that in the definition of a diagram made up of units we had the condition that the outside boundary of the diagram should not be in contained in just one unit. We now effectively remove this condition :

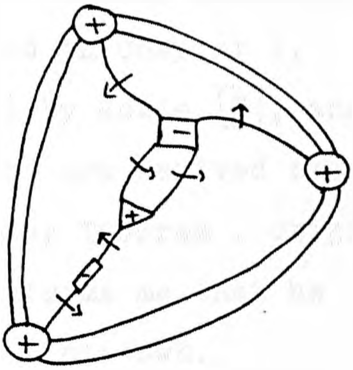
Theorem 3.6

Let D be a diagram representing a triple $\{G,r,w\}$, such that each disc has two axes of symmetry, and each disc of D is in a unit.

Then $\{G,r,w\}$ is not a counter-example.

Proof

By 3.4 we can assume that the outside of D is in just one unit which we shall assume is labelled +. If the outside relation is a word other than b^k , it occurs elsewhere in the same unit (by the symmetry of each disc). If the outside word is b^k then the same proof as 3.5 (mapping to the (d,d',n) triangle group) shows that d divides k and hence we do not have a counter-example.



CHAPTER 4

In this chapter we develop a method based on a paper of Lyndon [L2] which allows us in certain circumstances to relabel a special diagram; this will be used to prove the Freiheitssatz for locally indicable groups (4.9). This generalises both the result for subgroups of the additive group of reals (obtained by Lyndon in [L2]), and that for locally residually free groups due to Pride in [P]. As we noted in Chapter 1, this result has been obtained independently by Howie [H], and by a Russian, Brodskii [Br]; Howie's methods are derived from those used to prove Dehn's Lemma and the Loop Theorem , originally due to Papakyriakopoulos, and Howie informs me that he believes Brodskii's methods to be similar to his own.

We first define a potential function on a special diagram (4.1) which shall then be used to relabel the diagram. The principle examples of these functions are given in 4.2, and repeated use of these constructions is made later. The Lyndon Lemma (4.4), concerning the form such a potential function must have on a diagram representing a kind of failure of the Freiheitssatz for a class of groups, is then proved, and two important corollaries (4.5 and 4.6) are given. This is applied in 4.8 to show that G injects naturally into $\langle G,t;r\rangle$ if G is locally indicable, and this gives the Freiheitssatz for locally indicable groups almost immediately in 4.9 . In the final section we use the methods of the proof of 4.4 to show that G injects into $\langle G,t;r\rangle$ if $G*\langle s\rangle$ injects into $\langle G,s,t;r'\rangle$ and into $\langle G,s,t;r''\rangle$, where r' and r'' are words in $G*\langle s\rangle*\langle t\rangle$ which are obtained from r using a homomorphism from $\langle G,t;r\rangle$ to $\mathbb{Z}$.

Section 1 Potential Functions and the Lyndon Lemma

Our aim is to put a real-valued continuous function on a special diagram in order to allow a certain type of relabelling process to proceed. Recall here that a special diagram can be viewed (see 2.14) as formed by taking copies of a "disc with spokes" (each copy has label r) and copies of its mirror image (these copies are labelled r^{-1}) ; the free end-points of the spokes are then identified such that the transverse orientations are preserved (giving the t-edges) and, of course, such that the resulting diagram is planar.

Definition 4.1

A Potential Function on a special diagram D is a continuous p.l. map $\Phi : D \longrightarrow \mathbb{R}$ such that Φ is constant on any edge of D, and when B_1 and B_2 are two discs of D and $v_{1,i}$, $v_{2,i}$ are corresponding points on B_1 and B_2 , we have :

$\Phi(v_{1,i}) = \Phi(v_{2,i}) + \Phi(p_1) - \Phi(p_2)$ where p_1 and p_2 are the distinguished points on B_1 and B_2 respectively.

4.2 Construction of Potential Functions

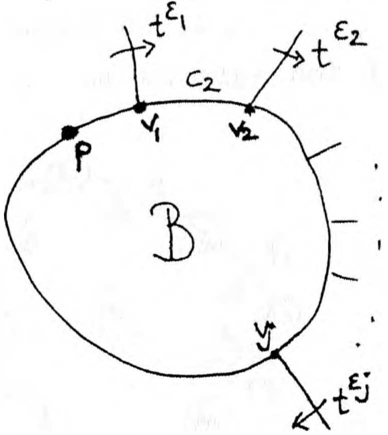
We now give examples of potential functions on special diagrams; these are arranged in a strict order, introducing step by step the procedures which give the essential construction of the final example 4.2(v), to be used in section 2.

4.2(i) Clearly a constant map, taking the entire diagram to zero, is a potential function (the trivial potential function).

4.2(ii) If we have a connected special diagram D defining a triple $\{G', r', w'\}$ with $\sigma_t(r') = 0$, then we can use the homomorphism $\Theta = 0 * \mathrm{id} : G' * \langle t \rangle \longrightarrow \mathbb{Z}$ to form a potential function as follows:

4.2(ii) (cont.) Let $r' = c_1 t^{\varepsilon_1} c_2 t^{\varepsilon_2} c_3 \dots c_n t^{\varepsilon_n}$, $\varepsilon_i = \pm 1$, and $c_i \in G'$ (possibly $=1$) .

We construct the potential function Φ by first defining a function of the desired type on the disc with spokes B used to construct the diagram. We then show how to extend the function as the discs with spokes B_i (the copies of B and its mirror image) are joined up. Let p be the distinguished point and let v_i be the vertex where the edge labelled t^{ε_i} meets B.



Define the function h on the vertices of B as :

$h(p) = 0$

$h(v_1) = \frac{1}{2}\varepsilon_1$

$h(v_2) = \varepsilon_1 + \frac{1}{2}\varepsilon_2$

$h(v_j) = \varepsilon_1 + \varepsilon_2 + \dots + \frac{1}{2}\varepsilon_j$

Linearly extend h to the rest of the circumference of the disc, and then radially extend to the interior; on the j-th spoke let $h = h(v_j)$.

Let $B_0, B_1, B_2, \dots, B_k$ be the discs with spokes occurring in D , ready equipped with a copy of the function h. Let $\{e_1, e_2, \dots, e_k\}$ be a set of edges in D forming a maximal tree (regarding each disc as a point), and suppose that e_1 joins B_0 to B_i, joining vertices $v_{0,j}$ and $v_{i,m}$. ($v_{i,j}$ is the vertex on B_i which corresponds to the vertex v_j on B)

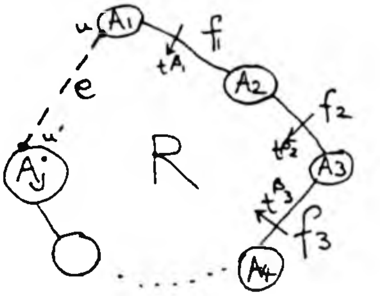
We define the function H on B_i using h as :

$H(w_0) = h(w_0)$ for any point w_0 on B_0 ;

$H(v_{i,m}) = H(v_{0,j})$ $(= h(v_j)\ ) = H(e_1)$ and let

$k_i = H(v_{i,m}) - h(v_m)$ and define $H(w_i) = h(w_i) + k_i$ for all points w_i on B_i ; note that $H(p_i) = k_i$.

4.2(ii) (cont.) As we noted earlier, for all points w on the edge e_1, we have $H(w) = H(v_{o,j}) = H(v_{i,m})$.
In this way we extend H over the maximal tree, from the discs where we already know the values of H, to the rest, proceeding outwards along the edges.
It remains to show that adding any edge to the maximal tree joins two vertices of D where H takes the same value, so that we can then extend H over these remaining edges of D.
Let e be an edge not in the maximal tree; adding e forms an interior region of D; call it R ; the boundary of R consists of segments of discs and a set of edges. Let $A_1, A_2, \ldots, A_j$ be the discs meeting ∂R, and let $f_1, f_2, \ldots, f_j$ (with $e = f_j$) be the edges occurring on ∂R, with f_i labelled t^{β_i} (inwards), in clockwise order.



Suppose that the endpoints of $e = f_j$ on A_1 and on A_j are u and u' respectively. We must show that $H(u) = H(u')$ in order to extend H along the edge e joining them. But we have :

$$H(f_1) = H(u) + \tfrac{1}{2}(\beta_j - \beta_1)$$

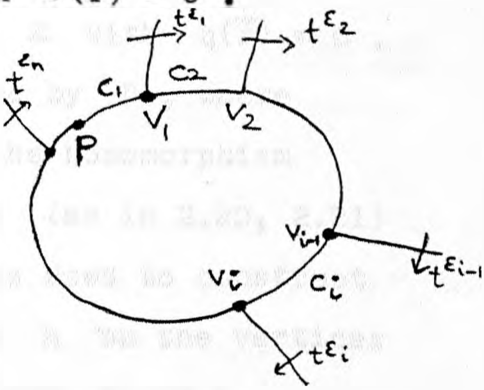
$$H(f_2) = H(f_1) + \tfrac{1}{2}(\beta_1 - \beta_2) = H(u) + \tfrac{1}{2}(\beta_j - \beta_2)$$

etc. proceeding clockwise round ∂R until

$$H(u') = H(f_{j-1}) + \tfrac{1}{2}(\beta_{j-1} - \beta_j) = H(u) .$$

Hence we can extend H over the edge e and hence over all the edges of the special diagram D ; as previously, extend radially over the regions (e.g. over R above) to give Φ of the required form.

4.2(iii) Let D be a connected special diagram defining the triple $\{G', r', w'\}$ with $r' = c_1 t^{\varepsilon_1} c_2 t^{\varepsilon_2} \ldots c_n t^{\varepsilon_n}$, $c_i \in G'$, and $\varepsilon_i = \pm 1$; as before, use $\overline{r}'$ to denote $c_1 c_2 \ldots c_n$. Suppose that $\sigma_t(r') \neq 0$ $(0 \neq \sum_{i=1}^{n} \varepsilon_i)$, and that there exists a homomorphism q' taking G' onto $\mathbb{Z}$ such that $q'(\overline{r}') \neq 0$ $(0 \neq \sum_{i=1}^{n} q'(c_i))$. After multiplying by a scaling factor (and thus regarding q' as a non-trivial homomorphism from G' to $\mathbb{R}^+$, the additive group of reals) we can suppose that $q'(\overline{r}') = -\sigma_t(r')$, so that we have a non-trivial homomorphism θ such that $\theta = q' * \mathrm{id} : G' * \langle t \rangle \to \mathbb{R}^+$ with $\theta(r') = 0$.



Let B be the disc with spokes used to construct D as in (ii) . Define the function h on the vertices of B (and at p) as :

$$h(p) = 0$$

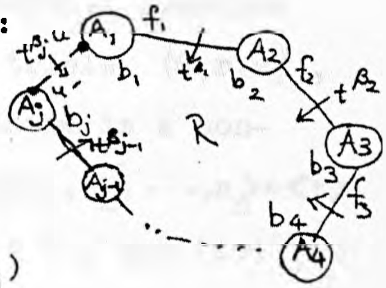
$$h(v_1) = q'(c_1) + \tfrac{1}{2}\varepsilon_1$$

$$h(v_2) = q'(c_1) + q'(c_2) + \varepsilon_1 + \tfrac{1}{2}\varepsilon_2$$

etc.

$$h(v_i) = q'(c_1) + \ldots + q'(c_i) + \varepsilon_1 + \varepsilon_2 + \ldots + \tfrac{1}{2}\varepsilon_i$$

We extend h to H on a maximal tree for D , as in (ii), such that each disc B_i has an associated constant k_i. To extend beyond the maximal tree, using the construction and notation of (ii), with b_i the label on $A_i \cap R$ (read anticlockwise on A_i) :



$$H(f_1) = H(u) + q'(b_1) + \tfrac{1}{2}(\beta_j - \beta_1)$$

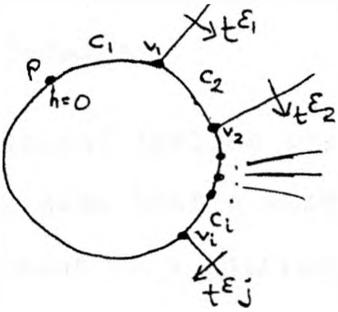
$$H(f_2) = H(f_1) + q'(b_2) + \tfrac{1}{2}(\beta_1 - \beta_2)$$

etc., so that :

$$H(u') = H(u) + q'(b_1) + q'(b_2) + \ldots + q'(b_j)$$

4.2(iii) (cont.) But the label on the boundary of R, read clockwise is the word $b_1 b_2 \ldots b_j$, and this is a relation in G' by the definition of G', so we have $q'(b_1 b_2 \ldots b_j) = q'(b_1) + q'(b_2) + \ldots + q'(b_j) = 0$ as q' is a homomorphism. So $H(u') = H(u)$ as required, and we can extend H as in (ii), firstly over the remaining edges, and then over the interior regions.

4.2(iv) Let D be a special diagram representing the triple $\{G, r, w\}$ where $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} \ldots a_n t^{\alpha_n}$, $a_i \in G - \{1\}$, $\alpha_i \neq 0$; suppose that there exists a homomorphism q taking $\text{Sbgp}_G\langle a_1, a_2, \ldots, a_n\rangle$ onto $\mathbb{Z}$ with $q(\overline{r}) = 0$. Let $\{G', r', w'\}$ be the triple defined by D, where $r' = c_1 t^{\varepsilon_1} c_2 t^{\varepsilon_2} \ldots c_m t^{\varepsilon_m}$, and $\emptyset$ the homomorphism from G' onto $\text{Sbgp}_G\langle a_1, a_2, \ldots, a_n\rangle$ (as in 2.20, 2.21) and let B be the disc with spokes used to construct D as in (ii). Define the function h on the vertices of B (and p) as :



$h(v_1) = q \circ \emptyset(c_1)$

$h(v_2) = q \circ \emptyset(c_1) + q \circ \emptyset(c_2)$

$h(v_i) = \sum_{j=1}^{i} q \circ \emptyset(c_j)$

$h(p) = 0$

Using this function as in (ii) and (iii), we can define a potential function on D.

4.2(v) It is now easy to see how to put a potential function on a special diagram representing the triple $\{G, r, w\}$, where $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} \ldots a_n t^{\alpha_n}$, when there is a non-trivial homomorphism θ taking $\text{Sbgp}_G\langle a_1, a_2, \ldots, a_n\rangle * \langle t\rangle$ to $\mathbb{R}^+$ such that $\theta(r) = 0$; if $\theta(t) = 0$, use (iv) ; else rescale θ so that $\theta(t) = 1$, and construct h as in (ii) or (iii) with $q' = \theta \circ \emptyset$.

Before proceeding to the statement of the Lyndon Lemma (4.4), several definitions are necessary; their relevance will become evident during the proof of 4.4, and they occur again and again during this final chapter.

Definition 4.3

(i) A potential function on a special diagram D is univalent if it takes just one value on the distinguished points of D : p_1, p_2 distinguished points of D implies $\Phi(p_1)=\Phi(p_2)$. A special diagram D is univalent if all potential functions defined on D are univalent.

(ii) A potential function on a special diagram D is nulvalent if it takes just one value on the t-edges of D . Otherwise the function is polyvalent.

A special diagram D is nulvalent if all potential functions defined on D are nulvalent (or alternatively, if there is no polyvalent function defined on D). Otherwise D is polyvalent.

Note that nulvalent implies univalent (for both functions and diagrams), and also that a univalent diagram is connected (else map each component to a (different) integer).

(iii) A class C of groups is free-product-closed (usually abbreviated to f-p-c) if the infinite cyclic group $\mathbb{Z}$ is an element of C , and for any pair of groups G_1 and G_2 in C , their free product $G_1 * G_2$ is also in C .

Examples of f-p-c classes are: the class of all free groups (this is a subclass of any f-p-c class), the class of all locally indicable groups (see section 2), and the class of all torsion-free groups.

Definition 4.3 (cont.)

(iv) Let C be a class of groups; a counter-example for C is a counter-example $\{G, r, w\}$ such that G is in C. A minimal counter-example for C is a counter-example for C, such that the length of r is minimal over all such counter-examples, and the number of t-occurrences in r is minimal over all such r of minimal length.

Note that a minimal counter-example for a class C is not uniquely defined. We make this definition as the method which we shall use to show that r has a solution over G for any G in a class of groups C, and for any r of cyclically reduced length greater than 1, is to show that there are no minimal counter-examples.

(v) Let $\{G_i\}$ be a collection of groups, and let $\{r_1, r_2, \ldots, r_m\}$ be a set of cyclically reduced words in $*_i G_i$. The set $\{r_j\}$ is staggered with respect to $\{H_k\}$ if $\{H_k\}$ is a subcollection of $\{G_i\}$, and there exist integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ such that $a_1 < a_2 < \ldots < a_m$, $b_1 < b_2 < \ldots < b_m$, $a_i \leq b_i$ for all i, and such that : each r_j contains occurrences from both H_{a_j} and H_{b_j}, and if $n \notin [a_j, b_j]$, then r_j contains no occurrence from H_n.

Note that $r \in G * \langle t \rangle$ trivially forms a staggered set with respect to $\langle t \rangle$, as long as r is cyclically reduced and has at least one t-occurrence.

We are now in a position to state and prove the Lyndon Lemma; essentially we use the methods of [L2], but the change from Dehn-diagrams to "diagrams", and the above definitions, help to give us a more general result.

Lemma 4.4 **The Lyndon Lemma** (Lyndon [L2])

Let C be an f-p-c class of groups.

Then either (i) there is a nulvalent diagram representing a counter-example for C ;

or (ii) for any subset $\{G_i\} \subset C$, and any set of words $\{r_j\}$ in $*_i G_i$ which is staggered with respect to some subset $\{H_k\} \subset \{G_i\}$, where $H_{a_k} \cong \mathbb{Z} \cong H_{b_k}$ for all k , and for any ordered subset $A \subset \{r_j\}$ $A = \{r_{j_1}, r_{j_2}, \ldots, r_{j_p}\}$ (with $j_1 < j_2 < \ldots < j_p$) we have that any consequence of A (but no proper subset of A) contains occurrences from H_{a_1} and from H_{b_p} .

Note that when $\{r_j\}$ contains just one element, which is the case which interests us, case (ii) says that r has a solution over G for all groups G in C. Hence if there is a counter-example for C , then case (i) holds; this is the form in which we shall use the lemma, and we formulate it as :

Corollary 4.5

Let C be a f-p-c class of groups.

If there is a counter-example for C, then there is a nulvalent diagram representing a counter-example for C.

Proof of 4.4

As this lemma is very important, we begin with a sketch description of the proof before proceeding to the technicalities which are required to establish the two main steps. In the construction of the second step we discover the reason behind the introduction of the potential functions.

Proof of 4.4 (cont.)

The failure of case (ii) for the class C implies the existence of certain diagrams; in the first part of the proof we show that a "minimal" such diagram in fact represents a counter-example for C (i.e. a failure of (ii) where A has just one element). In step 2 we show how to relabel this diagram using a polyvalent potential function: this relabelling produces a diagram representing a different counter-example for C, and the length of the new relation is less than that of the original. Repeating this relabelling process reduces the length of the new relator to a minimum, at which stage the diagram does not have a polyvalent potential function, and so the diagram is nulvalent as required.

Proof Proper :

Suppose that case (ii) fails; we shall shew that case (i) holds. There is set A of words $\{r_j\}$ in $*_i G_i$ staggered with respect to $\{H_k\}$ (with the additional conditions given in (ii)) and there is a diagram D representing a consequence w of A where w is not a consequence of a proper subset of A, and such that w omits occurrences from H_q, where $q=\max b_j$ or $q=\min a_j$, where j varies over all values such that r_j is in A. Without loss of generality we can suppose that $q=\max b_j$. (The condition that w is not a consequence of a proper subset of A implies that each word in A appears as a label on a disc in D.) We suppose in addition that $\{G_i\}$, $\{r_j\}$ and D are chosen such that d(D), the number of discs in D labelled in A, is minimal over all failures of case (ii) for the given class C.

Proof of 4.4 (cont.)

Claim 1 - Such a minimal D represents a counter-example for C.

If not, A contains more than one word and there is more than one label from A occurring on a disc in D : in this case we shall find another diagram D' contained inside D , such that D' also represents a failure of case (ii) for the class C , and $d(D')<d(D)$, contradicting the choice of D .

If k is the maximum index such that r_k is in A (recall that the staggering imposes a strict order on the elements of A) then $q=\max b_j$ implies that $q=b_k$; as w omits occurrences from H_q , no edge labelled in H_q meets ∂D.

Let M' be a closed submanifold in Int D such that :

(a) M' is the closure of a 2-ball with some 2-balls removed from its interior : M' is a "disc with holes" ;

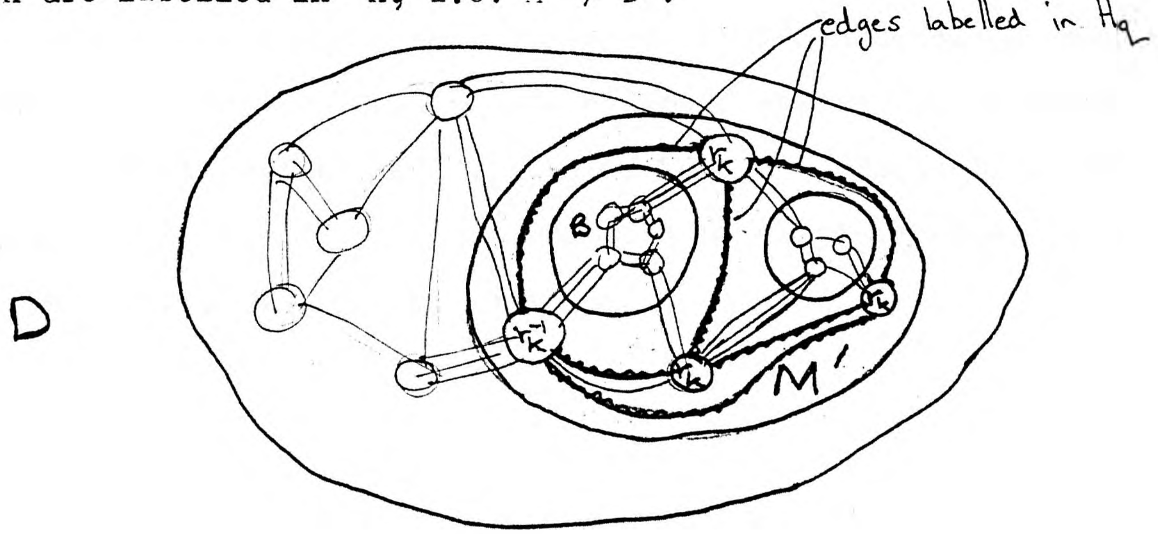
(b) Int M' contains a disc labelled $r_v^{\pm 1}$ iff $v=k$;

(c) $\partial M'$ meets no edge from H_g for $g<a_k$;

(d) $\partial M'$ meets no discs, and meets an edge in at most one point;

Condition (d) and the fact that $\partial M'$ does not meet ∂D (as M' is in Int D) give that an edge labelled in H_q does not meet $\partial M'$ (as such an edge does not meet $\partial M'$ again, and it does not meet either ∂D or a disc labelled in $A - \{r_k\}$).

If A contains more than one word, there are discs in D-M' which are labelled in A, i.e. $M' \neq D$.



Proof of 4.4 (cont.)

If M' has no holes, i.e. if M' is a 2-ball, then M' is a diagram representing $w'=1$ in $\dfrac{\ast_i G_i}{N(r_{j_k})}$ for $w' \in \ast_i G_i$; as we noted earlier, no edge labelled in H_q meets $\partial M'$; so $w' \in G' = \ast_{i \neq j} G_i$, where H_q is G_j; also $w' \neq 1$ in G' , else replace M' in D by a diagram omitting discs labelled in A, thus reducing d(D), which was chosen to be minimal. Regarding H_q as $\langle t \rangle$, M' represents a counter-example, and as C is f-p-c and $\{G_i\} \subset C$, we have that $\ast_{i \neq j} G_i \in C$, and M' represents a failure of case (ii). By supposition, $d(M') < d(D)$,contradicting the minimality of D. The case when M' has holes remains, i.e. when $\partial M'$ has more than one component. In this case we examine $cl(D - M') = M''$; let B be a component of M" which is a 2-ball; if B contains no discs labelled in A, then replace M' by $(M' \cup B)$; a quick check of conditions (a)-(d) on M' shows that this change of M' does not violate any of them, as no discs labelled in A are introduced, and the new boundary is a subset of the old. We are left with the case when B contains discs labelled in A; we wish to show that B is a diagram representing a failure of case (ii) of the lemma. Let p be the smallest index such that r_p is in A, and $r_p^{\pm 1}$ appears as a label on a disc in B. Then by condition (c) and the fact that $a_p < a_q$, no edge labelled in H_{a_p} meets $\partial M'$ and hence does not meet $\partial B \subset \partial M'$; hence B represents a failure of case (ii) as claimed, and as B contains none of the discs labelled $r_k^{\pm 1}$, $d(B) < d(D)$, contradicting the minimality of D. Hence D - M' contains no discs labelled in A, and the proof of Claim 1 is complete.

Proof of 4.4 (cont.)

We have shown that D, our minimal failure of (ii), represents a counter-example for C, say $\{G,r,w\}$ with G in C, and r a cyclically reduced word in $G*\langle t\rangle$ of length not less than 2. By ignoring all edges labelled in G, we can regard D as a special diagram representing a counter-example for C.

Step 2 - If D is polyvalent then D can be relabelled.

Let Φ be a polyvalent potential function on D; if no such potential function exists, then D is nulvalent as required. Relabel the edge e with the label $t_{\Phi(e)}$, and replace all the G-edges and G-discs which we removed to regard D as a special diagram. We now have a diagram D' where the edges are labelled in G or in $P_v = \langle t_v\rangle$, where v varies over the values taken by Φ on the edges of D. The discs of D (which carried the label r) are labelled r'_j in D', where these labels are obtained from replacing the t-occurrences in r by appropriate t_v-occurrences; there are as many different r'_j as there are different values taken by Φ on the distinguished points of D, and the conditions on potential functions ensure that $\{r_j\}$ is a staggered set with respect to $\{P_v\}$. Note that if b_j and a_j are the maximum and minimum values taken by Φ on a disc labelled r_j in D', then $b_j - a_j = b_i - a_i$ for any disc labelled r_i; if $b_j - a_j = 0$, then Φ was nulvalent.

By the above, we now have that D' represents a failure of case (ii) for the same class C of groups, as C contains all finitely generated free groups and their free products with elements of G. Also $d(D') = d(D)$, so that Claim 1 tells us that the set $A = \{r_j\}$ has just one element, that is Φ is univalent on D, and in D' there is just one label obtained from r, call it r'.

Proof of 4.4 (cont.)

Let a, b be the minimum and maximum values attained by $\bar{\Phi}$ on the edges at an r-disc. In D' replace the label on the edges labelled t_b by the label t ; D'' now represents a counter-example $\{G*(\ast_{v\neq b}P_v), r'', w\}$ and the fact that C is a f-p-c class ensures that $G*(\ast_{v\neq b}P_v)$ is in C . The new relator r'' is obtained from r' by replacing occurrences of t_b by t; therefore the cyclically reduced length of r'' , regarded as a word in $G''*\langle t\rangle$, $L(r'')$ is less than $L(r')$, and is greater than 2. D'' thus represents a counter-example for C .

If D'' is not nulvalent, we can repeat the above proceedure, and in a finite number of steps we arrive at a diagram which is nulvalent, giving the required example for case (i).

This completes the proof of 4.4.

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The principal form in which we shall use the Lyndon Lemma in the next section is a corollary of the method of proof :

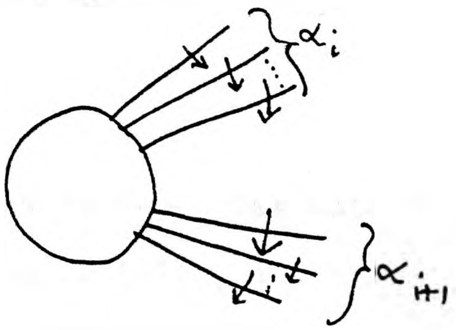
Corollary 4.6

Let $\{G,r,w\}$ be a minimal counter-example for a f-p-c class of groups C.
Then there is a nulvalent diagram for a counter-example $\{G,r,w'\}$ and if $r = a_1t^{\alpha_1}a_2t^{\alpha_2}\dots a_nt^{\alpha_n}$ with $a_i \in G-\{1\}$ and $\sigma_t(r) = 0$, then $\alpha_{i+1} = -\alpha_i$ and $|\alpha_i| = 1$.

Proof

Let D be a diagram representing $\{G,r,w\}$; if there is a polyvalent potential function on D, we can relabel the edges of D as in the second half of the proof of 4.4 to obtain another counter-example for C in which the new relation is shorter than r , or the new relation is essentially the same as r , and the diagram is not connected - the polyvalency being due to differently-valued constant potential function on each component. In the second case replace D by a connected component of D which represents a counter-example $\{G,r,w'\}$ (as in 2.21) . As we chose a minimal counter-example, relabelling as described is not possible, thus the first part of the theorem holds.

If $\sigma_t(r) = 0$, then we can use the method of 4.2(v) and (ii) to put a potential function on D ; if $|\alpha_i|>1$ or α_i has the same sign as α_{i+1} , there is a pair of adjacent edges on a disc with spokes in D with the same orientation. Then the construction of 4.2(ii) gives a potential function with different values on these two edges ; thus the theorem holds.



Section 2 The Freiheitssatz for Locally Indicable Groups

The main application of our potential functions is, as earlier advertised, to prove the Freiheitssatz for the class of locally indicable groups. The proof is in two steps : first we show (4.8) that any new relator r is soluble for a locally indicable group G, and then we use this result to obtain (4.9) almost immediately the required version of the Freiheitssatz.

Definition 4.7

A group is locally indicable if any finitely generated subgroup has $\mathbb{Z}$ as a homomorphic image.

Note that a subgroup of a free product is a free product of (conjugates of) subgroups together with a free group (by the Kuros subgroup theorem), and that each factor is finitely generated if the whole is finitely generated (by Grushko's theorem). This means that:(Defn.4.3(iii))

the class of all locally indicable groups is f-p-c.

This means that we can use the results of the previous section, in particular 4.4, 4.5 and 4.6 .

Proposition 4.8

Let G be a locally indicable group, and let r be a cyclically reduced word in $G * \langle t \rangle$ with t-occurences.

Then G injects into $\langle G, t; r \rangle$.

Proof

The class C of locally indicable groups is f-p-c (as noted above), so let us suppose that the proposition is false, and that $\{G, r, w\}$ is a minimal counter-example for C (Defn.4.3(iv)), represented by a nulvalent diagram D, given by Corollary 4.6.

Proof of 4.8 (cont.)

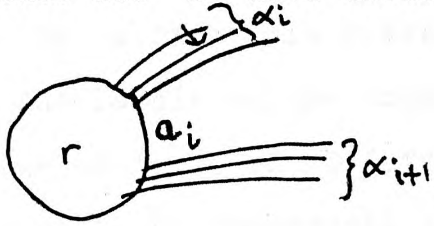
The proof now splits into two cases, depending on whether the exponent sum of t in r, $\sigma_t(r)$, is or is not equal to zero. Let r be the word $a_1t^{\alpha_1}a_2t^{\alpha_2}\dots a_nt^{\alpha_n}$ with $a_i \in G - \{1\}$ $\alpha_i \in Z - \{0\}$.

In case 1a) and case 2 we show how to construct a polyvalent potential function on the diagram D, contradicting the nulvalency. In case 1b) we use the nulvalency to obtain another counter-example for C which contradicts the minimality of $\{G, r, w\}$.

Case 1 $\sigma_t(r) \neq 0$.

Subcase a) Suppose that there is a homomorphism q taking $\mathrm{Sbgp}_G\langle a_1, a_2, \dots, a_n\rangle$ onto $\mathbb{Z}$ such that $\sum_i q(a_i)$ is equal to zero.

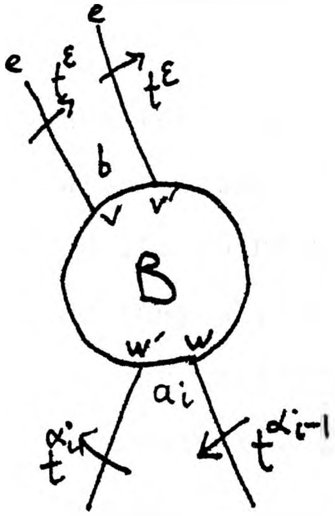
Here we can put a potential function on D as in 4.2(iv) using $q * 0 : \mathrm{Sbgp}_G\langle a_1, a_2, \dots, a_n\rangle * \langle t\rangle \twoheadrightarrow \mathbb{Z}$. For some i, $q(a_i) \neq 0$ as q is onto, and hence the two edges adjacent to the segment labelled a_i on a disc with spokes for D have different potentials.



Subcase b) Suppose that no such q as above exists. Then take some $q : \mathrm{Sbgp}_G\langle a_1, a_2, \dots, a_n\rangle$ goes onto $\mathbb{Z}$, such that $\sum_i q(a_i) = z \neq 0$; suppose $\sigma_t(r) = -s \neq 0$. Using the notation of 4.2(v) and 4.2(iii), we use $\theta = s.q * z.\mathrm{id} : \mathrm{Sbgp}_G\langle a_1, a_2, \dots, a_n\rangle * \langle t\rangle \twoheadrightarrow \mathbb{Z}$; so that $\theta(r) = 0$, and $q' = q \circ \phi$; with $\theta' = s.q' * z.\mathrm{id}$ we put a potential function on D as in 4.2(iii).

Proof of 4.8 (cont.)

subcase 1b) (cont.) Let B be a disc with spokes used to construct D; if for some i we have that $|\alpha_i| \neq 1$, then on D there are two adjacent edges e, e' with the same t-orientation. These edges are seperated by a segment of ∂B labelled b such that $\phi(b) = 1$; that is, b is trivial in G. Therefore $q'(b) = q(\phi(b)) = 0$, and $h(v') = h(v) + q'(b) + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = h(v) + \varepsilon$ where $\varepsilon = \pm 1$ is the orientation of the edges e and e'.



But D is nulvalent, so by the above $|\alpha_i| = 1$ for all i. Also when $\alpha_{i-1} = \alpha_i$, we must have that $q(a_i) = -\alpha_i$ (as $h(w') = h(w) + q(a_i) + \alpha_i$:see picture) and similarly $q(a_i) = 0$ if and only if $\alpha_{i-1} = -\alpha_i$.

As the exponent sum of t in r is non-zero, there is a subword ta_jt (or $t^{-1}a_jt^{-1}$) in r; by the above, a_j is non-trivial in G. We now relabel the t-edges of D by $a_j^{-1}t$; this gives a new diagram D' in which the labels on the boundaries of the regions are the same as those in D (after trivial cancellations), so that D' represents the counter-example $\{G, r', w\}$ where r' is obtained from r by replacing t (resp. t^{-1}) by $a_j^{-1}t$ (resp. $t^{-1}a_j$); in particular the subword ta_jt becomes $a_j^{-1}t^2$ so that r' is strictly shorter than r, contradicting the assumed minimality of $\{G, r, w\}$ (see defn 4.3(iv)).

Proof of 4.8 (cont.)

Case 2 $\sigma_t(r) = 0$.

As $\{G, r, w\}$ is a minimal counter-example for C and as $\sigma_t(r) = 0$, we can apply Corollary 4.6 to give that r has the form $a_1ta_2t^{-1}a_3ta_4 \ldots ta_{2m}t^{-1}$. As r has this alternating form, for each region of D, all the t-edges on its boundary have their transverse orientation inwards or they all have their orientation pointing outwards. We can therefore regard D as a special diagram for the counter-example $\{G * G', r', w'\}$, where G' is an isomorphic copy of G, and we regard a_{2i} as being in G' for all i ; because of the restriction on the orientations of the t-edges on the boundary of a region, the relations corresponding to the regions of D (that is the labels on the boundaries of regions) are words in G or in G'. As C is a f-p-c class, $G * G'$ is an element of C; also the length of r is equal to the length of r', so that $\{G * G', r', w'\}$ is also a minimal counter-example for C . But as G is a locally indicable group, there are homomorphisms q_1 , q_2 such that $\mathrm{sbgp}_G\langle a_1, a_3, \ldots, a_{2m-1}\rangle$ and $\mathrm{sbgp}_{G'}\langle a_2, a_4, \ldots, a_{2m}\rangle$ go onto $\mathbb{Z}$.

If $\sum_{i=0}^{m-1} q_1(a_{2i+1}) = 0$ then use the homomorphism $\theta = q_1 * 0 * 0$ from $G * G' * \langle t\rangle$ onto $\mathbb{Z}$ to define a potential function on D, as in 4.2(iv) : this function is polyvalent. The same construction can be used if $\sum_{i=1}^{m} q_2(a_{2i}) = 0$. If neither of these exist, then rescale each of q_1 and q_2 and use $\theta = q_1 * q_2 * 0 : G * G' * \langle t\rangle \twoheadrightarrow \mathbb{Z}$ (scaled so that $\theta(r') = 0$) in the construction of 4.2(iv). In all cases the potential function resulting is polyvalent as for some j , $\theta(a_j) \neq 0$; this contradicts the nulvalency of D.

Proof of 4.8 (cont.)

We have therefore shown that there does not exist a nulvalent diagram representing a minimal counter-example for the class of locally indicable groups, and therefore there is no counter-example for this class, and the proposition holds.

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Theorem 4.9 The Freiheitssatz for Locally Indicable Groups

Let G_1 and G_2 be locally indicable groups, and r a cyclically reduced word in $G_1 * G_2$ of length ≥ 2. Then G_i injects into $\dfrac{G_1 * G_2}{N\langle r \rangle}$ for $i = 1,2$.

Proof

Suppose that the theorem fails and G_1 does not inject naturally into $\dfrac{G_1 * G_2}{N\langle r \rangle}$ where $r = a_1 b_1 a_2 b_2 \ldots a_n b_n$, $a_i \in G_1 - \{1\}$, $b_i \in G_2 - \{1\}$.

Replacing each occurrence of b_i by $t b_i t^{-1}$, we have that G_1 does not inject into $\dfrac{G_1 * G_2 * \langle t \rangle}{N\langle r' \rangle}$, where $r' = a_1 t b_1 t^{-1} \ldots a_n t b_n t^{-1}$.

But $G'' = G_1 * G_2$ is locally indicable and r' has length greater than 2 (that is free-product length in $G'' * \langle t \rangle$) , so that this contradicts proposition 4.8 .

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Section 3 Exponent-Sum Zero Results

In 4.2(v), we saw how to put a potential function on a diagram representing a counter-example when we are given a homomorphism from $\langle G,t;r\rangle$ onto the integers; the case when $\sigma_t(r) = 0$ has a particular such homomorphism associated, as in 4.2(ii); in this case, the potential function depends only on the distribution of t-occurrences in r.

Using these potential functions and the relabelling techniques of section 1, we shall show (4.11) that the existence of a counter-example $\{G,r,w\}$ depends upon the existence of another counter-example $\{G*\langle s\rangle, r^{\circ}, w'\}$ where r° is one of two words obtained from r (defn 4.10) which are usually "simpler" than r, in the sense of having fewer t-occurrences.

We then use this result to obtain some results concerning torsion and added relators of a certain form (4.13).

Definition 4.10

Let r be a cyclically reduced word in $G*\langle t\rangle$, and let θ be a homomorphism from $\langle G,t;r\rangle$ to $\mathbb{Z}$.

Writing $r = c_1t^{\varepsilon_1}c_2t^{\varepsilon_2}\ldots c_mt^{\varepsilon_m}$ where $c_i \in G$ (possibly $c_i = 1$ in G) and $\varepsilon_i = \pm 1$, define :

$$h_j = \left(\sum_{i=1}^{j} (\theta(c_i) + \varepsilon_i)\right) - \frac{1}{2}\varepsilon_j \quad , \quad \text{for each } j \ , \ m \geq j \geq 1 .$$

Define r_{max} (respectively r_{min}) as the elements of the group $G*\langle s\rangle*\langle t\rangle$ obtained from r by :

replace each occurrence of t^{ε_i} by s^{ε_i}, except where h_j achieves its maximum (resp. minimum).

N.B. For a given word r in $G*\langle t\rangle$, it is to be noted that r_{max} and r_{min} depend upon the particular homomorphism chosen.

Example : Let $r = a_1ta_2ta_3t^{-1}a_4t^{-1}$ with $a_i \in G - \{1\}$.

Using the usual homomorphism $0 * id : G*\langle t\rangle \to \mathbb{Z}$,

we have that $h_1 = \frac{1}{2}$, $h_2 = \frac{3}{2}$, $h_3 = \frac{3}{2}$, $h_4 = \frac{1}{2}$.

Thus $r_{max} = a_1sa_2ta_3t^{-1}a_4s^{-1}$

$r_{min} = a_1ta_2sa_3s^{-1}a_4t^{-1}$

We now use this definition to give the general version of the principal theorem of this section; in practice we shall use only the special case when the exponent-sum of t in r is zero, when a homomorphism as required in 4.10 exists naturally, as we remarked earlier.

Theorem 4.11

Suppose that G does not naturally inject into $\langle G,t;r\rangle$ and let θ be a homomorphism from $\langle G,t;r\rangle$ to $\mathbb{Z}$. Then $G*\langle s\rangle$ does not naturally inject into $\langle G*\langle s\rangle,t;r^\circ\rangle$ where r° is one of r_{min}, r_{max} .

Proof

Let D be a special diagram representing the counter-example $\{G,r,w\}$. If θ is trivial, then $r = r_{min} = r_{max}$, and the theorem holds trivially.

We can therefore suppose that θ is onto. As in 4.2(v) we use θ to define a potential function Φ on D, and as in the proof of 4.4 we relabel the t-edges using Φ to obtain the diagram D'.

Suppose that Φ was not univalent on D.

Let $v_m, v_{m+1}, .., v_M$ be the values attained by Φ on the t-edges of D, with $v_m < v_{m+1} < .... < v_M$. Where e is an edge of D, and $\Phi(e) = v_j$, we relabel e by t_{v_j} .

Proof of 4.11 (cont.)

In D' the discs which were labelled r, are now labelled in $A = \{r_i'\}$, where each r_i' is obtained from r by replacing the t-labels by the relevant t_{v_j}-labels; from the definition of potential functions, we see that the elements of A can be ordered by the lowest (or respectively the highest) index a_i (resp. b_i) such that t_{a_i} (resp. t_{b_i}) occurs in r_i'. Thus r_1' contains $t_{a_1} = t_m$. We now have that A is a set of words staggered with respect to $\{F(t_{v_i})\}$, and D' represents $w \in G - \{1\}$ such that $w = 1$ in $\dfrac{G * (\overset{M}{\underset{i=m}{*}} F(t_{v_i}))}{N\langle A \rangle}$.

As in 4.4, let M' be a "two-ball with holes" contained in the interior of D' containing in its interior all those discs which are labelled r_1' and no other r_i' discs, such that each edge of D' meets $\partial M'$ in at most one point etc.. Then no edge labelled t_m meets $\partial M'$; if M' is a two-ball ("has no holes") then we may take this to be a diagram D"; else, as in 4.4, take a two-ball region in D'-M' and apply the same process, this time taking M" containing all the discs which are labelled r_j' where j is the highest index on the discs in the two-ball region which are labelled in A (and none of the others etc.). Continuing in this way, we shall eventually find a subdiagram D" containing just one label from A, r_x' say, such that no edge labelled t_y meets $\partial D''$, where $y = a_x$ or b_x (the minimum and maximum values of the subscripts on the r_x' discs in D"); which of the two is relevant depends upon whether we find the diagram D" after an odd or an even number of steps. D" represents a genuine counter-example, and not just a triple, as otherwise we could replace this region of D' by a region without r_j' discs, and hence in D we could replace the corresponding region by one without r-discs.

Proof of 4.11 (cont.)

It remains to replace all the t_{v_i}-labels by s except for the t_y-labels, which we relabel t; this converts D" into a diagram representing a counter-example of the type claimed.

■

Recall that (definition 1.5) when $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} \ldots a_n t^{\alpha_n}$ with $a_i \in G - \{1\}$, $\alpha_i \in \mathbb{Z} - \{0\}$, the t-shape of r , denoted $[r]$, is the ordered n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

We say that a t-shape $[r]$ is soluble in a class of groups C, if G injects into $\langle G, t; u \rangle$, for all groups G in C , and for all words u such that $[u] = [r]$.

Corollary 4.12

Let C be a f-p-c class of groups, and $[r]$ a t-shape such that $\sigma_t(r) = 0$.

Let $r_{\min}$ and $r_{\max}$ be obtained using $0 * \mathrm{id} : G * \langle t \rangle \to \mathbb{Z}$.

If $[r_{\min}]$ and $[r_{\max}]$ are soluble in C , then $[r]$ is soluble in C .

Proof

Suppose that the corollary is not true, and let $\{G, r, w\}$ be a counter-example with $G \in C$ and $\sigma_t(r) = 0$, let D be a special diagram representing $\{G, r, w\}$. Then applying 4.11, there exists a counter-example $\{G * \langle s \rangle, r^{\circ}, w'\}$, where r° is one of $r_{\min}$, $r_{\max}$; this is not possible by assumption.

■

Let $r \in G * \langle t \rangle$, $r = a_1 t^{\alpha_1} a_2 t^{\alpha_2} \dots a_n t^{\alpha_n}$ with $a_i \in G - \{1\}$, $\alpha_i \in \mathbb{Z} - \{0\}$ such that $\sigma_t(r) = 0$. Suppose that there exists a homomorphism from $\langle G, t; r \rangle$ to $\mathbb{Z}$ such that $r_{\min}$ and $r_{\max}$ contain just two t-occurrences each :

$$r_{\min} = a_1 s^{\alpha_1} a_2 s^{\alpha_2} \dots . a_{j-1} s^{\beta_1} t a_j t^{-1} s^{\beta_2} a_{j+1} \dots . a_n s^{\alpha_n}$$

$$r_{\max} = a_1 s^{\alpha_1} a_2 s^{\alpha_2} \dots .. a_{k-1} s^{\beta_3} t^{-1} a_k t s^{\beta_4} a_{k+1} \dots a_n s^{\alpha_n}$$

say, where $\beta_1 = \alpha_{j-1} - 1$, $\beta_2 = \alpha_j + 1$, $\beta_3 = \alpha_{k-1} + 1$, $\beta_4 = \alpha_k - 1$.

Theorem 4.13

Suppose that G does not inject naturally into $\langle G, t; r \rangle$, with r as described above.

Then i) one of a_j , a_k has finite order in G ,

and ii) if r has a cyclically conjugate form $a_j v a_k v^{-1}$ then a_j and a_k have different orders in G.

Proof

The proof is an application of 4.11 : suppose that we have a counter-example $\{G, r, w\}$, with r as described.

The theorem 4.11 gives a counter-example $\{G * \langle s \rangle, r^\circ, w'\}$ where r° is one of $r_{\min}$, $r_{\max}$; without loss of generality, we may suppose that $r^\circ = r_{\min}$.

Let $b = s^{\beta_2} a_{j+1} s^{\alpha_{j+1}} \dots a_n s^{\alpha_n} a_1 s^{\alpha_1} \dots a_{j-1} s^{\beta_1}$; then r° is cyclically conjugate to $b t a_j t^{-1} = r'$, and $G * \langle s \rangle$ injects into $\langle G * \langle s \rangle, t; r' \rangle$ if and only if a_j and b have the same order in $G * \langle s \rangle$, as we have an HNN extension (see 2.25).

But b has finite order in $G * \langle s \rangle$ if and only if b has the form $b_1 a b_1^{-1}$ and a has finite order in G (see for instance [MKS] 4.1.4) ; as $r_{\max}$ contains just two t-occurrences, b must have the form $u a_k u^{-1}$.

Proof of 4.13 (cont.)

In order that $\{G * \langle s\rangle, r°, w'\}$ be a counter-example, we must therefore have that either a_j or b has finite order in $G * \langle s\rangle$, that is, a_j has finite order in G, or b has the form ua_ku^{-1} and a_k has finite order in G.

In either case we see that i) holds, and in the second case we see by returning the s-occurrences in u to t-labels that r is cyclically conjugate to $u'a_ku'^{-1}ta_jt^{-1}$, and so ii) holds.

An immediate corollary of this is :

Corollary 4.14

Let G be a torsion-free group, and r as in 4.13. Then G injects into $\langle G,t;r\rangle$.

Note that the conditions of 4.13 depend to a large extent on the t-shape of the word r, as the condition that $\sigma_t(r) = 0$ assures the existence of the homomorphism $0 * \mathrm{id} : G * \langle t\rangle \to \mathbb{Z}$ in which case the t-shapes $[r_{\min}]$ and $[r_{\max}]$ depend solely upon the t-shape $[r]$; of course other homomorphisms may exist in particular cases.

Corollary 4.14 is a somewhat weakened form of a result of Schiek [Sch 1] :

Schiek's Theorem

Let $r = t^{s_1}a_1t^{-s_1}t^{s_2}a_2t^{-s_2}\ldots\ldots t^{s_n}a_nt^{-s_n}$, where $a_i \in G-\{1\}$ and $s_i = s_j$ if and only if $i = j$.

Then G injects into $\langle G,t;r\rangle$ if the order of a_i in G is the same as that of $a_{i+1}\ldots a_na_1\ldots a_{i-1}$ in G for all i.

It was originally claimed that we could improve on this result of Schiek, but this is not possible as the letter s may appear in the word w' given by the minimal diagram representing the counter-example $\{G * \langle s \rangle, r_{min}, w'\}$ say.

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