Title: Some remarks on functional analysis

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Some remarks on functional analysis

Overview

Functional analysis is a key tool in the study of partial differential equations which helps to answer key questions such as existence, well-posedness and the class in which a solution should belong. We begin these remarks by introducing normed spaces and Banach spaces and then bounded linear operators in normed spaces. Next, we define Hilbert spaces and consider aspects relating to linear operators on Hilbert spaces. With this structure we are able to consider well-posed of problems and describe the notions of Hölder and Lyapunov stability and Hadamard well-posedness. Finally, we describe how some thermal stress problems can be formulated using abstract operator notation.

Introduction

Many of the theories involving thermal stresses, whether this be in thermoelasticity, in heat conducting fluids, in viscoelastic materials, or more exotic substances like auxetic foams, are based on partial differential equations (PDEs). To develop a model for a real life situation one needs to know the physical domain of the body under consideration, the PDE or the system of PDEs governing the behaviour of the body, the boundary conditions, the initial conditions, and then one needs to understand what class of function is acceptable as a solution to the boundary value problem, or boundary initial value problem which arises.

Henceforth, when we speak of a "problem" we shall mean a PDE or system of PDEs together with associated boundary conditions and (if needed) initial conditions.

To give an example, let us consider the evolution of temperature, θ, in a rigid body $\Omega \subset \mathbb{R}^3$. The balance of energy equation is

$$\rho_0 \dot{U} = -q_{i,i} \tag{1}$$

where ρ_0, U and q_i are density, internal energy, and heat flux, a superposed dot denotes partial differentiation with respect to time, a subscript $_{,i}$ denotes $\partial/\partial x_i$ and the Einstein summation convention is employed. If we adopt Fourier's law then

$$q_i = -k\theta_{,i} \tag{2}$$

where k is the thermal conductivity, which we take as constant. Then, since $U = U(\theta)$, equations (1) and (2) reduce to

$$\dot{\theta} = \kappa \Delta \theta \tag{3}$$

where $\kappa = k/\rho_0 c$, $c = \partial U/\partial \theta$ being the specific heat. Equation (3) is defined on $\Omega \times (0, T)$ for some positive time, T, and we assume we are given boundary and initial conditions as

$$\theta(\mathbf{x},t) = \theta_\Gamma(\mathbf{x},t), \quad \mathbf{x} \in \Gamma, \quad t \in (0,T) \tag{4}$$

$$\theta(\mathbf{x},0) = \theta_0, \quad \mathbf{x} \in \Omega. \tag{5}$$

Here Γ is the boundary of Ω.

We now require to find the solution to the problem (3)–(5). However, one may ask what we mean by a solution? How smooth is this solution, can the solution have a discontinuity in $\theta_{,i}$? Is it possible for a shock wave to form, i.e. can θ itself become discontinuous? Shock waves are highly important in thermal stresses, cf. Dafermos [6]. Perhaps the solution may not exist for all time? Given the functions θ_Γ and θ_0, how do they influence the class of solutions which might arise? The last topic is one of regularity of the solution. To answer questions such as those of regularity, or even the question of does a particular solution exist, one needs a mathematical definition of the classes of solution one may expect, or the space of functions in which a solution may lie. To develop these ideas requires the use of functional analysis. The remainder of this article looks at some of the elementary ideas of functional analysis appropriate to thermal stresses.[1]

It is impossible in a short article to include anything like the vast amount of material on functional analysis which is necessary for one to undertake a complete study of analytical properties of PDEs. For instance, Smirnov's book alone covers 631 pages. However, the interested reader can find much more relevant material in the books by Dafermos [6]; Evans [7]; Smirnov [14]; Naimark [13]; Gilbarg and Trudinger [8]; Grisvard [9]; Brown and Page [4]; Bennett and Sharpley [3].

[1] Within the context of linear thermoelasticity questions of existence, regularity and decay were first addressed in the fundamental work of Dafermos [5], and many developments followed, both in classical thermoelasticity and in thermoelastic theories allowing for heat waves. This development is detailed in pages 163–165 of the book by Straughan [15].

Normed spaces and Banach spaces

Suppose V is a vector space over the complex field $\mathbb{C}$. A *norm* on V is defined to be a real-valued function $\|x\|$, defined for $x \in V$, with properties:

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$,
3. $\|x + y\| \leq \|x\| + \|y\|$,

for any $x, y \in V$, and any $\alpha \in \mathbb{C}$.

The space V with the associated norm $\|\cdot\|$ is called a normed space.

The norm defines a metric on V and thus allows us to define distances so we may discuss convergence of sequences, convergence of series, Cauchy sequences, completeness.

Let $\{x_n\}$ be a sequence in V. Then $\{x_n\}$ is a *Cauchy sequence* if and only if, given $\epsilon(> 0)$, there exists $N(\epsilon) \in \mathbb{N}$, such that $\|x_m - x_n\| < \epsilon$ whenever $m, n \geq N$.

A space is *complete* if every Cauchy sequence in the space converges to some limit in the space.

A complete normed space is called a *Banach space*.

Examples

1. For $a, b \in \mathbb{R}$, $a < b$, the set $C(a, b)$ of all complex-valued continuous functions on $[a, b]$ is a complex vector space, with a norm defined by

$$\|f\| = \sup_{a \leq t \leq b} |f(t)|, \quad f \in C(a, b).$$

2. Define l_1 to be the space of all complex sequences $(x_1, x_2, \cdots)$ such that $\sum_{n=1}^{\infty} |x_n| < \infty$. Then l_1 is a complex vector space with norm

$$\|\mathbf{x}\| = \sum_{n=1}^{\infty} |x_n|,$$

$\mathbf{x} = (x_1, x_2, \cdots)$.

3. Define l_∞ to be the set of all bounded sequences $(x_1, x_2, \cdots)$ of complex numbers. Then l_∞ is a complex vector space with norm

$$\|\mathbf{x}\| = \sup_{n\geq 1} |x_n|.$$

4. Let $L_1(\mathbb{R})$ denote the set of all complex-valued Lebesgue integrable functions on $\mathbb{R}$. Then $L_1(\mathbb{R})$ is a complex vector space with norm[2]

$$\|f\| = \int_{\mathbb{R}} |f(t)|\, \mathrm{d}t.$$

Once can show that the normed spaces $C(a, b)$, l_1, l_∞ and $L_1(\mathbb{R})$ as defined above are Banach spaces.

Linear operators in normed spaces

Let V and W be normed spaces. A *linear operator* from V into W is a mapping $T : V \longrightarrow W$ such that $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$ whenever $v, w \in V$ and $\alpha, \beta \in \mathbb{C}$. Often one writes Tv rather than $T(v)$. One says that T is a *bounded* linear operator if there is a constant $c \in \mathbb{R}$ such that

$$\|Tv\| \leq c\|v\|, \quad \forall\ v \in V.$$

Examples

1. Suppose $V = W = C(0, 1)$ and let $f \in C(0, 1)$. Define T as a mapping from V to W by

$$(Tf)(t) = \int_0^t f(\eta)\, \mathrm{d}\eta.$$

Then $T : V \longrightarrow W$ is a linear operator, and

[2] Strictly, one should define an equivalence relation on $L_1(\mathbb{R})$ by writing $f \equiv g$ if $f(t) = g(t)$ almost everywhere. Then one denotes by $[f]$ the equivalence class for $f \in L_1(\mathbb{R})$. The correct vector space is then $\mathcal{L}_1(\mathbb{R}) = \{[f] | f \in L_1(\mathbb{R})\}$.

$$
\begin{aligned}
|TF(t)| &= \left| \int_0^t f(\eta)\,\mathrm{d}\eta \right| \\
&\leq \int_0^t |f(\eta)|\,\mathrm{d}\eta \\
&\leq \int_0^t \|f\|\,\mathrm{d}\eta \\
&= t\,\|f\| \\
&\leq \|f\|\,.
\end{aligned}
$$

Whence $\|Tf\| \leq \|f\|$, and so in this case T is bounded and $\|T\| \leq 1$.

T as defined above is an integral operator, cf. Talenti [16].

2. Let $V = W = C(0,1)$ and let k be a continuous complex-valued function on $[0,1] \times [0,1]$, i.e. on the set $\{(s,t) \mid 0 \leq s \leq 1,\ \ 0 \leq t \leq 1\}$. For $f \in C(0,1)$ one defines Tf on $[0,1]$ by

$$(Tf)(\eta) = \int_0^1 k(\eta,t)f(t)\,\mathrm{d}t.$$

Then T is a linear operator, in fact, a bounded linear operator.

Let V be a normed vector space. A linear operator from V into $\mathbb{C}$ is said to be a *linear functional*.

With the aid of the Hahn-Banach theorem, one may show that if V is a normed space, v is a non-zero element in V, then there exists a bounded linear functional h on V such that $\|h\| = 1$ and $h(v) = \|v\|$.

Hilbert spaces

Let V be a complex vector space. An *inner product* on V is a complex valued function (x,y), defined for all $x, y \in V$, which satisfies

1. $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$,
2. $(y,x) = \overline{(x,y)}$,
3. $(x,x) \geq 0$ with $(x,x) = 0$ if and only if $x = 0$,

for all $x, y \in V$ and for all $\alpha, \beta \in \mathbb{C}$, where the overbar denotes complex conjugate. The space V with the inner product $(\cdot, \cdot)$ is called an inner product space. In fact the equation

$$\|x\| = (x, x)^{1/2} \tag{6}$$

defines a norm on V.

The inner product of x, y satisfies the Cauchy-Schwarz inequality

$$|(x, y)| \leq \|x\| \, \|y\|.$$

The inner product space V is a *Hilbert space* if it is complete with respect to the norm defined by equation (6).

Examples

1. Define l_2 to be the set of complex sequences of the form $(s_1, s_2, \cdots)$ with $\sum_{n=1}^{\infty} |s_n|^2 < \infty$. Then with the inner product given by

   $$(s, t) = \sum_{n=1}^{\infty} s_n \overline{t_n}$$

   l_2 is a Hilbert space.

2. Denote by $L_2(\mathbb{R})$ the set of all complex-valued Lebesgue square measurable functions on $\mathbb{R}$, i.e. $|f|^2$ is Lebesgue measurable. Then with the inner product

   $$(f, g) = \int_{\mathbb{R}} f(s)\overline{g(s)} \, \mathrm{d}s$$

   L_2(R) is a Hilbert space.[3]

Linear operators on Hilbert spaces

Suppose H is a Hilbert space and let $B(H)$ denote the set of all bounded linear operators taking H into H. Then, for a given $T \in B(H)$ one may show there is a unique linear operator T^* in $B(H)$ such that

[3] Again, one should work with the space $\mathcal{L}_2(\mathbb{R})$ of equivalence classes of f.

$$(Tx, y) = (x, T^*y), \quad \forall\, x, y \in H.$$

Furthermore, $(T^*)^* = T$ and $\|T^*\| = \|T\|$.

The operator T^* is called the *adjoint* of T. When $T = T^*$ the operator is said to be *self-adjoint*.

Linear operators play a very important role in the theory of thermal stresses. In particular, the adjoint operator and self-adjoint operators have a key role.

If a linear operator is not bounded then we say it is *unbounded*. The theory of unbounded linear operators is more difficult than the bounded case, cf. Lions [12]; Smirnov [14]. However, one may still proceed. The key is to have Hilbert spaces V and H with V dense in H and the (unbounded) linear operator T mapping V into H. This requirement is necessary in order to define the adjoint operator uniquely, cf. Naimark [13], and the argument may also be found in Ames and Straughan [2], p. 8.

Additional references for functional analysis and function spaces are Adams [1]; Kufner et al [11]; Zenisek [17] and the references cited therein.

Well-posed problems

To define the notion of a well-posed problem, we need the concept of continuous dependence on the data. We follow the notation of John [10].

Consider a problem with a set, U, of solutions and a set, F, of data. Let A be the mapping from the set of data to the set of solutions. In order that the difference of two data terms or solutions is meaningful, we assume that the solutions to the problem are defined on a subset of U, say R, and the data is defined on a subset of F, say G, such that R and G are normed linear spaces with $\|\cdot\|_R$ and $\|\cdot\|_G$ as the norms on R and G respectively.

For any subset S, of R, with norm $\|\cdot\|_S$, if $u_1, u_2 \in U$, and $f_1, f_2 \in F$, such that $u_1 = Af_1$, $u_2 = Af_2$, we make the following definition. The mapping A is *Hölder continuous* at f_1, if and only if

$$\sup_{u_2 \in S} \|u_1 - u_2\|_S < M\epsilon^\alpha \text{ whenever } \|f_1 - f_2\|_G < \epsilon$$

where M, α are positive constants depending on U and S.

In a thermal stress problem, we let $R = \{u(t) \in U \mid t \in [0, T)\}$, where $[0, T)$ refers to an interval of time, $S = \{u(t) \in U \mid t \in [0, T_1),\ T_1 \le T\}$, and $G = \big\{f \in F \mid$ such that there exists $u \in U$ with $u(0) = f\big\}$.

The definition of Hölder continuity now leads to a way to discuss continuous dependence of the solution on the initial data of the problem. We shall refer to the phenomenon of continuous dependence of the solution on the initial data with respect to time as *stability*.

The solution $u_1(\cdot, t)$ is *Hölder stable* on the interval $0 \le t \le T$, if and only if, given $\epsilon (> 0)$, then

$$\sup_{0 \le t \le t_1 < T} \|u_1(\cdot, t) - u_2(\cdot, t)\|_t < C\epsilon^\alpha \text{ whenever } \|u_1(\cdot, 0) - u_2(\cdot, 0)\|_0 < \epsilon$$

for all $u_2(\cdot, 0) \in F$, where $\alpha \in (0, 1]$, $\|\cdot\|_t$ and $\|\cdot\|_0$ are norms defined on the solutions at the time t and initially, respectively, and C is a positive constant independent of ϵ.

This is a weaker definition of stability than that of Lyapunov. For completeness we include a definition of Lyapunov stability.

The solution $u_1(\cdot, t)$ is *stable in the sense of Lyapunov*, if and only if, given $\epsilon (> 0)$, there exists $\delta(\epsilon)$, such that

$$\sup_{t \in [0, \infty)} \|u_1(\cdot, t) - u_2(\cdot, t)\|_t < \epsilon \text{ whenever } \|u_1(\cdot, 0) - u_2(\cdot, 0)\|_0 < \delta$$

for all $u_2(\cdot, 0) \in F$.

The initial value problem for the backward heat equation

To illustrate the concepts of Hölder and Lyapunov stability, we consider the equation for temperature in a rigid body as discussed in the Introduction, but now in one space dimension, and for *the backward in time problem*. It is sufficient to transform t to $-t$ in (3) and then consider the problem

$$\frac{\partial \theta}{\partial t} + \frac{\partial^2 \theta}{\partial x^2} = 0, \; x \in [0, l], \quad t \in [0, T],$$
$$u(0,t) = u(l,t) = 0,$$
$$u(x,0) = \frac{C}{m} \sin\left(\frac{m\pi x}{l}\right).$$

One may show the unique solution to this problem is

$$u(x,t) = \frac{C}{m} \exp\left(\frac{m^2\pi^2 t}{l^2}\right) \sin\left(\frac{m\pi x}{l}\right).$$

Observe that as $m \longrightarrow \infty$, $|u(x,0)| \longrightarrow 0$, but $|u(x,t)| \longrightarrow \infty$. Hence, the solution does not depend continuously on the data in the sense of Lyapunov. However, one may demonstrate that the solution does depend Hölder continuously on the initial data on compact subintervals of $[0, T)$.

Well-posed problems in the sense of Hadamard

This definition was introduced by the French mathematician J.A. Hadamard in the early part of the twentieth century.

A problem is said to be *well-posed in the sense of Hadamard* if there exists a set F of data and a set U of solutions such that for every $f \in F$,

1. There exists a corresponding $u \in U$,
2. u is the unique member of U corresponding to f,
3. u depends continuously on f.[4]

[4] This could be Lyapunov or Hölder continuous dependence, depending on what one wants to achieve.

The requirement of a well-posed problem is a very important one in the theory of thermal stresses.

Abstract operator equations

It is sometimes desirable in some thermal stress problems to treat an abstract operator equation. In this way, one may be able to assess questions of well-posedness directly from an abstract equation which encompasses many particular problems in thermal stresses.

Smirnov [14], p. 549, shows that the linear operator $-\Delta = \sum_{i=1}^{3} \partial^2/\partial x_i^2$ is an unbounded operator on the Hilbert space $L_2(\Omega)$. The domain of this operator will be densely defined in $L_2(\Omega)$, where here Ω is a bounded domain in $\mathbb{R}^3$.

We now let V be a densely defined subset of a Hilbert space H and consider the equation

$$Au_{tt} = Lu + f \tag{7}$$

where A and L are, in general, unbounded linear operators taking V into H, and $f : H \longrightarrow H$ is a source term.

As a specific example of equation (7) in the theory of thermal stresses we might consider the equations for a linear thermoelastic body of type II in the sense of Green and Naghdi. The equations for such a thermoelastic body in the isotropic case are given by, e.g. Straughan [15], p. 54, as

$$\left.\begin{aligned} \rho_0 \ddot{u}_i &= \rho_0 b_i - E_1 \theta_{,i} + \mu \Delta u_i + (\lambda + \mu) u_{j,ij} \\ \rho_0 c \ddot{\theta} &= \rho_0 \dot{r} + \kappa \Delta \theta - \theta_0 E_1 \ddot{u}_{i,i} \end{aligned}\right\} \tag{8}$$

Here standard indicial notation is employed, u_i denotes displacement, θ the temperature, ρ_0 the density, c the specific heat, b_i and r are the externally supplied body force and temperature, Δ is the Laplacian, θ_0 and E_1 are constants, λ and μ are the

Lamé constants and κ is the thermal conductivity. A superposed dot denotes $\partial/\partial t$ and equations (8) are defined on the domain $\Omega \times [0,T)$, Ω being a bounded domain in $\mathbb{R}^3$, with T a number. Equation (8) must be supplemented with boundary and initial conditions and we take these as

$$\left.\begin{aligned} u_i(\mathbf{x},t) &= u_i^{\Gamma}(\mathbf{x},t), \\ \theta(\mathbf{x},t) &= \theta^{\Gamma}(\mathbf{x},t), \end{aligned}\right\} \quad \mathbf{x} \in \Gamma,\ t > 0, \tag{9}$$

and

$$\left.\begin{aligned} u_i(\mathbf{x},0) &= u_i^{0}(\mathbf{x}), \\ \theta(\mathbf{x},0) &= \theta^{0}, \end{aligned}\right\} \quad \mathbf{x} \in \Omega \tag{10}$$

where Γ is the boundary of Ω, and u_i^{Γ}, θ^{Γ}, u_i^0, θ^0 are prescribed functions.

To identify (8)–(10) as a special case of (7) we put $\mathbf{u} = (u_1, u_2, u_3, \theta)^T$ and $\mathbf{f} = (\rho_0 b_1, \rho_0 b_2, \rho_0 b_3, \rho_0 \dot{r})^T$. We then define the space H to be $(L_2(\Omega))^4$ and the domain V to be the set

$$\left\{(\mathbf{u},\theta) \in C^2(\Omega) \mid \mathbf{u} = \mathbf{u}^{\Gamma} \text{ on } \Gamma, \theta = \theta^{\Gamma} \text{ on } \Gamma\right\}.$$

The linear operators A and L are then matrix operators

$$A = \begin{pmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & \rho_0 & 0 & 0 \\ 0 & 0 & \rho_0 & 0 \\ \theta_0 E_1 \frac{\partial}{\partial x} & \theta_0 E_1 \frac{\partial}{\partial y} & \theta_0 E_1 \frac{\partial}{\partial z} & \rho_0 c \end{pmatrix}$$

and

$$L = \begin{pmatrix} \mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x^2} & (\lambda+\mu)\frac{\partial^2}{\partial x \partial y} & (\lambda+\mu)\frac{\partial^2}{\partial x \partial z} & -E_1 \frac{\partial}{\partial x} \\ (\lambda+\mu)\frac{\partial^2}{\partial x \partial y} & \mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial y^2} & (\lambda+\mu)\frac{\partial^2}{\partial y \partial z} & -E_1 \frac{\partial}{\partial y} \\ (\lambda+\mu)\frac{\partial^2}{\partial x \partial z} & (\lambda+\mu)\frac{\partial^2}{\partial y \partial z} & \mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial z^2} & -E_1 \frac{\partial}{\partial z} \\ 0 & 0 & 0 & \kappa\Delta \end{pmatrix}$$

Another useful example taken from thermal stresses is to consider a thermoelastic body of type III, as given by Green and Naghdi, see e.g. Straughan [15], p. 57.

For this class of material the linearized isotropic equations are similar to those of (8) but have an extra dissipation term in θ present.[5] For thermoelasticity of type III, the vector equation (8), still holds. However, (8_2) is replaced by

$$\rho_0 c\ddot{\theta} = \rho_0 \dot{r} + \kappa \Delta\theta + \kappa^* \Delta\dot{\theta} - \theta_0 E_1 \ddot{u}_{i,i}. \tag{11}$$

Here κ^* is a positive constant. The boundary-initial value problem comprised of (8_1), (11) together with (9) and (10) may be regarded as an example of the abstract operator equation

$$Au_{tt} + Bu_t = Lu + f.$$

To see this V and H are as before, as are u, A, L and f. The linear operator B is

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\kappa^* \Delta \end{pmatrix}.$$

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[5] In fact, equations (8) are sometimes said to describe thermoelasticity without dissipation.

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