MORPHIC HEIGHTS AND PERIODIC POINTS

M. EINSIEDLER, G. EVEREST, AND T. WARD

Number theory (New York, 2003), 167–177, Springer, New York, 2004

ABSTRACT. An approach to the calculation of local canonical morphic heights is described, motivated by the analogy between the classical height in Diophantine geometry and entropy in algebraic dynamics. We consider cases where the local morphic height is expressed as an integral average of the logarithmic distance to the closure of the periodic points of the underlying morphism. The results may be thought of as a kind of morphic Jensen formula.

1. INTRODUCTION

Let $\phi : \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{P}^1(\overline{\mathbb{Q}})$ be a morphism of degree d, defined over the rationals. Call, Goldstine and Silverman (see [3],[4]) have associated to ϕ a canonical global *morphic height* $\hat{\lambda}_\phi$ on $\overline{\mathbb{Q}}$, with the properties that

(1) $\hat{\lambda}_\phi(\phi(q)) = d\hat{\lambda}_\phi(q)$ for any $q \in \mathbb{P}^1(\overline{\mathbb{Q}})$;
(2) q is pre-periodic if and only if $\hat{\lambda}_\phi(q) = 0$.

A point q is called *pre-periodic* under ϕ if the orbit $\{\phi^{(n)}(q)\}$ is finite (write $f^{(n)}$ for the nth iterate of a map f). The global height decomposes into a sum of local canonical morphic heights $\lambda_{\phi,v}$:

$$\hat{\lambda}_\phi(q) = \sum_v n_v \lambda_{\phi,v}(q).$$

Here v runs over all the valuations (both finite and infinite) of the number field generated by q and the n_v denote the usual normalising constants. In the special case that ϕ takes the form $\phi[x,y] = [y^d f(x/y), y^d]$ for a polynomial f of degree d, Call and Goldstine [3] prove that

$$\lambda_{\phi,v}(q) = \lim_{n\to\infty} \frac{1}{d^n} \lambda_v(\phi^{(n)}(q)), \tag{1}$$

1991 *Mathematics Subject Classification.* 11S05, 37F10.

The first author acknowledges the support of EPSRC postdoctoral award GR/M49588, the second thanks Jonathan Lubin and Joe Silverman for the AMS Sectional meeting on Arithmetic Dynamics at Providence, RI, 1999.

where λ_v is the local projective height $\lambda_v(q) = \log^+ |q|_v$, and that the local height $\lambda_{\phi,v}(q)$ vanishes if and only if $|\phi^{(n)}(q)|_v$ is bounded for all n, and finally, that q is pre-periodic if and only if

$$\lambda_{\phi,v}(q) = 0 \text{ for all } v. \tag{2}$$

Example 1.1. (1) Let $f(z) = z^d$ with $d > 1$. Here the canonical morphic heights and the projective heights agree. Jensen's formula ([13, Theorem 15.18]) gives

$$\int_{\mathbb{S}^1} \log |y - q| \mathrm{d}m(y) = \log^+ |q|,$$

where m is the Haar measure on the circle $\mathbb{S}^1$. The circle is also the Julia set for this morphism on $\mathbb{C}$, and it is the closure of the set of non-zero periodic points, which are all roots of unity. In the p-adic case, the Julia set is empty but it is still true that the local height is the Shnirel'man integral of the logarithmic distance from the closure of the set of periodic points. For all v, finite and infinite, the following holds:

$$\log^+ |q|_v = \lim_{n\to\infty} \frac{1}{d^n} \sum_{\zeta \neq q : \zeta^{d^n - 1} = 1} \log |\zeta - q|_v$$

For finite v this may be seen, for instance, using Section 3. Alternatively, it follows from the Diophantine estimate

$$|q^{d^n} - q|_v > C(q)|n|_v$$

provided the left hand side is non-zero (cf. Remark 3.6). For $v|\infty$ a result from transcendence theory (in this case, Baker's theorem) is needed.

(2) Suppose $a, b \in \mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$ and let

$$f(z) = \frac{z^4 - 2az^2 - 8bz + a^2}{z^3 + az + b}.$$

Then f gives rise to a morphism of degree 4 which describes the duplication map on an elliptic curve. The global and local morphic heights coincide with the usual notions of height on the curve. For the infinite valuations, the local height is again the integral average of the logarithmic distance to points on the Julia set, which is the closure of the set of periodic points. At the singular reduction primes it is still true that the local height is the integral average of the logarithmic distance to the periodic points. Both of these assertions are proved in [6] where this morphism was used to construct a dynamical system which interprets these heights in dynamical terms. The proofs require elliptic transcendence theory to show that a rational point cannot approximate a periodic point too closely (cf. Proposition 3.8). This is part of a much broader analogy between heights in Diophantine geometry and entropy in algebraic dynamics (see [5], [6], [7], [10]).

In this paper, our purpose is to describe a family of examples where the local canonical morphic height can be expressed as a limiting integral over periodic points of the underlying morphism. The finite and infinite cases require different approaches. In both, we consider the special class of morphisms corresponding to affine polynomial maps and in the former case, we assume good reduction in the sense of Morton and Silverman [12].

There are two directions in which this work can be made more sophisticated that are not pursued here. The first is to give a more formal interpretation of the limiting process using Shnirel'man integrals (see [14], or [8] for a modern treatment); the second is to extend the arguments to other morphisms.

2. Complex case

Assume that $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a morphism of degree d, with $\phi[x,y] = [y^d f(x/y), y^d]$ for a polynomial f of degree d. For basic definitions of complex dynamics, consult [1]. The following theorem expresses the local morphic height as an integral over the Julia set $J(f)$ of the polynomial f, and standard results from complex dynamics show that this is in turn a limiting integral over periodic points.

Theorem 2.1. *If $f(z) = az^d + \cdots + a_0$ is a polynomial, then for any $q \in \mathbb{C}$,*

$$\lambda_{\phi,\infty}(q) = \frac{1}{d-1}\log|a| + \int_{J(f)} \log|x-q| \mathrm{d}m(x), \tag{3}$$

where m is the maximal invariant measure for f on $J(f)$.

Proof. Assume first that q is in the domain of attraction of ∞ for f. The zeros of the polynomial $f_n(x) = f^{(n)}(x) - x$ are precisely the solutions of the equation $f^{(n)}(x) = x$. Note that $d_n = \deg(f_n) = d^n$, where $d = \deg(f)$. Since $|f^{(n)}(q)| \to \infty$, $\frac{1}{d_n}\log|f_n(q)|$ is approximately $\frac{1}{d^n}\log|f^{(n)}(q)|$, which converges to $\lambda_{\phi,\infty}(q)$. Since q lies in the open Fatou set, $\log|x-q|$ is continuous on $J(f)$. Now

$$\frac{1}{d_n}\log|f_n(q)| = \frac{1}{d_n}\sum_{f^{(n)}(x)=x}\log|x-q| + \frac{1}{d_n}\log|B_n|, \tag{4}$$

where the sum is over the nth 'division points' and

$$B_n = a^{1+d+d^2+\cdots+d^{(n-1)}}$$

is the leading coefficient of $f^{(n)}(x)$. Thus

$$\frac{1}{d_n}\log|B_n| = \frac{1}{d^n}\left(\frac{d^n-1}{d-1}\right)\log|a| \to \frac{1}{d-1}\log|a|. \tag{5}$$

Now it is known that

$$\frac{1}{d_n}\sum_{f^{(n)}(x)=x}\log|x-q| \to \int_{J(f)}\log|x-q|\mathrm{d}m(x),$$

where m is the maximal invariant measure for f restricted to the Julia set (see [11]; [9]).

In the following it is convenient to assume that a=1, we can ensure this by conjugating by a linear map. Now $|x-f(q)| = \prod_{f(t)=x}|t-q|$, so

$$\begin{aligned}\int_{J(f)}\log|x-f(q)|\mathrm{d}m(x) &= \int_{J(f)}\sum_{f(t)=x}\log|t-q|\mathrm{d}m(x)\\ &= d\int_{J(f)}\log|x-q|\mathrm{d}m(x) \qquad (6)\end{aligned}$$

(the last equality follows from [11] or [9, Theorem (d)]).

Let now $q\notin J(f)$ have bounded orbit. Since $J(f)$ is closed, we can find $\epsilon>0$ such that $B_\epsilon(q)\cap J(f)=\emptyset$. If $|f^n(q)-q|>\epsilon/2$ for almost all n, then

$$\frac{1}{d_n}\log|f^n(q)-q|\to 0.$$

We can argue as in the first case to get

$$\int_{J(f)}\log|x-q| = 0 = \lambda_{\phi,\infty}(q).$$

Assume now $|f^{n_j}(q)-q|\le\epsilon/2$ for some sequence $n_j\to\infty$. Then $|f^{n_j}(q)-x|>\epsilon/2$ for $x\in J(f)$. However, since $J(f)$ and $f^{n_j}(q)$ are bounded, we have also an upper bound

$$\log\frac{\epsilon}{2}\le\int_{J(f)}\log|x-f^{n_j}(q)|\le M.$$

Together with Equation (6) we get

$$1/d^{n_j}\log\frac{\epsilon}{2}\le\int_{J(f)}\log|x-q|\le 1/d^{n_j}M$$

which concludes the proof in this case.

It remains to show that the formula holds for $q\in J(f)$. Since $J(f)$ has no interior, there is a sequence $q_n\to q$ with $q_n\notin J(f)$. Then

$\log|x-q_n| \to \log|x-q|$ for all $x \in J(f)\backslash\{q\}$. Since $J(f)$ is bounded, $\log|x-q_n|$ and $\log|x-q|$ are uniformly bounded above by M say for $x \in J(f)\backslash\{q\}$. So by Fatou's lemma

$$0 = \lim_{n\to\infty} \int_{J(f)} \log|x-q_n| \mathrm{d}m(x) \leq \int_{J(f)} \log|x-q| \mathrm{d}m(x) \leq M. \tag{7}$$

This shows that $x \mapsto \log|x-q|$ is in $L^1(m)$. If

$$\int_{J(f)} \log|x-q| \mathrm{d}m(x) > 0,$$

then Equation (6) contradicts (7). □

3. The p-adic case

Let $\mathbb{C}_p$ denote the usual completion of the algebraic closure of the p-adic numbers $\mathbb{Q}_p$, and use $|\cdot|$ to denote the extension of the p-adic norm to $\mathbb{C}_p$. Write $\mathcal{O}_p$ for the ring of integral elements in $\mathbb{C}_p$, and $\mathcal{P}$ for the maximal ideal of $\mathcal{O}_p$. In this section we assume that $\phi : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$ is a morphism of degree d corresponding to an affine polynomial f of degree d with coefficients in $\mathcal{O}_p$ and leading coefficient in $\mathcal{O}_p^*$. Notice that these assumptions are, for polynomials, equivalent to the assumption that the map ϕ has good reduction in the sense of [12]: ϕ induces a morphism of schemes over $\mathrm{Spec}(\mathcal{O}_p)$. The Julia set is empty in this setting (see [2], [12]), so a direct analogue of (3) is not possible.

The main result expresses the local canonical morphic height as a limiting integral over periodic points for the polynomial f.

Theorem 3.1. *If ϕ has good reduction and is defined by a polynomial f of degree d, then*

$$\lambda_{\phi,p}(q) = \log^+|q| = \lim_{n\to\infty} \frac{1}{d^n} \sum_{\xi \neq q: f^{(n)}(\xi)=\xi} \log|\xi - q|$$

where the sum is taken with multiplicities.

Notice first that for $q \in \mathbb{C}_p\backslash\mathcal{O}_p$ this is clear, so from now on we assume that $q \in \mathcal{O}_p$. Despite the simple resulting value of the height, the convergence involved requires an argument. The main issue is to produce lower bounds on the size of $|\xi - \zeta|$ for distinct periodic points ξ and ζ.

The *least period* of a periodic point ξ is the cardinality of the orbit of ξ. The points of period n are the solutions to the polynomial equation

$$f^{(n)}(x) - x = 0, \tag{8}$$

and are therefore all elements of $\mathcal{O}_p$. Following [12], for a periodic point ξ define $a_n(\xi)$ to be the multiplicity of ξ in (8), with the obvious convention that ξ has multiplicity zero in an equation that it does not satisfy. Notice that $a_n(\xi) \neq 0$ if and only if n is a multiple of the least period of ξ. Define $a_n^*(\xi)$ by

$$a_n^*(\xi) = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_n(\xi),$$

where μ is the Möbius function. Increases in the multiplicity of the periodic point ξ along the sequence of multiples of its least period are recorded by $a_n^*(\xi)$. The periodic point ξ is an *essential n-periodic point* if $a_n^*(\xi) > 0$.

The following proposition is a special case of [12, Prop. 3.2].

Proposition 3.2. *Let K be an algebraically closed field of characteristic $p \geq 0$, and f a polynomial over K with degree $d \geq 2$. Fix a periodic point $\zeta \in K$ with least period m, and let r denote the multiplicative order of $(f^{(m)})'(\zeta)$ in K^* or ∞ if $(f^{(m)})'$ is not a root of unity. Then for $n \geq 1$, $a_n^*(\zeta) \geq 1$ if and only if one of the following conditions hold.*

(1) *$n = m$;*
(2) *$n = mr$;*
(3) *$p > 0$ and $n = p^e mr$ for some $e \geq 1$.*

When $K = \mathbb{C}_p$, this proposition will also be applied to the polynomial $\bar{f}$ induced by reduction mod $\mathcal{P}$. Notice that the sum of the multiplicities of the points of period n under f lying in one residue class gives the multiplicity of the image point as a point of period n under $\bar{f}$.

For the proof of Theorem 3.1 the following proposition is needed; this will be proved later.

Proposition 3.3. *Suppose ξ is a periodic point with least period n for a polynomial f of good reduction. Then for any fixed $q \in \mathcal{O}_p$, $|q - \xi| \to 1$ as $n \to \infty$, provided $q \neq \xi$.*

Proof. (of Theorem 3.1) By Proposition 3.2 applied to the field $\mathbb{C}_p$ with characteristic zero, if ξ is a periodic point with least period m, then the multiplicity of ξ viewed as a periodic point of period $m, 2m, 3m, \ldots$ is uniformly bounded. Fix $q \in \mathcal{O}_p$ and $s \in (0, 1)$. Proposition 3.3 says that the number of periodic points in the metric ball $D_s(q)$ is finite. It follows that

$$\liminf_{n\to\infty} \frac{1}{d^n} \sum_{\xi \neq q : f^{(n)}(\xi) = \xi} \log|\xi - q| \geq \log s.$$

On the other hand, each term in the sum is non-positive, so letting $s \to 1$ proves the theorem. □

All that remains is to prove Proposition 3.3, for which we need some lemmas.

Lemma 3.4. *Assume that $f(0) = 0$, and let ζ be a periodic point of f. Then $|\zeta| = |f^{(n)}(\zeta)|$ for all $n \geq 1$.*

Proof. The spherical metric used in [12] coincides with the usual metric in $\mathcal{O}_p$, and f has good reduction. So by [12, Prop. 5.2],

$$|f(x) - f(y)| \leq |x - y|$$

for $x, y \in \mathcal{O}_p$. The lemma follows at once. □

Lemma 3.5. *Assume that $f(0) = 0$ and $n > 1$ is fixed.*

(1) *If $|f'(0)| < 1$ then $\prod_{\xi \neq 0} |\xi|^{a_n^*(\xi)} = 1$.*

(2) *If $|f'(0) - 1| < p^{-1}$ then* $\prod_{\xi \neq 0} |\xi|^{a_n^*(\xi)} = \begin{cases} 1/p & \text{if } n \text{ is a power of } p, \\ 1 & \text{if not.} \end{cases}$

Proof. In case 1., $\bar{f}'(0) = 0$ in the algebraically closed field $\mathcal{O}_p/\mathcal{P}$, so Proposition 3.2 may be applied with $\zeta = 0 + \mathcal{P}$, $m = 1$ and $r = \infty$. It follows that for $n > 1$ $a_n^*(0 + \mathcal{P}) = 0$, so there cannot be an essential n-periodic point ξ for f with $|\xi| < 1$.

In case 2., $m = 1$ and $r = 1$ for the point $\zeta = 0 + \mathcal{P}$ in Proposition 3.2. It follows that only values of n of the form p^k are relevant. Notice that

$$\prod_{\xi \neq 0} |\xi|^{a_{p^k}^*(\xi)} = \left| \left(\frac{f^{(p^k)}(x) - x}{f^{(p^{k-1})}(x) - x} \right)_{x=0} \right| \tag{9}$$

If $f'(0) \neq 1$, then the right-hand side of (9) is given by

$$\left| \frac{(f'(0))^{p^k} - 1}{(f'(0))^{p^{k-1}} - 1} \right| = \frac{1}{p}$$

by the binomial theorem. If $f'(0) = 1$, write $f(x) = x + x^e g(x)$ with $e > 1$ and $g(0) \neq 0$, then a simple induction argument shows that

$$f^{(k)}(x) = x + kx^e g(x) + O(x^{2e-1}).$$

It follows that (9) is equal to p again.

□

Proof. (of Proposition 3.3) Let ζ be any periodic point, with least period ℓ. The first step is to prove the proposition for $q = \zeta$. Let ξ have least period n under f. The multiplicity of $\zeta + \mathcal{P}$, which has least period m for some $m|\ell$ must increase at ℓ (because $a_\ell^*(\zeta) > 1$). It follows by Proposition 3.2 that ℓ is equal to m, mr, or mrp^e for some $e \geq 1$. Assume first that n is not of one of those forms; then $|\xi - \zeta| = 1$ because the multiplicity of $\zeta + \mathcal{P}$ cannot increase at n in $\mathcal{O}_p/\mathcal{P}$.

In the remaining cases, we may assume for large n that $\ell | n$. Then ξ is a periodic point with least period n/ℓ under $f^{(\ell)}$. Applying the conjugation $x \mapsto x - \zeta$ means that 0 is a fixed point of g, where $g(x) = f^{(\ell)}(x + \zeta) - \zeta$.

If $|g'(0)| < 1$, then by Lemma 3.5 applied to g, $|\xi - \zeta| = 1$.

If $|g'(0)| = 1$, let t be the order of $g'(0) + \mathcal{P}$ in $\mathcal{O}_p/\mathcal{P}$. Then

$$|(g'(0))^t - 1| < 1.$$

There exists a $c \geq 1$ such that

$$|(g'(0))^{tp^c} - 1| < 1/p.$$

As before, we may assume that $tp^c | n$, so Lemma 3.5 may be applied to the map $h = g^{(tp^c)}$ to give

$$\prod_{j=1,\dots,n/tp^c\ell} |h^{(j)}(\xi - \zeta)| \geq 1/p.$$

It follows by Lemma 3.4 that $|\xi - \zeta| \geq p^{-tp^c\ell/n}$. Since t, c, ℓ depend only on ζ, $|\xi - \zeta| \to 1$ as $n \to \infty$. The ultrametric inequality in $\mathbb{C}_p$ now gives the result for any $q \in \mathcal{O}_p$. $\square$

Remark 3.6. Notice that the discussion above also gives a quantitative version of Proposition 3.3. This Diophantine result may be of independent interest. If f is a polynomial of good reduction, then

$$|f^{(n)}(q) - q| > C(f, q)|n|$$

for all $n \geq 1$, provided the left hand side is non-zero.

Example 3.7. To see the different cases that are possible in Proposition 3.3, consider the following examples.

(1) Let $f(x) = g(x^p) + ph(x)$ be a monic polynomial with coefficients in $\mathcal{O}_p$. Then $|f'(q)| < 1$ for any $q \in \mathcal{O}_p$, so in Lemma 3.5 only the first case is ever used. It follows that in Proposition 3.3, $|\zeta - \xi| = 1$ for any distinct periodic points ζ, ξ.

(2) Let $f(x) = x^2 - (1 + a)x$ for some small a. Then 0 is a fixed point, and $|f'(0)| = 1$. Now

$$f^{(2)}(x) - x = x^4 - (2a + 2)x^3 + (a^2 + a)x^2 + (a^2 + 2a)x,$$

so $(f^{(2)}(x) - x)/(f(x) - x)$ has constant term $(a^2+2a)/(-a-2) = -a$. Therefore there must be two non-zero points of period 2 that are close to the fixed point 0.

If the polynomial f has coefficients outside $\mathcal{O}_p$, then in contrast to Proposition 3.3, there may be sequences of periodic points converging to a periodic point. For example, $f(x) = x^2 + \frac{1}{2}x$ on $\mathbb{C}_2$ has this property. Therefore, to recover Theorem 3.1 in greater generality (for polynomials of bad reduction or rational functions) some kind of Diophantine approximation results are needed. In Example 1.1.2 these tools are provided by elliptic transcendence theory.

Proposition 3.8. *Let ζ be a periodic point with least period ℓ under f. Assume that $|(f^{(\ell)})'(\zeta)| > 1$. Then there are periodic points $\xi \neq \zeta$ arbitrarily close to ζ.*

Proof. Define $g = f^{(\ell)}$ and $a = (f^{(\ell)})'(\zeta)$. Without loss of generality we can assume that $\zeta = 0$. Then

$$g(x) = ax + bx^e + O(x^{e+1}) \text{ with } b \neq 0$$

and

$$g^{(2)}(x) = a^2x + (ab + a^e b)x^e + O(x^{e+1}).$$

Define $b_2 = (ab + a^e b)$, then $|b_2| = |b||a|^e$. By induction one can see that

$$g^{(k)}(x) = a^k x + b_k x^e + O(x^{e+1})$$

with $|b_k| = |b||a|^{ke}$.

Therefore the Newton polygon of $g^{(k)}(x) - x$ starts with a line with slope $s \leq -k + c$ for a fixed c (depending on b and e). So there exists a periodic point ξ with $|\xi| = p^s \leq p^{-k+c}$. $\square$

4. Tchebycheff polynomials

Example 4.1. Consider the Tchebycheff polynomial of degree d, $f(z) = T_d(z) = \cos(d \arccos(z))$. The Julia set is the interval $J(f) = [-1, 1]$. The map $\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = \frac{1}{2}(z + z^{-1})$ is a semi-conjugacy from $g : z \mapsto z^d$ onto $z \mapsto f(z)$, in other words, $f(\phi(z)) = \phi(z^d)$. Write ψ for the branch of the inverse of ϕ defined on $\{z \in \mathbb{C} \mid |z| > 1\}$. The canonical morphic height at the infinite place is (for $q \notin J(f)$)

$$\begin{aligned}
\lambda_{\phi,\infty}(q) &= \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^{(n)}(q)| \\
&= \lim_{n\to\infty} \frac{1}{d^n} \log^+ |\phi g^{(n)} \psi(q)| \\
&= \lim_{n\to\infty} \frac{1}{d^n} \log^+ |\frac{1}{2}\left(g^{(n)}\psi(q) + \frac{1}{g^{(n)}\psi(q)}\right)| \\
&= \lim_{n\to\infty} \max\left\{0, \frac{1}{d^n} \log |g^{(n)}\psi(q)|\right\} \\
&= \log^+ |\psi(q)|.
\end{aligned}$$

For $q \in J(f)$, the same formula holds since $\lambda_{\phi,\infty}(q) = 0$ there by [3] and $\log^+ |\psi(q)| = 0$ there by a direct calculation.

Now by Jensen's formula, for any $q \in \mathbb{C}$,

$$\begin{aligned} \log^+ |\psi(q)| &= \log 2 + \int_{\mathbb{S}^1} |\phi(y) - q| \mathrm{d}y \\ &= \log 2 + \int_{J(f)} \log |t - q| \mathrm{d}m(t) \end{aligned}$$

since m is the image under ϕ of the maximal measure (Lebesgue) on the circle. That is,

$$\lambda_{\phi,\infty}(q) = \log 2 + \int_{J(f)} \log |t - q| \mathrm{d}m(t). \tag{10}$$

The constant $\log 2$ in $\lambda_\infty(q)$ may be explained in accordance with Theorem 2.1. The leading coefficient of T_d is 2^{d-1}, so $\frac{1}{d-1} \log |a|$ in this case is exactly $\log 2$.

A similar approach can be adopted in the case of polynomials with connected Julia sets. There the local conjugacy near ∞ extends to the whole domain of attraction of ∞, which is the complement of the filled Julia set.

Example 4.2. As before, let $f(x) = T_d(x) = \cos(d \arccos(x))$ be the Tchebycheff polynomial of degree d and let ϕ be the corresponding morphism. We would like to use Theorem 3.1, but f does not satisfy the assumptions since it is not monic.

Let $g(x) = 2f(\frac{x}{2})$. Notice that f is defined uniquely by the property $f(\frac{1}{2}(z + z^{-1})) = \frac{1}{2}(z^d + z^{-d})$. It follows that g is characterized by the property $g(z + z^{-1}) = (z^d + z^{-d})$, which shows that $g \in \mathbb{Z}[x]$ is a monic polynomial. Let ψ be the morphism defined by g, then by Theorem 3.1 we have for $q \in \mathbb{C}_p$

$$\lambda_{\psi,p}(q) = \log^+ |q| = \lim_{n\to\infty} \frac{1}{d^n} \sum_{\xi \neq q : g^{(n)}(\xi) = \xi} \log |\xi - q|. \tag{11}$$

Since $g(x) = 2f(\frac{x}{2})$, we have that $\lambda_{\phi,p}(q) = \lambda_{\psi,p}(2q)$ and on the right hand side of (11) that $f^{(n)}(\xi) = \xi$ if and only if $g^{(n)}(2\xi) = 2\xi$. Therefore

$$\lambda_{\phi,p}(q) = \log^+ |2q| = \log |2| + \lim_{n\to\infty} \frac{1}{d^n} \sum_{\xi \neq q : f^{(n)}(\xi) = \xi} \log |\xi - q|,$$

which is again analogous to Equation (10) in Example 4.1.

Example 4.2 works because the Tchebycheff polynomial can be conjugated to a polynomial of good reduction; a similar approach can be adopted for any polynomial that is conjugate to one of good reduction.

References

1. A. Beardon, *Iteration of Rational Functions*, Springer, New York, 1991.
2. R. Benedetto, *Reduction, dynamics, and Julia sets of rational functions*, J. Number Theory, to appear.
3. G.S. Call and S.W. Goldstine, *Canonical heights on projective space*, J. Number Theory **63** (1997), 211–243.
4. G.S. Call and J.H. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. **89** (1993), 163–205.
5. P. D'Ambros, G. Everest, R. Miles, and T. Ward, *Dynamical systems arising from elliptic curves*, Colloq. Math. (to appear) (2000).
6. M. Einsiedler, G. Everest, and T. Ward, *Entropy and the canonical height*, Preprint.
7. G. Everest and T. Ward, *A dynamical interpretation of the global canonical height on an elliptic curve*, Experiment. Math. **7** (1998), 305–316.
8. N. Koblitz, *p–adic analysis: a short course on recent work*, LMS Lecture Notes 46, Cambridge Univ. Press, 1980.
9. A. Freire, A. Lopes, and R. Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil. Mat. **14** (1983), no. 1, 45–62.
10. D.A. Lind and T. Ward, *Automorphisms of solenoids and p-adic entropy*, Ergodic Theory Dynam. Systems **8** (1988), 411–419.
11. M.Y. Lyubich, *Entropy of analytic endomorphisms of the Riemann sphere*, Funktsional. Anal. i Prilozhen. **15** (1981), 83–84.
12. P. Morton and J.H. Silverman, *Periodic points, multiplicities, and dynamical units*, J. Reine Angew. Math. **461** (1995), 81–122.
13. W. Rudin, *Real and Complex Analysis*, McGraw–Hill, New York, 1974.
14. L.G. Shirel'man, *On functions in normed algebraically closed division rings*, Izv. Akad. Nauk. SSSR Ser. Mat. **2** (1938), 487–498 (Russian).

(M.E.) Mathematical Institute, University of Vienna, Strudlhofgasse 4, A-1090 Wien, Austria.

E-mail address: `manfred@mat.univie.ac.at`

(G.E. & T.W.) School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK.

E-mail address: `g.everest@uea.ac.uk`

E-mail address: `t.ward@uea.ac.uk`