

# RELATIVE CAPITAL ACCUMULATION WITH HETEROGENEOUS INDIVIDUALS\*

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The purpose of this paper is to show how differences in individuals' labour productivities cause differences in their accumulation of capital, and thereby analysing the evolution of the income distribution. There are three cases of interest: (i) the high productive accumulate relatively more capital [growing inequality], (ii) no individual accumulates relatively more capital [neutrality], (iii) the low productive accumulate relatively more [diminishing inequality]. Which of these cases is generated depends on the price dynamics (the growth rate of wages and the level of the interest rate relative to the rate of time preference), together with the preferences for consumption. The exact conditions for the price dynamics to generate (i), (ii) and (iii) are derived. Furthermore, since the price dynamics is endogenous in general equilibrium, we find the conditions for preferences and technology that determine relative capital accumulation. We find (in general equilibrium) that growing economies typically cause the high productive to accumulate more capital than the low productive if preferences are Decreasing Absolute Risk Aversion, and shrinking economies cause the less productive to accumulate more (i.e. decumulate less). The relations are reversed for Constant and Increasing Relative Risk Aversion. The final part of the paper analyses the effects of capital taxation on the income distribution.

*Keywords:* Consumer heterogeneity, income distribution dynamics, relative capital accumulation, taxation.

*JEL classification codes:* E25, D31, D33, D91, E62, H31.

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# 1. INTRODUCTION

The aim of this paper is to analyse individuals' capital accumulation and thereby the evolution of the income distribution when individuals differ in labour productivity, i.e. in income earning abilities. An individual with higher labour productivity would, *ceteris paribus*, have a different consumption profile and thereby accumulate different amount of savings, from an individual with lower productivity. However, it is not possible to state a priori *which* individuals will accumulate more capital relative to others. On one hand a high productive individual will earn more life-time income and therefore would like to smooth this income evenly over her life-time. However, since the income-earning differences persist it is not clear that the individual would save more. In fact, as we will show in this paper, if market wages are sufficiently increasing over the time-horizon, a high-productive individual would earn even more in the future and therefore would save less. We will show that it is the *price dynamics* (changes in wages and the level of interest rates) that will determine which individuals will accumulate relatively more capital.

We may think of three cases regarding relative capital accumulation: (i) the high skilled accumulate relatively more capital (ii) no individual accumulates relatively more capital (iii) the low skilled accumulate relatively more capital. Case (i) could be labelled *growing inequality*. As the economy evolves, presumably converging to a steady state, the high skilled would not only earn more income from their labour supply but also more income from their capital endowments. Thus as the economy moves to a steady state the inequality in total income increases. Case (ii) could be labelled *neutrality*. Inequality in labour income persists, but no inequality in capital income is generated. Case (iii) produces *diminishing inequality*. The low skilled would earn less from labour but more from their capital endowments. Such an economy reduces the inherited inequality.

In this paper we will derive the exact conditions for (i), (ii), or (iii) to hold. The conditions turn out to be restrictions on the price dynamics. Since the price dynamics is endogenous in general equilibrium we will go further and derive the results in terms of preferences and technologies only.

We shall also analyse the effects of taxes on the income distribution. It is often argued that a capital income tax can reduce the inequality in income.<sup>1</sup> However, we will show that

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<sup>1</sup> E.g. Atkinson and Stiglitz (1980), p. 266.

in general equilibrium this need not be the case. If the production technology has an elasticity of substitution greater than unity, a capital income tax increases inequality in a neighbourhood of the steady state. If the elasticity is less than unity the tax decreases inequality, and with unity elasticity (Cobb-Douglas) the tax has no effect on the distribution.

The only previous study on capital accumulation and heterogeneity was conducted by Chatterjee (1994).<sup>2</sup> In Chatterjee's framework individuals are *identical* except with respect to their *initial levels* of capital. Chatterjee focuses on how the initial distribution of capital determine the distribution in the future. However, we would expect that the distribution of capital is endogenous, depending on the underlying fundamental differences among individuals. No study has analysed the income distribution when individuals differ in some fundamental aspects, like labour productivity.

The optimal dynamic taxation literature only focuses on the tax structure, not on the resulting distribution among the individuals.<sup>3</sup> No study has successfully analysed the effects of taxes on the distribution. Stiglitz (1978) and Atkinson and Stiglitz (1980) *assume* a consumption function (i.e. it is not derived from optimising behaviour) in their analysis on the income distribution.

This paper is structured as follows. We begin (in Section 2) with a two-period framework in order to illustrate, in a simple way, some results that generalise to a multi-period framework. Section 3 introduces a continuous-time infinite-horizon economy. We derive the *exact* price dynamics that is required to produce *growing inequality*, *neutrality*, and *diminishing inequality*. Section 4 endogenises the prices, and thereby we are able to give conditions on technology and preferences for the different cases of the evolution of inequality. Sections 1-4 treat an economy with fixed (exogenous) labour supply. Section 5 introduces endogenous labour supply and derive the *exact* price-dynamics conditions, which only slightly differ from the ones in Section 3. Next the price dynamics for endogenous labour supply is solved for. Section 6 introduces fiscal policy and analyses the effects on the distribution of imposition of distortionary taxes. The findings of the paper are summarised in Section 7.

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<sup>2</sup> Stiglitz (1969) conducted a study with an assumed savings function (thus no individual optimisation).

<sup>3</sup> The "classical" result is that a zero capital-income tax is optimal in steady state (and even out-of steady state for iso-elastic utilities): Judd (1985) and Chamley (1986).

## 2. A TWO-PERIOD CHARACTERISATION

Individuals receive utility from consumption only. Labour supply is taken to be exogenously given and constant. In the first period they start with equal amounts of initial capital (possibly zero), receive a wage income, and make their first period consumption and savings decisions. In the second period they receive the period two wage income and the return from their savings. Everything is consumed in the second period. Individuals solve

$$\max_{c_1, c_2} u(c_1) + \frac{1}{1+\theta} u(c_2) \quad (2.1)$$

*s.t.*

$$\left. \begin{aligned} k_0 + w_1 &= c_1 + k_1 \\ (1+r)k_1 + w_2 &= c_2 \end{aligned} \right\} \quad (2.2)$$

The first order condition

$$u'(c_1) - \frac{1+r}{1+\theta} u'(c_2) = 0 \quad (2.3)$$

together with the budget equations (2.2) describe  $c_1$  and  $c_2$  (and thereby  $k_1$ ) uniquely in terms of the *prices*  $w_1, w_2, r$ , i.e.  $c_1=c^1(w_1, w_2, r)$  and  $c_2=c^2(w_1, w_2, r)$ . Individuals differ in productivity and are assumed to be perfect substitutes in production. Then the wage difference will become linear. Let the wage difference be represented by a productivity parameter  $\gamma^i$  such that

$$w_t^i = \gamma^i w_t, \quad \forall i, t.$$

Individuals have the same utility function  $u(\cdot)$  and the same discount rate  $\theta$ .

### ***2.1 General utility specification***

In order to answer the question about relative capital accumulation the relation between  $\gamma$  and savings must be established, i.e. the derivative  $dk_1/d\gamma$  must be sought. The first order condition together with the budget equations form an implicit function in  $\gamma$  and  $k_1$

$$u'(k_0 - k_1 + \gamma w_1) - \frac{1}{1+\theta} (1+r) u'((1+r)k_1 + \gamma w_2) = 0 \quad (2.3')$$

Differentiating through this implicit function with respect to  $\gamma$  and  $k_1$  gives the sought derivative

$$\frac{dk_1}{d\gamma} = \frac{u''(c_1)w_1 - \frac{1+r}{1+\theta}w_2u''(c_2)}{u''(c_1) + \frac{(1+r)^2}{1+\theta}u''(c_2)} \quad (2.4)$$

or by dividing through by the first order condition (2.3)

$$\text{sign}\left(\frac{dk_1}{d\gamma}\right) = \text{sign}\left(A(c_1) - A(c_2)\frac{w_2}{w_1}\right) \quad (2.5)$$

where  $A(c) \equiv -\frac{u''(c)}{u'(c)}$  is the Arrow-Pratt measure Absolute Risk Aversion.<sup>4</sup>

We will refer to Decreasing Absolute Risk Aversion (DARA) when  $A'(c)<0$ , Constant Absolute Risk Aversion (CARA) when  $A'(c)=0$ , and Increasing Absolute Risk Aversion (IARA) when  $A'(c)>0$ . Here we should think of  $A(c)$  as a measure of *intertemporal substitution*, i.e. how willing an individual is to defer consumption to the future. A wealthier individual is willing to defer consumption more if  $A'(c)<0$ , and less if  $A'(c)>0$ .<sup>5</sup>

Whether consumption is increasing or decreasing depends on the interest rate relative to the discount rate. By (2.3) we have

$$\text{sign}(c_2 - c_1) = \text{sign}(r - \theta). \quad (2.6)$$

Clearly the *price dynamics* together with the curvature of the utility function (as represented by  $-u''/u'$ ) will determine the sign of  $dk_1/d\gamma$ . We shall look at a particular example in order to illustrate this.

### Example 2.1

Suppose  $w_1 = w_2$  and that  $A(c)$  is decreasing in  $c$  (i.e. Decreasing Absolute Risk Aversion DARA). Since  $A'(c)<0$  we have

$$\text{sign}(A(c_1) - A(c_2)) = \text{sign}(r - \theta)$$

Then  $w_1 = w_2$  implies  $\text{sign}\left(\frac{dk_1}{d\gamma}\right) = \text{sign}(r - \theta)$ .

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<sup>4</sup> It seems not appropriate to talk in terms of risk aversion in an economy without risk. However, it turns out to be the Arrow-Pratt measure of the *curvature* of the utility function that determines the relative capital accumulation. We should think of this as a measure of the rate of *intertemporal substitution*.

<sup>5</sup> This follows from the Euler equation (2.3). If we write  $\Delta = c_2 - c_1$ , then  $u'(c_1) - (1+r)(1+\theta)^{-1}u'(c_1+\Delta) = 0$ , and we have  $\text{sign}(\partial\Delta/\partial c_1) = \text{sign}(A(c_1) - A(c_2))$ . If  $c_2 - c_1 > 0$  then  $A'(c) < 0$  implies  $A(c_1) > A(c_2)$  and thereby  $\partial\Delta/\partial c_1 > 0$ . A wealthier individual will have higher initial consumption  $c_1$ .

In this example if the interest rate is greater than the discount rate the *high skilled* would accumulate *relatively more capital* than the low skilled. On the other hand if the interest rate is lower it would be the low skilled that would accumulate relatively more capital.

Using (2.5) and (2.6) we may make a systematic account of the relationship between the prices in the economy and the relative capital accumulation. We should then distinguish between  $A'(c)=0$ ,  $A'(c)<0$ , and  $A'(c)>0$ . The example above shows the procedure and the results are reported in Table 2.1.

**Table 2.1** Relative Capital Accumulation - Two-Period Economy

DARA/CARA/IARA	$r > \theta$	$r = \theta$	$r < \theta$
$w_2 > w_1$	?/-/-	-/-/-	-/-/?
$w_2 = w_1$	+/0/-	0/0/0	-/0/+
$w_2 < w_1$	+/?/+	+/?/+	?/?/+

The table entry signs (?,-,+,0) refers to the sign of the derivative  $dk_1/dy$ .

The table is read in the following way . If  $w_2 > w_1$  and  $r > \theta$  the table entry is ?/-/-, this implies that relative capital accumulation is ambiguous in the DARA-case and that the high skilled would accumulate less capital in the CARA- and IARA-cases.

From the table we have several results. If wages are increasing and the interest rate is greater than the rate of time preference, then if utility is IARA the low skilled accumulate relatively more capital than the high skilled. If wages are decreasing and the interest rate is smaller than the rate of time preference, the high skilled accumulate relatively more capital. For CARA preferences, only the wage dynamics determines relative capital accumulation. If wages are increasing then the low skilled accumulate relatively more capital, if wages are decreasing the high skilled accumulate relatively more. The question marks in the table denote ambiguity. This is the case if the utility is DARA and if either wages are increasing and the interest rate is greater than the rate of time preference, or if wages are decreasing and the interest rate is smaller than the rate of time preference.

This ambiguity is due to the fact that there are two effects at work (sometimes working in opposite directions). One related to the *rate of change in wages*, and another related to the willingness of individuals to defer consumption, i.e. *intertemporal substitution*, as measured by the Absolute Risk Aversion.

(i) If wages are increasing over time the high skilled will earn proportionately more wage income in the future and therefore they need to *save less* in physical capital. This may be seen in equation (2.5), the greater  $w_2/w_1$  the smaller  $dk_1/d\gamma$ .

(ii) If the interest rate is greater than the rate of time preference, individuals choose an increasing consumption path, i.e.  $c_2 > c_1$  by the Euler equation (2.3). The question is how the increase in consumption,  $c_2 - c_1$ , differs across individuals. If individuals have DARA utility functions, then individuals with higher initial consumption,  $c_1$ , would choose a greater increase in future consumption, and thus *saving more*. The individuals who can attain a higher level of initial consumption are those with greater life-time wealth, i.e. those with higher skills. Clearly for DARA preferences the intertemporal substitution effect works against the effect of increasing wages. For IARA preferences the relation is reversed and (i) and (ii) work in the same direction. For CARA preferences the intertemporal substitution effect is zero ( $c_2 - c_1$  is independent of  $c_1$ ) and therefore only the effect of changes in the wage is present.<sup>6</sup>

In the most plausible cases, where utility belongs to DARA and either ( $w_2 > w_1$ ,  $r > \theta$ ) or ( $w_2 < w_1$ ,  $r < \theta$ ),<sup>7</sup> these two effects work in the opposite direction and therefore the relative capital accumulation is *ambiguous*.<sup>8</sup> To get any further more explicit assumptions have to be made about the utility function (to discern which effect that dominates). Ambiguity is also due to the fact that no restrictions on the prices have been made, i.e. the economy studied can be thought of as an open economy with exogenous prices. By closing the economy and introducing a production specification the ambiguity problem can partially be solved. In such

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<sup>6</sup> For CARA-preferences, because the intertemporal substitution effect is absent, only the *wage dynamics* determines the relative capital accumulation and the interest rate becomes irrelevant (see Table 2.1).

<sup>7</sup> In general equilibrium for plausible production functions (where  $f_{kl} > 0$ ) wages are increasing (decreasing) when the interest rate is greater (smaller) than the rate of time preference. DARA utility functions are more plausible empirically.

<sup>8</sup> When the economy is in "steady state", i.e. the wage is constant and the interest rate equals the discount rate, then the skill level does not affect individuals' capital accumulation.

a *general equilibrium* context not all prices are consistent with the utility specification. By confining the analysis to equilibrium prices more can be said about relative capital accumulation. This is done in Section 4.

### 3. THE INFINITE-HORIZON ECONOMY

#### 3.1 Assumptions

##### A1 Individuals' preferences

Each individual chooses a consumption path,  $c(t)$ , for  $t \in [0, \infty)$  so as to maximise her life-time utility

$$J(k_0) = \max_{c(t)} \int_0^{\infty} e^{-\theta t} u(c(t)) dt \quad (3.1)$$

where  $u(\cdot)$  is strictly concave.

##### A2 Individuals' constraints

Individual  $i$  owns capital  $k^i(t)$  and earns interest at a rate  $r(t)$ . Individuals differ linearly in productivity, but supply (exogenously) the same hours of work, normalised to unity. The productivity parameter  $\gamma^i$  is distributed according to  $F(\gamma^i)$ , which is continuous. For each *efficient* unit of supplied labour, i.e. in proportion to  $\gamma^i$ , each individual earns the wage rate  $w(t)$ . An individual's budget constraint is therefore

$$\dot{k}^i(t) = r(t)k^i(t) + w(t)\gamma^i l - c^i(t) \quad (3.2)$$

Individuals are assumed to be endowed with the same amount of capital at date zero, i.e.  $k_0^i = k_0, \forall i$ .

##### A3 Prices

The price paths  $r(t)$ ,  $w(t)$ , for  $t \in [0, \infty)$  are exogenous.



We will relax some of these assumptions later on. In Section 4 we will introduce a production technology in order to endogenise the prices, in Section 5 we will introduce endogenous labour supply, and in Section 6 we will introduce fiscal policy.

### 3.2 Economic equilibrium

Since individuals differ only in their labour productivity we shall suppress the superscripts  $i$ , and write the individual specific quantities in terms of  $\gamma$ . Individuals solve (3.1) subject to (3.2). The Euler equation for this problem<sup>9</sup>

$$\dot{c}(t, \gamma) = \frac{r(t) - \theta}{A(c(t, \gamma))} \quad (3.3)$$

together with the budget equation (3.2) and the transversality condition

$$\lim_{t \rightarrow \infty} k(t, \gamma) e^{-\int_0^t r(s) ds} = 0 \quad (3.4)$$

uniquely describe the optimal paths  $c(t, \gamma)$ ,  $k(t, \gamma)$ , for  $t \in [0, \infty)$  given any price paths  $r(t)$ ,  $w(t)$ , for  $t \in [0, \infty)$ . Integrate the budget equation from some  $T \in [0, t)$  to  $t$  and use (3.4) to obtain

$$\int_T^\infty c(t, \gamma) e^{-\int_T^t r(s) ds} dt = k(T, \gamma) + \int_T^\infty w(t) \gamma e^{-\int_T^t r(s) ds} dt \quad (3.5)$$

For the special case where  $T=0$  (3.5) becomes

$$\int_0^\infty c(t, \gamma) e^{-\int_0^t r(s) ds} dt = k_0 + \int_0^\infty w(t) \gamma e^{-\int_0^t r(s) ds} dt \quad (3.6)$$

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<sup>9</sup> Since  $\partial \dot{c} / \partial c = -\dot{c} A'(c)$ , we can confirm that if individuals have DARA preferences the wealthier individuals (those with higher  $c$ ) will defer consumption more (if consumption is increasing over time). For IARA preferences the relation is reversed, and for individuals with CARA preferences the level of consumption does not matter.

### 3.3 Relative capital accumulation

We wish to find conditions under which an individual endowed with a higher labour productivity will accumulate more (or less) than individuals with lower labour productivity (and vice versa). We will begin by deriving "local" results, concerning relative capital accumulation of two individuals that are close in their productivities, and then derive results that hold globally. For local results we wish to find  $\partial k(T, \gamma) / \partial \gamma$ . If the derivative is positive (negative) an individual with marginally higher productivity than that of another individual will accumulate more (less) capital than the individual with lower productivity. The result would be a global result if the sign of  $\partial k(T, \gamma) / \partial \gamma$  is independent of  $\gamma$ .

First, differentiate (3.5) and (3.6) with respect to  $\gamma$ , then we have

$$\int_T^\infty \frac{\partial c(t, \gamma)}{\partial \gamma} e^{-\int_T^t r(s) ds} dt = \frac{\partial k(T, \gamma)}{\partial \gamma} + \int_T^\infty w(t) e^{-\int_T^t r(s) ds} dt \quad (3.7)$$

$$\int_0^\infty \frac{\partial c(t, \gamma)}{\partial \gamma} e^{-\int_0^t r(s) ds} dt = \int_0^\infty w(t) e^{-\int_0^t r(s) ds} dt \quad (3.8)$$

Next, differentiate (3.7) with respect to  $\gamma$

$$\frac{d}{dt} \frac{\partial c(t, \gamma)}{\partial \gamma} = -\frac{r(t) - \theta}{A(c(t, \gamma))^2} A'(c(t, \gamma)) \frac{\partial c(t, \gamma)}{\partial \gamma} \quad (3.9)$$

and integrate with respect to  $t$

$$\frac{\partial c(t, \gamma)}{\partial \gamma} = \frac{\partial c(0, \gamma)}{\partial \gamma} e^{-\int_0^t \frac{r(s) - \theta}{A(c(s, \gamma))^2} A'(c(s, \gamma)) ds} \quad (3.10)$$

Substitute (3.10) into (3.8) and we have

$$\frac{\partial c(\mathbf{0}, \gamma)}{\partial \gamma} = \frac{\int_0^{\infty} w(t) e^{-\int_0^t r(s) ds} dt}{\int_0^{\infty} e^{-\int_0^t r(s) + \frac{r(s) - \theta}{A(c(s, \gamma))^2} A'(c(s, \gamma)) ds} dt} \quad (3.11)$$

Substituting (3.10) and (3.11) into (3.7) gives

$$\frac{\partial k(T, \gamma)}{\partial \gamma} = \frac{\int_0^{\infty} w(t) e^{-\int_0^t r(s) ds} dt \int_T^{\infty} e^{-\int_T^t r(s) + \frac{r(s) - \theta}{A(c(s, \gamma))^2} A'(c(s, \gamma)) ds} dt}{\int_0^{\infty} e^{-\int_0^t r(s) + \frac{r(s) - \theta}{A(c(s, \gamma))^2} A'(c(s, \gamma)) ds} dt e^{\int_0^T r(s) + \frac{r(s) - \theta}{A(c(s, \gamma))^2} A'(c(s, \gamma)) ds} dt} - \int_T^{\infty} w(t) e^{-\int_T^t r(s) ds} dt \quad (3.12)$$

We need to evaluate (3.12), and its sign is ambiguous at first sight (since the first term is positive and the second negative). We see, however, that the price dynamics together with the individual's absolute risk aversion determine the sign. Define  $\varepsilon(t)$  as a function of time satisfying

$$\frac{\dot{w}(t)}{w(t)} = -\frac{r(t) - \theta}{A(c(t, \gamma))^2} A'(c(t, \gamma)) + \varepsilon(t) \quad (3.13)$$

Then the following theorem gives sufficient conditions for (3.12) to take on either sign.

**Theorem 3.1** *Assume A1, A2, and A3, then the following is true:*

- (i) *If  $\varepsilon(t) > 0$  for  $t \in [0, \infty)$  then  $\partial k(T, \gamma) / \partial \gamma < 0$  for  $T \in (0, \infty)$ .*
- (ii) *If  $\varepsilon(t) = 0$  for  $t \in [0, \infty)$  then  $\partial k(T, \gamma) / \partial \gamma = 0$  for  $T \in [0, \infty)$ .*
- (iii) *If  $\varepsilon(t) < 0$  for  $t \in [0, \infty)$  then  $\partial k(T, \gamma) / \partial \gamma > 0$  for  $T \in (0, \infty)$ .*

for  $\varepsilon(t)$  defined as in (3.13).

*Proof:* Define  $g$  and  $h$  as follows

$$\mathbf{g}(t, T) \equiv e^{-\int_T^t r(s) + \frac{r(s) - \theta}{A(c(s, \gamma))^2} A'(c(s, \gamma)) ds}, \quad h(t, T) \equiv e^{\int^t e(s) ds} - 1 \quad (3.14)$$

Integrating (3.13) from  $v$  to  $t$ , using for the definitions in (3.14), and substituting for  $w(t)$  in the present value of wages, we obtain

$$\int_v^\infty e^{-\int_v^t r(s) ds} w(t) dt = w(v) \int_v^\infty \mathbf{g}(t, v) [h(t, v) + 1] dt \quad (3.15)$$

Substituting (3.15) with  $v$  evaluated at 0 and  $T$  into (3.12), and also using the definitions (3.14), gives

$$\frac{\partial k(T, \gamma)}{\partial \gamma} = \frac{w(0) \int_0^\infty \mathbf{g}(t, 0) [h(t, 0) + 1] dt \int_T^\infty \mathbf{g}(t, T) dt}{\frac{1}{\mathbf{g}(T, 0)} e^{-\int_0^T r(s) ds} \int_0^\infty \mathbf{g}(t, 0) dt} - w(T) \int_T^\infty \mathbf{g}(t, T) [h(t, T) + 1] dt \quad (3.16)$$

Substitute for  $w(0)$ , premultiply by the integral in the denominator, notice that the +1 terms cancel, then we have

$$\int_0^\infty \mathbf{g}(t, 0) dt \frac{\partial k(T, \gamma)}{\partial \gamma} \frac{1}{w(T)} = \int_0^\infty \mathbf{g}(t, 0) h(t, T) dt \int_T^\infty \mathbf{g}(t, T) dt - \int_0^\infty \mathbf{g}(t, 0) dt \int_T^\infty \mathbf{g}(t, T) h(t, T) dt \quad (3.17)$$

Split the integrals on the right-hand side that go from 0 to infinity into two parts (from 0 to  $T$  and from  $T$  to infinity), then two terms cancel and we have

$$\int_0^\infty \mathbf{g}(t, 0) dt \frac{\partial k(T, \gamma)}{\partial \gamma} \frac{1}{w(T)} = \int_0^T \mathbf{g}(t, 0) h(t, T) dt \int_T^\infty \mathbf{g}(t, T) dt - \int_0^T \mathbf{g}(t, 0) dt \int_T^\infty \mathbf{g}(t, T) h(t, T) dt \quad (3.18)$$

or

$$\int_0^{\infty} \mathbf{g}(t, \mathbf{0}) dt \frac{\partial k(T, \gamma)}{\partial \gamma} = w(T) \int_T^{\infty} \mathbf{g}(t, T) \int_0^T \mathbf{g}(s, \mathbf{0}) [h(s, T) - h(t, T)] ds dt \quad (3.19)$$

Notice that  $s \in [0, T)$  and  $t \in [T, \infty)$ , then  $h(s, T) - h(t, T) < (>) 0$  if  $\epsilon(t) > (<) 0$  for  $t \in [T, \infty)$ . QED

Theorem 3.1 is a "local" result in the sense that the sign  $\partial k(T, \gamma) / \partial \gamma$  may depend on the level of  $\gamma$ . However, for some combinations of prices and if the sign of the derivative of absolute risk aversion is the same for all individuals, some global results follow.

**Corollary 3.1** *Assume A1, A2, and A3, then the following is true.*

(i) *If preferences are Decreasing Absolute Risk Aversion for all individuals along their consumption paths, then a high productive individual accumulates less (more) capital than a low productive individual if either  $\dot{w}(t) > 0$  and  $r(t) \leq \theta$  ( $\dot{w}(t) < 0$  and  $r(t) = \theta$ ), or  $\dot{w}(t) = 0$  and  $r(t) < \theta$  ( $\dot{w}(t) \leq 0$  and  $r(t) > \theta$ ) for  $t \in [0, \infty)$ .*

(ii) *If preferences are Increasing Absolute Risk Aversion for all individuals along their consumption paths, then a high productive individual accumulates less (more) capital than a low productive individual if either  $\dot{w}(t) \geq 0$  and  $r(t) > \theta$  ( $\dot{w}(t) \leq 0$  and  $r(t) < \theta$ ) or  $\dot{w}(t) > 0$  and  $r(t) = \theta$  ( $\dot{w}(t) < 0$  and  $r(t) = \theta$ ) for  $t \in [0, \infty)$ .*

(iii) *If preferences are Constant Absolute Risk Aversion for all individuals along their consumption paths, then a high productive individual accumulates less (more) capital than a low productive individual if  $\dot{w}(t) > 0$  ( $\dot{w}(t) < 0$ ) for  $t \in [0, \infty)$ . If  $\dot{w}(t) = 0$  for  $t \in [0, \infty)$  then all individuals accumulate the same amount of capital.*

*Proof:* From Theorem 3.1 we know that only the sign of  $\epsilon(t)$   $t \in [0, \infty)$  determines relative capital accumulation. If prices are such that  $\text{sign}(\epsilon(t))$  is independent of  $\gamma$ , we have a global result. For the prices stated as conditions in Corollary 3.1 this is the case (by inspection of equation (3.13)). QED

We may summarise Corollary 3.1 in the following table.

**Table 3.1** Relative Capital Accumulation - Infinite-horizon economy

DARA/CARA/IARA	$r(t) > \theta$	$r(t) = \theta$	$r(t) < \theta$
$\dot{w}(t) > 0$	?/-/-	-/-/-	-/-/?
$\dot{w}(t) = 0$	+/0/-	0/0/0	-/0/+
$\dot{w}(t) < 0$	+/+/?	+/+/+	?/+/+

The table entry signs (?,-,+,0) refers to the sign of the derivative  $\partial k(t,\gamma)/\partial \gamma$ .

The table is read in the following way . If  $\dot{w}(t) > 0$  and  $r(t) > \theta$  the table entry is ?/-/-, this implies that relative capital accumulation is ambiguous in the DARA-case and that the high skilled would accumulate less capital in the CARA- and IARA-cases.

To conclude, the results from the two-period economy directly carried over to the infinite horizon economy.

One may suspect that the greater the absolute value of  $\varepsilon(t)$  is the greater is the magnitude of relative capital accumulation. The following theorem verifies this.

**Theorem 3.2** *Assume A1, A2, and A3, and two different price paths:  $\varepsilon(t)$  and  $\hat{\varepsilon}(t)$  (defined as in (3.13)). Let  $k(T,\gamma)$  be generated by  $\varepsilon(t)$ , and  $\hat{k}(T,\gamma)$  by  $\hat{\varepsilon}(t)$ , then the following is true:*

- (i) *If  $\hat{\varepsilon}(t) > \varepsilon(t) > 0$  for  $t \in [0, \infty)$  then  $\partial \hat{k}(T,\gamma)/\partial \gamma < \partial k(T,\gamma)/\partial \gamma < 0$  for  $T \in (0, \infty)$ .*
- (ii) *If  $\hat{\varepsilon}(t) < \varepsilon(t) < 0$  for  $t \in [0, \infty)$  then  $\partial \hat{k}(T,\gamma)/\partial \gamma > \partial k(T,\gamma)/\partial \gamma > 0$  for  $T \in (0, \infty)$ .*

*Proof:* Follows directly by (3.19).

QED

We see that if  $\varepsilon > (<) 0$  a high productive individual accumulates a capital stock which at all dates is marginally smaller (larger) than the capital stock of an individual with marginally lower productivity. However, so far we cannot say if the distance between two individuals increase or decrease (or both) over time. To give an answer we need to find the time derivative  $\partial^2 k(T,\gamma)/\partial T \partial \gamma$ .

**Theorem 3.3** Assume A1, A2, and A3, and that  $\lim_{t \rightarrow \infty} \epsilon(t) = 0$ , then the following is true

$$\left. \frac{\partial^2 k(T, \gamma)}{\partial T \partial \gamma} \right|_{T=0} < 0, \quad \lim_{T \rightarrow \infty} \frac{\partial^2 k(T, \gamma)}{\partial T \partial \gamma} > 0 \Leftrightarrow \epsilon(t) > 0, \quad t \in [0, \infty) \quad (3.20)$$

and consequently there is at least one  $T^*$  such that  $\left. \frac{\partial^2 k(T, \gamma)}{\partial T \partial \gamma} \right|_{T=T^*} = 0$ . Sufficient for  $T^*$  to be unique is that

$$[r(t) + \epsilon(t)] \left( \frac{g(s, t)}{r(t)} + 1 \right) \geq \frac{\dot{r}(t)}{r(t)} + \frac{\dot{w}(t)}{w(t)} \quad (3.21)$$

*Proof:* See the appendix.

Theorem 3.3 shows that if inequality is generated ( $\epsilon < 0$ ), then inequality is increasing in the beginning, and decreasing at the end. If in addition the condition in 3.21 holds we have a "classical" Kuznets curve.

As mentioned earlier, the exact price conditions generating the sign and size of relative capital accumulation are local results (except for the results in Corollary 3.1). The reason is that individual consumption units enter equation (3.13). There is an important case when Theorem 3.1, 3.2, and 3.3 become global results, valid for comparison between any individuals. We have the following

**Theorem 3.4** Assume A1, A2, and A3, then necessary and sufficient for Theorem 3.1, 3.2, and 3.3 to hold independent of  $\gamma$  is that the instantaneous utility function,  $u(\cdot)$ , belongs to the Hyperbolic Absolute Risk Aversion (HARA) class:

$$u(c(t)) = \frac{1}{\delta - 1} \left( a + \delta c(t) \right)^{\frac{\delta - 1}{\delta}} \quad (3.22)$$

*Proof:* Necessary and sufficient is that (3.13) is independent of individual consumption. That is  $A'(c)A(c)^{-2} = \text{constant}$ , or  $d(1/A(c))/dc = \text{constant}$ . So, necessary and sufficient is that  $1/A(c)$

is a linear function of  $c$ , say  $1/A(c) = a + \delta c$ , and this precisely characterises the HARA-class (Linear Risk Tolerance). That condition can further be integrated twice to obtain (3.22) QED

The HARA class will play an important role in the general equilibrium analysis in section 4. The function has the following properties:  $u(\cdot)$  is DARA if  $\delta > 0$ , CARA if  $\delta = 0$ , and IARA if  $\delta < 0$ .<sup>10</sup>

## 4. GENERAL EQUILIBRIUM

### A3' Production

There is a large number of competitive firms operating under a constant-returns-to-scale technology in capital and labour:  $f(k,l) = f_k(k,l)k + f_l(k,l)l$ .

Since labour is exogenous we may suppress  $l$ . The factor returns in competitive equilibrium are  $r=f'(k)$  and  $w=f(k)-f'(k)k$ .

**Lemma 4.1** *Assume A1, A2, and A3', then necessary and sufficient for the general equilibrium price paths  $r(t)$ ,  $w(t)$  to be independent of the distribution of  $\gamma$  is that the instantaneous utility function of the individuals belongs to the HARA class.*

*Proof:* See the extension of Pollack (1971) in Marsiliani and Renström (2009).

Lemma 4.1 is an aggregation result. The general equilibrium is as if there is only one individual in the economy. This is because the HARA class generates linear Engel curves. We may then represent the competitive equilibrium by only two equations

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<sup>10</sup> The HARA specification incorporates a number of commonly used utility functions as special cases:

Iso-elastic:  $a=0$ ,  $\delta > 0$ .

Logarithmic:  $a=0$ ,  $\delta = 1$ .

Negative exponential:  $a>0$ ,  $\delta = 0$

Quadratic:  $a>c^{\max}$ ,  $\delta = -1$

Stone-Geary:  $a=-\delta c_0$ ,  $\delta > 0$  [ $c_0$  being subsistence level of consumption]



$$\dot{c}(t) = (a + \delta c(t))[f'(k(t)) - \theta] \quad (4.1)$$

$$\dot{k}(t) = f(k(t)) - c(t) \quad (4.2)$$

The steady-state of this economy is obviously saddle-path stable.

Since the dynamical system is saddle-path stable, we have only two possibilities for out-of-steady-state dynamics, either  $(\dot{c}>0, \dot{k}>0)$  or  $(\dot{c}<0, \dot{k}<0)$ . Since  $\dot{w} = -f''(k)k\dot{k}$  we have

$$\text{sign}(\dot{w}) = \text{sign}(\dot{k}) = \text{sign}(\dot{c}) = \text{sign}(f'(k) - \theta).$$

That is, the wage rate can only be increasing (decreasing) if the interest rate is greater (smaller) than the rate of time preference. This leaves us with only the diagonal elements in Table 3.1 (all other cases can be ruled out in general equilibrium). The only entries to be studied are DARA preferences in a growing economy  $(\dot{c}>0, \dot{k}>0)$  and in a declining  $(\dot{c}<0, \dot{k}<0)$ . It is quite difficult to study the entire path in a general equilibrium context. We shall therefore begin in a neighbourhood of a steady state, and thereafter establish the sufficient conditions for the dynamics in this neighbourhood to be preserved anywhere outside the steady state.

The pricing relations we are interested in are those of equation (3.13), i.e. the conditions for  $\dot{w}/w$  to exceed, to be equal to, or to be less than  $\delta[f'(k) - \theta]$ . Alternatively the conditions for their ratio being greater, equal, or smaller than unity. This ratio is

$$\begin{aligned} \delta \frac{r(t) - \theta}{\dot{w}(t)/w(t)} &= \delta \frac{f(k) - f'(k)k}{-f''(k)k} (f'(k) - \theta) \\ &= \delta \frac{f(k) - f'(k)k}{-f''(k)k} \frac{f'(k) - \theta}{f(k) - c(k)} \end{aligned} \quad (4.3)$$

Denote the steady state capital stock as  $k^*$ . Then we have

**Lemma 4.1** *Assume A1, A2, and A3', the ratio (4.3) in steady state is*

$$\lim_{k \rightarrow k^*} \delta \frac{r(t) - \theta}{\dot{w}(t)/w(t)} = \delta \frac{f(k^*)/k^* - \theta}{c'(k^*) - \theta} \quad (4.4)$$

*Proof:* As  $k \rightarrow k^*$  both numerator and denominator in (4.3) go to zero. Using l'Hôpital's rule (4.4) is obtained. QED

**Lemma 4.2** *Assume A1, A2, and A3', the aggregate consumption function,  $c'(k^*)$ , in steady state is*

$$c'(k^*) - \theta = \theta [\sqrt{1 + 4m} - 1] / 2, \text{ where } m \equiv -f''(k^*)(a + \delta f(k^*))\theta^{-2} \quad (4.5)$$

*Proof:* First

$$c'(k) = \frac{\dot{c}}{\dot{k}} = \delta \frac{f'(k) - \theta}{f(k) - c(k)} (a + \delta c(k)) \quad (4.6)$$

and using l'Hôpital's rule

$$c'(k^*) = \lim_{k \rightarrow k^*} c'(k) = \frac{f''(k^*)}{f'(k^*) - c'(k^*)} (a + \delta c^*) \quad (4.7)$$

Rearranging (4.7) to obtain a quadratic equation

$$[c'(k^*)]^2 - \theta c'(k^*) + f''(k^*)(a + \delta f(k^*)) \quad (4.8)$$

The positive solution is (4.5). QED

**Theorem 4.1** *Assume A1, A2, A3', and that the instantaneous utility function for all individuals is DARA, and that the production technology is CES, and that  $a \leq 0$  in the HARA specification (or "small") then*

- (i) *a growing economy ( $r > \theta, \dot{w} > 0$ ) always make the high skilled accumulating more capital than the low skilled,*
- (ii) *a declining economy ( $r < \theta, \dot{w} < 0$ ) always make the low skilled accumulating more capital than the high skilled,*

*in a neighbourhood of the steady state.*

*Proof:* Combining Lemma 4.1 and 4.2 we obtain

$$\lim_{k \rightarrow k^*} \delta \frac{r(t) - \theta}{\dot{w}(t)/w(t)} = 2\delta \frac{f(k^*)/(\theta k^*) - \theta}{\sqrt{1 + 4m} - 1} \quad (4.9)$$

which is positive for DARA preferences. We are interested in the condition

$$2\delta \frac{f(k^*)/(\theta k^*) - \theta}{\sqrt{1 + 4m} - 1} \quad (4.10)$$

which alternatively may be written as

$$\delta^2 [f(k^*)/(\theta k^*) - 1]^2 + \delta [f(k^*)/(\theta k^*) - 1] - m \underset{<}{\underset{>}{\geq}} 0 \quad (4.11)$$

Further rearrangement gives

$$\delta^2 [f(k^*)/(\theta k^*) - 1]^2 + H + a f'(k^*) f(k^*) \theta^{-2} \underset{<}{\underset{>}{\geq}} 0 \quad (4.12)$$

where

$$H \equiv \delta [f(k^*)/(\theta k^*) - 1 + f'(k^*) f(k^*) \theta^{-2}] \quad (4.13)$$

For CES production function  $H \geq 0$ , (with Cobb-Douglas  $H = 0$  as special case). For  $a$  sufficiently small (and always if  $a \leq 0$ ) equation (4.12) is always positive, therefore the ratio (4.3) is always greater than unity. QED

Thus a growing economy would cause *growing inequality*, while a declining economy would produce *diminishing inequality*. Of course the other types of preferences (CARA and IARA) retain their properties from Section 3, (i.e. reverse to Theorem 4.2). We should keep in mind that there is no razor's edge case between DARA and CARA. We could find neutrality for the DARA case if we choose  $a$  in the utility specification properly (i.e. such that (4.12) equals zero). For iso-elastic utility ( $a=0$ ) and Stone-Geary utility ( $a<0$ ) the theorem obviously holds. Constant elasticity of substitution production function is only a sufficient condition for Theorem 4.2 to hold, it is not necessary. Theorem 4.2 is a local result, but may extend globally. For example Theorem 4.2 holds globally (with respect to time) in a growing economy if  $f'(k) \leq c'(k) = \dot{c}/k$ , and in a shrinking economy if  $f'(k) \geq c'(k)$ , however it is very

difficult to find the conditions for these inequalities to hold. It can be verified, though, that it holds within a range outside the steady state (not just in a neighbourhood).

We may now summarise the general-equilibrium results in a table similar to 3.1. In general equilibrium we can rule out the price dynamics off the diagonal entries. We should also remember that we cannot find a clear cut-off point between DARA and CARA in the general equilibrium analysis. We only report the DARA specification where  $a$  is sufficiently small for Theorem 4.2 to hold.

**Table 4.1** Relative Capital Accumulation - Infinite-horizon economy

DARA*/CARA/IARA	$r(t) > \theta$	$r(t) = \theta$	$r(t) < \theta$
$\dot{w}(t) > 0$	+/-/-	NP	NP
$\dot{w}(t) = 0$	NP	0/0/0	NP
$\dot{w}(t) < 0$	NP	NP	-/+/+

The table entry signs (?,-,+,0) refers to the sign of the derivative  $\partial k(t,\gamma)/\partial \gamma$ .

\* Either Constant Relative Risk Aversion(iso-elastic utility) or the HARA specification when  $a \leq 0$  (or  $a$  is "small"), and the production function is CES.

## 6 FISCAL POLICY

### 6.1 Capital income taxation

Capital income taxes are often motivated by the argument of distributional concern, it is sometimes believed that they can reduce the inequality.<sup>11</sup> At first sight, by inspection of (4.3), with a purely redistributive<sup>12</sup> capital income tax  $\tau^k$  we have

$$\begin{aligned} \delta \frac{r(t) - \theta}{\dot{w}(t)/w(t)} &= \delta \frac{f(k) - f'(k)k}{-f''(k)k} [(1 - \tau^k)f'(k) - \theta] \\ &= \delta \frac{f(k) - f'(k)k}{-f''(k)k} \frac{(1 - \tau^k)f'(k) - \theta}{f(k) - c(k)} \end{aligned} \quad (5.1)$$

<sup>11</sup> Atkinson and Stiglitz (1980) p.266.

<sup>12</sup> The tax is purely redistributive, so we abstract from public consumption, therefore we still have  $\dot{k} = f(k) - c$ .

it looks as if the tax makes the left hand side of (6.1) more likely to be smaller than unity, and thus more likely to make the low skilled accumulate more capital. This conclusion, however, is wrong. The reason is that the price dynamics is not independent of the tax. To provide an answer we have to make a general equilibrium analysis. We can do so by employing the same steps as in Section 4.

**Theorem 4.1** *Assume A1, A2, A3', and that the instantaneous utility function for all individuals is iso-elastic ( $a = 0$ ), and that the production technology is CES, and that there is a purely redistributive capital income tax, then*

- (i) *in a growing economy ( $r > \theta$ ,  $\dot{w} > 0$ ), if the elasticity of substitution of the production function is greater (equal/less) than unity the capital income tax increases (does not affect / reduces) inequality,*
- (ii) *in a declining economy ( $r < \theta$ ,  $\dot{w} < 0$ ), if the elasticity of substitution of the production function is greater (equal/less) than unity the capital income tax decreases (does not affect / increases) inequality,*

*in a neighbourhood of the steady state.*

*Proof:* See the appendix.

Thus when the economy is growing and the elasticity of substitution in production is greater than unity a higher capital income tax increases the differences in capital accumulation and increases inequality. This is because the tax implies a greater ratio on the left-hand side of (6.1) in general equilibrium (in a growing economy). For this to occur the general equilibrium price effects have to be strong enough, which they are if the elasticity of substitution is large (greater than unity). On the other hand, if the elasticity is small (smaller than one) then the capital income tax decreases inequality. We should note that for Cobb-Douglas technology the capital income tax has no effect on the distribution.

Paradoxically, if we care about the distribution of income then we should subsidize capital if the elasticity is greater than unity and the economy is growing.<sup>13</sup>

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<sup>13</sup> This conclusion does not hold within an *optimal-tax* framework, with a Bergson-Samuelson welfare function, since that welfare function only cares about the individuals' *utilities* (not the distribution of income itself).

## 7 SUMMARY AND CONCLUSIONS

The main focus of this paper is relative capital accumulation when individuals differ in productivity. We showed that the price dynamics together with the curvature of the utility function of consumption determine whether high skilled accumulates more, equal or less amount of capital relative to the low skilled, i.e. whether there is growing inequality, neutrality, or diminishing inequality. We showed that (in partial equilibrium) the first (second/third) case is obtained if wages grow at a smaller (equal/greater) rate than the difference between the interest rate and the rate of time preference divided by a parameter reflecting the curvature of the utility function. Next we endogenised the prices in general equilibrium. We were then able to rule out some cases from the partial equilibrium analysis.

The main findings were that growing economies typically<sup>14</sup> produce growing inequality when preferences are Decreasing Absolute Risk Aversion (DARA) (this seems to be in accordance with what economists believe), and always diminishing inequality if preferences are Constant or Increasing Relative Risk Aversion (CARA or IARA). For shrinking economies the relations are reversed. Neutrality is a razor's edge case which is very unlikely to obtain out of the steady state.

The contribution of this paper is to show exactly how inequality is produced and developing the methodology for the analysis. It is important to know whenever we assume special classes of preferences how heterogeneous consumers would behave and how the distribution would evolve.

Finally, as the analysis of fiscal policy showed, it is not clear whether a capital income tax reduces inequality. It does so in a growing (shrinking) economy if the elasticity of substitution in production is less (greater) than unity. However if the economy is growing (shrinking) and the elasticity is greater (smaller) than unity, the capital income tax increases inequality. For the Cobb-Douglas case the tax has no effect on the evolution of the distribution.

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<sup>14</sup> "Typically" because we needed to assume a CES production function and either iso-elastic utility or retaining HARA but with the parameter  $a$  sufficiently "small". Clearly, with the "right" value of  $a$  we could, at least for an interval of time, produce neutrality.

## References

- Atkinson, Anthony B., and Joseph E. Stiglitz (1980), *Lectures on Public Economics*, McGraw-Hill.
- Becker, Robert A. (1980), "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households." *Quarterly Journal of Economics* 95(2), 375-382.
- Chamley, Christophe (1986), "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54(3): 607-622.
- Chatterjee, Satyajit (1994), "Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model," *Journal of Public Economics*, 54(1): 97-119.
- Judd, Kenneth L. (1985), "Redistributive Taxation in a Simple Perfect Foresight Model," *Journal fo Public Economics*, 28: 59-83.
- Lucas, Robert E., Jr. (1990), "Supply-Side Economics: An Analytical Review," *Oxford Economic Papers*, 42: 293-316.
- Marsiliani, Laura, and Thomas I. Renström (2009), "Aggregation in Dynamic Economies," Durham Business School Working Paper 117.
- Pollack, R. (1971), "Additive Utility Functions and Linear Engel Curves," *Review of Economic Studies*, 38: 401-413.
- Ramsey, Frank P. (1928), "A Mathematical Theory of Saving." *Economic Journal*, 38, 543-559.
- Stiglitz, Joseph E. (1969), "Distribution of Income and Wealth Among Individuals," *Econometrica*, 37: 382-397.
- Stiglitz, Joseph E. (1978), "Notes on Estate Taxes, Redistribution, and the Concept of Balanced Growth Path Incidence," *Journal of Political Economy*, 86: S137-S150.