Angular asymptotics for random walks

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Abstract

We study the set of directions asymptotically explored by a spatially homogeneous random walk in d-dimensional Euclidean space. We survey some pertinent results of Kesten and Erickson, make some further observations, and present some examples. We also explore links to the asymptotics of one-dimensional projections, and to the growth of the convex hull of the random walk.

Key words: Random walk; recurrent set; spherical asymptotics; asymptotic direction; convex hull; exceptional projections.

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1 Introduction

In this paper we examine some aspects of the way in which a random walk in d dimensions explores space, specifically through the limit points of the trajectory projected onto the sphere, and related questions concerning the growth of the convex hull of the walk. We ask, roughly speaking, in which directions does the walk grow without bound?

Let $d \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables in $\mathbb{R}^d$, and define the associated random walk $(S_n; n \in \mathbb{Z}_+)$ by $S_0 := \mathbf{0}$ and $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$; here and subsequently $\mathbf{0}$ is the origin in $\mathbb{R}^d$ and $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. We suppose throughout that S_n is genuinely d-dimensional, i.e., $\operatorname{supp} X$ is not contained in a $(d-1)$-dimensional subspace of $\mathbb{R}^d$.

Denote by $\mathbf{x} \cdot \mathbf{y}$ the Euclidean inner product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, and by $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^d$. Set $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$. For $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ define $\hat{\mathbf{x}} := \mathbf{x}/\|\mathbf{x}\|$; we also set $\hat{\mathbf{0}} := \mathbf{0}$. We view vectors in $\mathbb{R}^d$ as column vectors where necessary. Whenever the appropriate expectation exists, we write $\mu := \mathbb{E}\, X$, so $\mu \in \mathbb{R}^d$ is the mean drift vector of the random walk.

In Section 2 we look at the limit points in $\mathbb{S}^{d-1}$ of the sequence $\hat{S}_0, \hat{S}_1, \ldots$, drawing on closely related work of Kesten and Erickson [9–11, 19]. In particular, an adaptation of an idea of Kesten shows that the limit set is a.s. equal to a deterministic closed $\mathcal{D} \subseteq \mathbb{S}^{d-1}$ (see Theorem 2.1). In Section 3 we make more explicit the connection to the work of Kesten and Erickson [9–11, 19] on limit sets graded by particular speeds of growth. Section 4

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considers the special case where $\mathcal{D}$ has a single element, in which the walk is transient with a limiting direction. In Section 5 we make some observations about the case where the walk has increments with mean zero (zero drift). Section 6 presents an argument due to Erickson which shows that an arbitrary closed $\mathcal{D} \subseteq \mathbb{S}^{d-1}$ can be achieved as the limit set by constructing a random walk with suitable heavy-tailed increments (Theorem 6.1). In Section 7 we introduce some relevant convexity ideas. Section 8 turns to considering the asymptotics of the one-dimensional projections $S_n \cdot \mathbf{u}$, $\mathbf{u} \in \mathbb{S}^{d-1}$. Section 9 studies the convex hull of the trajectory, and draws some connections to the preceding sections. In Section 10 we present some examples. These illustrate, for instance, that while walks whose increments are symmetric and have zero mean must have $\mathcal{D} = \mathbb{S}^{d-1}$ when $d = 2$ (Proposition 5.2), for $d \geq 4$ the set $\mathcal{D}$ can have measure zero in $\mathbb{S}^{d-1}$ (Example 10.3).

We make a few historical comments. As observed by Blackwell, and Chung and Derman (see [15, p. 493] and [2, p. 658]), it is a consequence of the Hewitt–Savage zero–one law that $\mathbb{P}(S_n \in A \text{ i.o.}) \in \{0, 1\}$ for any Borel set $A \subseteq \mathbb{R}^d$. Those authors raised the question of classifying sets A accordingly for a given random walk (see e.g. [5, p. 447]). For bounded sets A containing the origin in their interior, the question is that of recurrence vs. transience, and is answered by Chung and Fuchs [6].

Attention focused on determining infinite sets A visited infinitely often by (transient) random walks on $\mathbb{Z}^d$, $d \geq 3$, most notably for the case where the random walk converges to Brownian motion, where a classification of recurrent sets A is available in the form of 'Wiener's test': for the case of simple symmetric random walk, see [3, 4, 17], for bounded and symmetric increments, see [23, §6.5], and for increments with zero mean and finite second moments, see [18, 30, 31]. Wiener's test and its generalizations [4, 22, 27] give analytic criteria in terms of the capacity of A or Green's functions of the walk. An early paper of Doney [7] showed that Wiener's test can yield very useful information, but, according to Spitzer, "in general the computations are prohibitively difficult" [30, p. 320]. The present paper addresses questions related to the transience or recurrence of sets A that are cones or half-spaces.

2 Recurrent directions

We say $\mathbf{u} \in \mathbb{S}^{d-1}$ is a *recurrent direction* for S_n if the sequence $\hat{S}_n$ has an accumulation point at $\mathbf{u}$, i.e., if $\hat{S}_n$ has $\mathbf{u}$ as a subsequential limit. Let L be the (random) set of all recurrent directions for S_n; equivalently,

$$L := \{\mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| = 0\}.$$

Note that in L the possible accumulation point at $\mathbf{0}$ is excluded. Also define

$$\mathcal{D} := \{\mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| = 0, \text{ a.s.}\},$$

i.e., the set of all a.s. recurrent directions for S_n.

For $d = 1$, ruling out the degenerate case where $\mathbb{P}(X = 0) = 1$, the well known trichotomy (see e.g. [8, Theorem 4.1.2]) states that either (i) $S_n \to +\infty$, a.s., (ii) $S_n \to -\infty$, a.s., or (iii) $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = +\infty$, a.s., corresponding to (i) $\mathcal{D} = \{+1\}$, (ii) $\mathcal{D} = \{-1\}$, and (iii) $\mathcal{D} = \{-1, +1\}$ (this latter case includes the case where S_n is recurrent). Our primary interest here is the case $d \geq 2$.

The following result is a consequence of a more general statement of Erickson [9] (see also §3 below), who pointed out that it can be obtained by adapting an argument of

Kesten [19] (see also Lemma 1 of [21] for a generalization attributed to Neidhardt). An alternative proof of the fact that L is deterministic could be obtained by appealing to a general zero–one result for random closed sets such as Proposition 1.1.30 of [26], having first established that L is closed.

Theorem 2.1. *The set $\mathcal{D}$ is a non-empty, closed subset of $\mathbb{S}^{d-1}$, and $\mathbb{P}(L = \mathcal{D}) = 1$.*

We work towards the proof of Theorem 2.1. For $\mathbf{u} \in \mathbb{S}^{d-1}$ and $r > 0$, define the set

$$C(\mathbf{u}; r) := \{\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\} : \|\hat{\mathbf{x}} - \mathbf{u}\| < r\}$$

and the event

$$A(\mathbf{u}; r) := \{S_n \in C(\mathbf{u}; r) \text{ i.o.}\}.$$

By the Hewitt–Savage zero–one law (see e.g. [8, Theorem 4.1.1]), $\mathbb{P}(A(\mathbf{u}; r)) \in \{0, 1\}$.

Let $B(\mathbf{x}; r) := \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\| < r\}$ denote the open Euclidean ball centred at $\mathbf{x} \in \mathbb{R}^d$ with radius $r > 0$, and for $\mathbf{u} \in \mathbb{S}^{d-1}$ let $B_s(\mathbf{u}; r) := \mathbb{S}^{d-1} \cap B(\mathbf{u}; r)$. For $A \subseteq \mathbb{R}^d$, we write $\operatorname{cl} A$ for the closure of A in $\mathbb{R}^d$ in the usual topology.

Lemma 2.2. *For any $\mathbf{u} \in \mathbb{S}^{d-1}$ and any $r > 0$, we have*

$$\{L \cap B_s(\mathbf{u}; r) \neq \emptyset\} \subseteq A(\mathbf{u}; r) \subseteq \{L \cap \operatorname{cl} B_s(\mathbf{u}; r) \neq \emptyset\}.$$

Proof. First note that

$$A(\mathbf{u}; r) = \{\hat{S}_n \in B_s(\mathbf{u}; r) \text{ i.o.}\}.$$

Hence $A(\mathbf{u}; r)$ implies that $\hat{S}_n \in \operatorname{cl} B_s(\mathbf{u}; r)$ i.o., and since $\operatorname{cl} B_s(\mathbf{u}; r)$ is compact, $\hat{S}_n$ must have an accumulation point in $\operatorname{cl} B_s(\mathbf{u}; r)$. On the other hand, if $\hat{S}_n$ has an accumulation point in $B_s(\mathbf{u}; r)$, then since $B_s(\mathbf{u}; r)$ is open in $\mathbb{S}^{d-1}$ we have $\hat{S}_n \in B_s(\mathbf{u}; r)$ i.o. □

The following continuity property is a key ingredient in the proof of Theorem 2.1.

Lemma 2.3. *Given any sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots \in \mathbb{S}^{d-1}$, and any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,*

$$\left| \limsup_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{u}) - \limsup_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{v}) \right| \leq \|\mathbf{u} - \mathbf{v}\|.$$

Proof. Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Then

$$\begin{aligned} \limsup_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{v}) &\leq \limsup_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{u}) + \limsup_{n\to\infty} (\mathbf{x}_n \cdot (\mathbf{v} - \mathbf{u})) \\ &\leq \limsup_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{u}) + \|\mathbf{v} - \mathbf{u}\|, \end{aligned}$$

since $\|\mathbf{x}_n\| = 1$. With a similar argument in the other direction, we get the result. □

Lemma 2.4. *The set $\mathcal{D}$ is closed in $\mathbb{S}^{d-1}$.*

Proof. Note that for any $\mathbf{u} \in \mathbb{S}^{d-1}$,

$$\|\hat{S}_n - \mathbf{u}\|^2 = (\hat{S}_n - \mathbf{u}) \cdot (\hat{S}_n - \mathbf{u}) = 1 + \mathbf{1}\{S_n \neq \mathbf{0}\} - 2\hat{S}_n \cdot \mathbf{u},$$

so that

$$\liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| = 0 \text{ if and only if } \limsup_{n\to\infty} (\hat{S}_n \cdot \mathbf{u}) = 1.$$

Thus

$$\mathbb{S}^{d-1} \setminus \mathcal{D} = \left\{\mathbf{u} \in \mathbb{S}^{d-1} : \limsup_{n\to\infty}(\hat{S}_n \cdot \mathbf{u}) < 1 \text{ a.s.}\right\}.$$

Consider $\mathbf{u} \in \mathbb{S}^{d-1} \setminus \mathcal{D}$. By the Hewitt–Savage theorem, $\limsup_{n\to\infty}(\hat{S}_n \cdot \mathbf{u}) = c$ a.s. for a constant $c < 1$. Lemma 2.3 shows that for any $\mathbf{v} \in \mathbb{S}^{d-1}$ with $\|\mathbf{u} - \mathbf{v}\| \leq \frac{1-c}{2}$, a.s.,

$$\limsup_{n\to\infty}(\hat{S}_n \cdot \mathbf{v}) \leq c + \frac{1-c}{2} = \frac{1+c}{2} < 1,$$

so that $\mathbf{v} \in \mathbb{S}^{d-1} \setminus \mathcal{D}$. Thus $\mathbb{S}^{d-1} \setminus \mathcal{D}$ is open in $\mathbb{S}^{d-1}$. □

Now we can complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We adapt, in part, an argument from the proof of Theorem 1 of [19]. We call a ball $B_s(\mathbf{u}; r)$ *rational* if $\mathbf{u} \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ and $r \in \mathbb{Q} \cap (0, \infty)$. Note that $\mathbb{S}^{d-1} \cap \mathbb{Q}^d$ is dense in $\mathbb{S}^{d-1}$, as follows from an argument based on stereographic projection (see e.g. [29]). Let $\mathcal{R}$ denote the (countable) set of all rational balls, and set

$$\mathcal{C} := \{B \in \mathcal{R} : \mathbb{P}(\hat{S}_n \in B \text{ i.o.}) = 1\}.$$

Then since $\mathcal{R}$ is countable, and, by the Hewitt–Savage theorem, $\mathbb{P}(\hat{S}_n \in B \text{ i.o.}) \in \{0, 1\}$ for any $B \in \mathcal{R}$, we have

$$\mathbb{P}(\hat{S}_n \in B \text{ i.o. for all } B \in \mathcal{C} \text{ but for no } B \in \mathcal{R} \setminus \mathcal{C}) = 1. \tag{2.1}$$

Observe that

$$\mathbf{u} \in L \text{ if and only if } \hat{S}_n \in B \text{ i.o. for every } B \in \mathcal{R} \text{ with } \mathbf{u} \in B, \tag{2.2}$$

and so $\mathbf{u} \in \mathcal{D}$ if and only if

$$\mathbb{P}(\hat{S}_n \in B \text{ i.o. for every } B \in \mathcal{R} \text{ with } \mathbf{u} \in B) = 1. \tag{2.3}$$

In particular, if $B \in \mathcal{R}$ contains some $\mathbf{u} \in \mathcal{D}$, then $B \in \mathcal{C}$. With (2.1), this means that

$$\mathbb{P}(\text{for all } \mathbf{u} \in \mathcal{D}, \hat{S}_n \in B \text{ i.o. for every } B \in \mathcal{R} \text{ with } \mathbf{u} \in B) = 1.$$

Together with (2.2), it follows that $\mathbb{P}(\mathcal{D} \subseteq L) = 1$.

Let $\mathcal{C}_k$ be the set of $B \in \mathcal{C}$ with $\operatorname{diam} B < 1/k$. Let $W_k := \cup\mathcal{C}_k$ and $W := \cap_{k\in\mathbb{N}} W_k$. Then it follows from (2.3) that $\mathbf{u} \in \mathcal{D}$ if and only if for every $k \in \mathbb{N}$ there exists some $B \in \mathcal{C}_k$ with $\mathbf{u} \in B$. That is, $\mathbf{u} \in \mathcal{D}$ if and only if $\mathbf{u} \in W$, i.e., $\mathcal{D} = W$.

Let $\mathcal{R}_k$ be the set of $B \in \mathcal{R}$ with $\operatorname{diam} B < 1/k$. Now let $M_k := \cup\{B \in \mathcal{R}_k : L \cap B \neq \emptyset\}$. Let $B \in \mathcal{R}$. Since B is open in $\mathbb{S}^{d-1}$, we have that $B \cap L \neq \emptyset$ implies that $\hat{S}_n \in B$ i.o. So $M_k \subseteq \cup\{B \in \mathcal{R}_k : \hat{S}_n \in B \text{ i.o.}\}$. Hence by (2.1) we have that $\mathbb{P}(M_k \subseteq \cup\mathcal{C}_k) = 1$, i.e., $\mathbb{P}(M_k \subseteq W_k) = 1$. It follows that $\mathbb{P}(\cap_{k\in\mathbb{N}} M_k \subseteq \mathcal{D}) = 1$. Note that if $\mathbf{u} \in L$, then for all $k \in \mathbb{N}$ we have $B \cap L \neq \emptyset$ for some $B \in \mathcal{R}_k$, so $\mathbf{u} \in M_k$ for all k; hence $L \subseteq \cap_{k\in\mathbb{N}} M_k$ a.s. Hence we conclude that $\mathbb{P}(L \subseteq \mathcal{D}) = 1$.

To prove that $\mathcal{D}$ is non-empty, taking $r = 2$ in Lemma 2.2 shows that $\hat{S}_n$ has at least one accumulation point in L, since $C(\mathbf{u}; 2) = \mathbb{R}^d \setminus \{\mathbf{0}\}$ and, since S_n is genuinely d-dimensional, $S_n \neq \mathbf{0}$ i.o., a.s. □

Here is an alternative characterization of the set $\mathcal{D}$.

Proposition 2.5. *We have that*

$$\mathcal{D} = \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbb{P}(A(\mathbf{u}; r)) = 1 \text{ for all } r > 0\}.$$

Proof. Define the set $\mathcal{D}' = \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbb{P}(A(\mathbf{u}; r)) = 1 \text{ for all } r > 0\}$. If $\mathbf{u} \in \mathcal{D}'$, then $\mathbb{P}(A(\mathbf{u}; 1/m)) = 1$ for all $m \in \mathbb{N}$, and so $\mathbb{P}(\cap_{m=1}^{\infty} A(\mathbf{u}; 1/m)) = 1$. In particular, $\mathbb{P}(\hat{S}_n \in B_s(\mathbf{u}; 1/m)$ i.o. for all $m \in \mathbb{N}) = 1$. In other words, a.s., $\liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| < 1/m$ for all $m \in \mathbb{N}$, and hence $\liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| = 0$, a.s., so $\mathbf{u} \in \mathcal{D}$. Thus $\mathcal{D}' \subseteq \mathcal{D}$.

On the other hand, suppose that $\mathbf{u} \in \mathbb{S}^{d-1} \setminus \mathcal{D}'$. Then there exists $r > 0$ such that $\mathbb{P}(A(\mathbf{u}; r)) < 1$, and, by the Hewitt–Savage theorem, in fact $\mathbb{P}(A(\mathbf{u}; r)) = 0$. Lemma 2.2 shows that $A(\mathbf{u}; r)^{\mathrm{c}} \subseteq \{L \cap B_s(\mathbf{u}; r) = \emptyset\}$ and hence $\mathbb{P}(L \cap B_s(\mathbf{u}; r) = \emptyset) = 1$. In particular, this means that $\mathbb{P}(\mathbf{u} \in L) = 0$ and so $\mathbf{u} \notin \mathcal{D}$. This shows that $\mathcal{D} \subseteq \mathcal{D}'$. □

We next show that the recurrent directions are determined solely by the behaviour of the walk at increasingly large distances from the origin. Define

$$L_\infty := \left\{ \mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty} \left(\frac{1}{1 + \|S_n\|} + \|\hat{S}_n - \mathbf{u}\| \right) = 0 \right\}, \tag{2.4}$$

and

$$\mathcal{D}_\infty := \left\{ \mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty} \left(\frac{1}{1 + \|S_n\|} + \|\hat{S}_n - \mathbf{u}\| \right) = 0, \text{ a.s.} \right\}.$$

In other words, $\mathbf{u} \in L_\infty$ if and only if there exists a (random) subsequence n_k of $\mathbb{Z}_+$ such that both $\lim_{k\to\infty} \|S_{n_k}\| = \infty$ and $\lim_{k\to\infty} \hat{S}_{n_k} = \mathbf{u}$. If $\mathbf{u} \in L_\infty$ we say that $\mathbf{u}$ is an *asymptotic direction* for the random walk. Clearly an asymptotic direction is a recurrent direction, so $\mathbb{P}(L_\infty \subseteq L) = 1$ and $\mathcal{D}_\infty \subseteq \mathcal{D}$.

Proposition 2.6. *If S_n is recurrent, then $\mathcal{D} = \mathcal{D}_\infty = \mathbb{S}^{d-1}$ and $\mathbb{P}(L = L_\infty = \mathbb{S}^{d-1}) = 1$.*

Proof. Suppose that S_n is recurrent. Since $\mathcal{D}_\infty \subseteq \mathcal{D}$ and $L_\infty \subseteq L$, it suffices to show that $\mathcal{D}_\infty = \mathbb{S}^{d-1}$ and $\mathbb{P}(L_\infty = \mathbb{S}^{d-1}) = 1$. Proposition A.1 shows that there is some $h \in (0, \infty)$ such that, a.s., for every $\mathbf{x} \in \mathbb{R}^d$, $S_n \in B(\mathbf{x}; h)$ i.o. But for every $\mathbf{u} \in \mathbb{S}^{d-1}$, every $r > 0$, and every $R \in (h, \infty)$, $C(\mathbf{u}; r)$ contains some $B(\mathbf{x}; h)$ with $\|\mathbf{x}\| > 2R$, so that, a.s., for every $\mathbf{u} \in \mathbb{S}^{d-1}$, every $r > 0$, and every $R \in (h, \infty)$, there is a subsequence n_k along which $\|\hat{S}_{n_k} - \mathbf{u}\| < r$ and $\|S_{n_k}\| > R$. This shows that $\mathbb{P}(L_\infty = \mathbb{S}^{d-1}) = 1$, and essentially the same argument implies that $\mathcal{D}_\infty = \mathbb{S}^{d-1}$. □

Corollary 2.7. *If $\mathcal{D} \neq \mathbb{S}^{d-1}$, then S_n is transient.*

The next result says that, a.s., the sets of recurrent and asymptotic directions coincide.

Theorem 2.8. *We have $\mathcal{D}_\infty = \mathcal{D}$, and $\mathbb{P}(L_\infty = \mathcal{D}) = 1$.*

Proof. The recurrent case is contained in Proposition 2.6; thus suppose that S_n is transient. Then since $\|S_n\| \to \infty$ a.s., we have that $\mathbb{P}(L = L_\infty) = 1$ and $\mathcal{D} = \mathcal{D}_\infty$. Combined with Theorem 2.1, this gives the result. □

Next we show how a distributional limit gives rise to recurrent directions. Here and elsewhere, '$\xrightarrow{\mathrm{d}}$' denotes convergence in distribution and 'supp' denotes the support of an $\mathbb{R}^d$-valued random variable.

Proposition 2.9. *(i) Suppose that there is a random vector $\zeta \in \mathbb{S}^{d-1}$ such that $\hat{S}_n \xrightarrow{\mathrm{d}} \zeta$ as $n \to \infty$. Then $\operatorname{supp} \zeta \subseteq \mathcal{D}$.*

(ii) Suppose there is a sequence a_n of positive real numbers and a random vector $\xi \in \mathbb{R}^d$ with $\mathbb{P}(\xi = \mathbf{0}) = 0$ such that $S_n/a_n \xrightarrow{\mathrm{d}} \xi$ as $n \to \infty$. Then $\operatorname{supp} \hat{\xi} \subseteq \mathcal{D}$.

Proof. For part (i), suppose that $\hat{S}_n \xrightarrow{\mathrm{d}} \zeta$. Then, for a given $\mathbf{u} \in \mathbb{S}^{d-1}$, for all but countably many $\varepsilon > 0$,

$$
\begin{aligned}
\mathbb{P}(\|\hat{S}_n - \mathbf{u}\| < \varepsilon \text{ i.o.}) &= \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{\|\hat{S}_m - \mathbf{u}\| < \varepsilon\} \right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m=n}^{\infty} \{\|\hat{S}_m - \mathbf{u}\| < \varepsilon\} \right) \\
&\geq \lim_{n\to\infty} \mathbb{P}(\|\hat{S}_n - \mathbf{u}\| < \varepsilon) \\
&= \mathbb{P}(\|\zeta - \mathbf{u}\| < \varepsilon),
\end{aligned}
$$

which is strictly positive provided $\mathbf{u} \in \operatorname{supp} \zeta$. It follows by the Hewitt–Savage theorem that if $\mathbf{u} \in \operatorname{supp} \zeta$, then $\mathbb{P}(\|\hat{S}_n - \mathbf{u}\| < \varepsilon \text{ i.o.}) = 1$ for all $\varepsilon > 0$, and hence $\mathbf{u} \in \mathcal{D}$.

For part (ii), we have that since $\mathbb{P}(\xi = \mathbf{0}) = 0$, and the function $\mathbf{x} \mapsto \hat{\mathbf{x}}$ is continuous on $\mathbb{R}^d \setminus \{\mathbf{0}\}$, the continuous mapping theorem implies that $\hat{S}_n \xrightarrow{\mathrm{d}} \hat{\xi}$, and then we may apply part (i). □

Here is a sufficient condition for $\mathcal{D} = \mathbb{S}^{d-1}$; if $d = 2$ the walk is recurrent and the result also follows from Proposition 2.6, while if $d \geq 3$ the walk is transient.

Corollary 2.10. *Suppose that $\mathbb{E}(\|X\|^2) < \infty$ and $\mu = \mathbf{0}$. Then $\mathcal{D} = \mathbb{S}^{d-1}$.*

Proof. By assumption and the central limit theorem, $n^{-1/2} S_n$ converges in distribution to a non-degenerate normal distribution. Proposition 2.9 then shows that $\mathbb{S}^{d-1} \subseteq \mathcal{D}$. □

3 Compactification and growth rates

Let $\overline{\mathbb{R}^d}$ denote the compactification of $\mathbb{R}^d$ obtained by adjoining the "sphere at ∞". More formally, $\overline{\mathbb{R}^d}$ is the compact metric space obtained by the completion of $\mathbb{R}^d$ with respect to the metric

$$
\rho(\mathbf{x}, \mathbf{y}) = \left\| \frac{\mathbf{x}}{1 + \|\mathbf{x}\|} - \frac{\mathbf{y}}{1 + \|\mathbf{y}\|} \right\|.
$$

Then we can represent $\overline{\mathbb{R}^d}$ as $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \mathbb{R}^d_\infty$ where $\mathbb{R}^d_\infty$ is in bijection to $\mathbb{S}^{d-1}$. We write elements of $\mathbb{R}^d_\infty$ as $\infty \cdot \mathbf{u}$ for $\mathbf{u} \in \mathbb{S}^{d-1}$. The metric ρ on $\mathbb{R}^d$ is equivalent to the Euclidean metric, and extended to $\overline{\mathbb{R}^d}$ it is such that $\mathbf{x}_n \in \mathbb{R}^d$ has $\mathbf{x}_n \to \infty \cdot \mathbf{u}$ for $\mathbf{u} \in \mathbb{S}^{d-1}$ if $\|\mathbf{x}_n\| \to \infty$ and $\hat{\mathbf{x}}_n \to \mathbf{u}$.

The set of accumulation points of $S_0, S_1, S_2, \ldots$, taken in $\overline{\mathbb{R}^d}$, thus consists of any accumulation points in $\mathbb{R}^d$ (a.s. there are none if S_n is transient) and accumulation points in $\mathbb{R}^d_\infty$ represented by the set $\mathcal{L}_\infty$ of asymptotic directions, as defined at (2.4).

Erickson [9], generalizing one-dimensional work of Kesten and himself [11, 19], considers a finer graduation of asymptotic directions. For $\alpha \in \mathbb{R}_+$, set

$$
\mathcal{L}^{>\alpha}_\infty := \left\{ \mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty} \left(\frac{n^\alpha}{1 + \|S_n\|} + \|\hat{S}_n - \mathbf{u}\| \right) = 0 \right\}.
$$

Then $\mathcal{L}_\infty^{>0} = \mathcal{L}_\infty$, while $\mathcal{L}_\infty^{>\alpha_2} \subseteq \mathcal{L}_\infty^{>\alpha_1}$ for any $0 \le \alpha_1 \le \alpha_2 < \infty$. Similarly, set

$$\mathcal{D}_\infty^{>\alpha} := \left\{ \mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty} \left(\frac{n^\alpha}{1 + \|S_n\|} + \|\hat{S}_n - \mathbf{u}\| \right) = 0, \text{ a.s.} \right\}.$$

Roughly speaking, the set $\mathcal{L}_\infty^{>\alpha}$ consists of those directions in which the walk grows at rate faster than n^α. Also for $\alpha > 0$ set

$$\mathcal{A}^\alpha = \left\{ \mathbf{x} \in \mathbb{R}^d : \liminf_{n\to\infty} \left\| n^{-\alpha} S_n - \mathbf{x} \right\| = 0 \right\}, \tag{3.1}$$

and $\mathcal{L}_\infty^\alpha = \{\hat{\mathbf{x}} : \mathbf{x} \in \mathcal{A}^\alpha \setminus \{\mathbf{0}\}\}$. Then $\mathcal{L}_\infty^\alpha \subseteq \mathcal{L}_\infty$ are those asymptotic directions in which the walk grows at rate precisely n^α.

Erickson [9,10] studies in detail $\mathcal{A}^\alpha$ and $\mathcal{L}_\infty^{>\alpha}$, with particular focus on the case $\alpha = 1$, which has some peculiar features. The version of Theorem 2.1 stated by Erickson [9, p. 802] is that $\mathbb{P}(\mathcal{L}_\infty^{>\alpha} = \mathcal{D}_\infty^{>\alpha}) = 1$, and $\mathcal{D}_\infty^{>\alpha}$ is a closed subset of $\mathbb{S}^{d-1}$.

For $d \ge 3$, the value $\alpha = 1/2$ is special, since a remarkable paper of Kesten [20] shows that $n^{-\alpha}\|S_n\| \to \infty$ for any $\alpha < 1/2$ and any genuinely d-dimensional random walk S_n in $\mathbb{R}^d$, $d \ge 3$. Thus for $d \ge 3$ we have $\mathcal{D}_\infty^{>\alpha} = \mathcal{D}_\infty$ for any $0 \le \alpha < 1/2$.

4 Limiting direction

By the Hewitt–Savage theorem, $\mathbb{P}(\lim_{n\to\infty} \hat{S}_n \text{ exists}) \in \{0,1\}$, and if the limit exists, then it is a.s. constant. If $\lim_{n\to\infty} \|S_n\| = \infty$ a.s. and $\lim_{n\to\infty} \hat{S}_n = \mathbf{u}$ a.s. for some $\mathbf{u} \in \mathbb{S}^{d-1}$, we say that S_n is *transient with limiting direction* $\mathbf{u}$.

Lemma 4.1. *Let* $\mathbf{u} \in \mathbb{S}^{d-1}$. *The following are equivalent.*

(i) $\mathcal{D} = \{\mathbf{u}\}$.

(ii) $\lim_{n\to\infty} \hat{S}_n = \mathbf{u}$, *a.s.*

(iii) S_n *is transient with limiting direction* $\mathbf{u}$.

Proof. The result will follow from the sequence of implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). That (iii) implies (ii) is trivial. If (ii) holds, then clearly $\mathbf{u} \in \mathcal{D}$, and for any $r > 0$ we have $\hat{S}_n \in B_s(\mathbf{u}; r)$ for all but finitely many n. For any $\mathbf{v} \in \mathbb{S}^{d-1} \setminus \{\mathbf{u}\}$, we may choose $r > 0$ sufficiently small so that $B_s(\mathbf{u}; r)$ and $B_s(\mathbf{v}; r)$ are disjoint, so that $\mathbb{P}(\hat{S}_n \in B_s(\mathbf{v}; r) \text{ i.o.}) = 0$, and hence Proposition 2.5 shows that $\mathbf{v} \notin \mathcal{D}$. Thus (i) holds.

Finally, suppose that (i) holds. Then Corollary 2.7 shows that S_n is transient, and in particular $S_n = \mathbf{0}$ only finitely often. By the Hewitt–Savage theorem, $\limsup_{n\to\infty} \|\hat{S}_n - \mathbf{u}\|$ is a.s. constant. If $\mathbf{u}$ is not a limiting direction for the walk, then this constant is strictly positive, so that, for some $\varepsilon > 0$, $\|\hat{S}_n - \mathbf{u}\| \ge \varepsilon$ i.o., a.s. Since the set $\{\mathbf{v} \in \mathbb{S}^{d-1} : \|\mathbf{v} - \mathbf{u}\| \ge \varepsilon\}$ is compact, it follows that $\hat{S}_n$ has an accumulation point $\mathbf{v} \neq \mathbf{u}$, and hence $\mathbf{v} \in \mathcal{D}$, which gives a contradiction. Hence (i) implies (iii). □

The following result is contained in Theorem 1.6.1(i) of [25].

Proposition 4.2. *Suppose that* $\mathbb{E}\|X\| < \infty$. *If* $\mu \neq \mathbf{0}$, *then* $\mathcal{D} = \{\hat{\mu}\}$.

Remark 4.3. If $\mu = \mathbf{0}$ there is no limiting direction: see Proposition 5.1 below.

Proof of Proposition 4.2. The strong law of large numbers (SLLN) shows that $n^{-1}S_n \to \mu$, a.s., and $n^{-1}\|S_n\| \to \|\mu\|$, a.s. If $\mu \neq \mathbf{0}$, then $\|S_n\| \to \infty$, so $S_n \neq \mathbf{0}$ for all but finitely many n, and then

$$\lim_{n\to\infty} \hat{S}_n = \lim_{n\to\infty} \frac{n^{-1}S_n}{n^{-1}\|S_n\|} = \hat{\mu}, \text{ a.s.}$$ □

5 The zero-drift case

In this section we turn to the case where the walk has zero drift, i.e., $\mu = \mathbf{0}$. If $d = 1$, then zero drift implies recurrence, and hence $\mathcal{D} = \{-1, +1\}$ (see e.g. [8, Theorem 4.2.7]). If $\mathbb{E}(\|X\|^2) < \infty$, then Corollary 2.10 shows that $\mathcal{D} = \mathbb{S}^{d-1}$. Thus the most interesting cases are when $d \geq 2$ and $\mathbb{E}(\|X\|^2) = \infty$. The following result contrasts with Proposition 4.2, and improves on Theorem 1.6.1(ii) of [25].

Proposition 5.1. *Suppose that $d \geq 2$, $\mathbb{E}\|X\| < \infty$, and $\mu = \mathbf{0}$. Then $\mathcal{D}$ is uncountable.*

In the case where $d = 2$, we can say more. For measurable $A \subseteq \mathbb{S}^{d-1}$ we write $|A|$ for the Haar measure of A. Write '$\stackrel{d}{=}$' for equality in distribution; $X \stackrel{d}{=} -X$ means that random variable $X \in \mathbb{R}^d$ has a centrally symmetric distribution.

Proposition 5.2. *Suppose that $d = 2$, $\mathbb{E}\|X\| < \infty$, and $\mu = \mathbf{0}$.*

(i) We have $|\mathcal{D}| \geq \frac{1}{2}|\mathbb{S}^1|$.

(ii) If $X \stackrel{d}{=} -X$, then $\mathcal{D} = \mathbb{S}^1$.

Remarks 5.3. (a) Example 10.2 below gives a walk with $d = 2$, $X \stackrel{d}{=} -X$, and $\mathbb{E}\|X\| = \infty$, for which $\mathcal{D}$ has only two elements, so the condition $\mathbb{E}\|X\| < \infty$ in Proposition 5.2 cannot be removed.

(b) Example 10.3 below gives a family of random walks in $\mathbb{R}^d$, $d \geq 4$, for which $\mu = \mathbf{0}$ and $X \stackrel{d}{=} -X$, but $\mathcal{D}$ is a set of measure zero, so in higher dimensions the hypotheses of Proposition 5.2 do not guarantee that $\mathcal{D}$ occupies a positive fraction of the sphere.

For further results in the zero-drift case, see Corollary 9.4 below. In the rest of this section we prove Propositions 5.1 and 5.2.

Lemma 5.4. *Suppose that $d \geq 2$, $\mathbb{E}\|X\| < \infty$, and $\mu = \mathbf{0}$. Then for every $\mathbf{u} \in \mathbb{S}^{d-1}$, there exists $\mathbf{v} \in \mathcal{D}$ with $\mathbf{u} \cdot \mathbf{v} = 0$.*

Proof. If S_n is recurrent, then the result follows from Proposition 2.6. So suppose that S_n is transient. Fix $\mathbf{u} \in \mathbb{S}^{d-1}$. For $\varepsilon > 0$, let $O_\varepsilon(\mathbf{u}) = \{\mathbf{v} \in \mathbb{S}^{d-1} : |\mathbf{v} \cdot \mathbf{u}| \leq \varepsilon\}$. Since $\mathbb{E}(X \cdot \mathbf{u}) = \mu \cdot \mathbf{u} = 0$, the random walk $S_n \cdot \mathbf{u}$ is recurrent, and $\liminf_{n \to \infty} |S_n \cdot \mathbf{u}| < \infty$. Since S_n is transient we have $\|S_n\| \to \infty$, so that $\liminf_{n \to \infty} |\hat{S}_n \cdot \mathbf{u}| = 0$. In other words, for every $\varepsilon > 0$ we have that for infinitely many $n \in \mathbb{N}$, $\hat{S}_n$ is in the compact set $O_\varepsilon(\mathbf{u})$. Hence $O_\varepsilon(\mathbf{u})$ must contain an element of $\mathcal{D}$. Thus there is a sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots \in \mathcal{D}$ with $|\mathbf{v}_j \cdot \mathbf{u}| \to 0$, and (since $\mathcal{D}$ is compact) this sequence has a subsequence which converges to $\mathbf{v} \in \mathcal{D}$ with $\mathbf{v} \cdot \mathbf{u} = 0$. □

Proof of Proposition 5.1. Suppose, for the purpose of deriving a contradiction, that $\mathcal{D}$ is countable. Set $O(\mathbf{u}) = \{\mathbf{v} \in \mathbb{S}^{d-1} : \mathbf{v} \cdot \mathbf{u} = 0\}$. Then $O = \cup_{\mathbf{u} \in \mathcal{D}} O(\mathbf{u})$ is a countable union of subsets of $\mathbb{S}^{d-1}$ of measure zero (since each $O(\mathbf{u})$ is a copy of $\mathbb{S}^{d-2}$). Thus O is measure zero, and so there exists $\mathbf{v} \in \mathbb{S}^{d-1} \setminus O$. This $\mathbf{v}$ has $\mathbf{v} \cdot \mathbf{u} \neq 0$ for all $\mathbf{u} \in \mathcal{D}$, which contradicts Lemma 5.4. Hence $\mathcal{D}$ cannot be countable. □

To prove Proposition 5.2, we need some additional notation. Let

$$\mathcal{D}_1 := \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbf{u} \in \mathcal{D},\ -\mathbf{u} \notin \mathcal{D}\},$$

$$\mathcal{D}_2 := \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbf{u} \in \mathcal{D},\ -\mathbf{u} \in \mathcal{D}\},$$
$$\mathcal{C}_1 := \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbf{u} \notin \mathcal{D},\ -\mathbf{u} \in \mathcal{D}\} = -\mathcal{D}_1,$$
$$\mathcal{C}_2 := \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbf{u} \notin \mathcal{D},\ -\mathbf{u} \notin \mathcal{D}\}.$$

Then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathbb{S}^{d-1} \setminus \mathcal{D} = \mathcal{C}_1 \cup \mathcal{C}_2$.

Lemma 5.5. *Suppose that $d = 2$, $\mathbb{E}\,\|X\| < \infty$, and $\mu = \mathbf{0}$. Then $\mathcal{C}_2 = \emptyset$.*

Proof. Lemma 5.4 shows that for every $\mathbf{u} \in \mathbb{S}^1$, there exists $\mathbf{v} \in \mathbb{S}^1$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \in \mathcal{D}$. As $\mathbf{u}$ runs over $\mathbb{S}^1$, the set of $\pm\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ runs over the whole of $\mathbb{S}^1$, and so in this case we conclude that for every $\mathbf{u} \in \mathbb{S}^{d-1}$, at least one of $\pm\mathbf{u}$ is in $\mathcal{D}$. Hence $\mathcal{C}_2 = \emptyset$. □

Proof of Proposition 5.2. Note that $|\mathcal{D}| = |\mathcal{D}_1| + |\mathcal{D}_2|$. If Lemma 5.5 applies, then we have $|\mathbb{S}^1 \setminus \mathcal{D}| = |\mathcal{C}_1| = |\mathcal{D}_1|$. Hence $|\mathbb{S}^1| = 2|\mathcal{D}_1| + |\mathcal{D}_2|$, and part (i) follows. If $X \stackrel{d}{=} -X$, then $\mathcal{D} = -\mathcal{D}$, so $\mathcal{D}_1 = \mathcal{C}_1 = \emptyset$. Thus $\mathbb{S}^{d-1} = \mathcal{D}_2 \cup \mathcal{C}_2$. If Lemma 5.5 applies, then $\mathcal{D} = \mathcal{D}_2 = \mathbb{S}^1$, giving part (ii). □

6 An arbitrary set of recurrent directions

We know from Theorem 2.1 that the set $\mathcal{D}$ is closed. The aim of this section is to show that there are, in general, no other restrictions on $\mathcal{D}$: it can be an arbitrary closed subset of the sphere. This result is essentially due to Erickson [10, pp. 508–510]; we reproduce the argument here.

Theorem 6.1. *Let A be a non-empty closed subset of $\mathbb{S}^{d-1}$. Suppose that the increment distribution of the random walk is given by $X = Q\xi$ where $Q \in \mathbb{S}^{d-1}$ and $\xi \in \mathbb{R}_+$ are independent, $\mathbb{P}(\xi > 0) > 0$, and $\operatorname{supp} Q = A$. Let $\xi_1, \xi_2, \ldots$ be independent copies of ξ, and suppose that*

$$\lim_{n\to\infty} \frac{\max_{1\le i\le n} \xi_i}{\sum_{i=1}^n \xi_i} = 1, \ \text{a.s.} \tag{6.1}$$

Then the recurrent directions of the random walk $S_n = \sum_{i=1}^n X_i$ are $\mathcal{D} = A$.

Remarks 6.2. (a) Pruitt, in Theorem 2 of [28], shows that (6.1) holds if and only if $\sum_{k\ge1} u_k^2 < \infty$, where $u_k = \mathbb{P}(2^k < \xi \le 2^{k+1})/\mathbb{P}(2^k < \xi)$. Examples that work have very heavy tails, and include $\mathbb{P}(\xi > r) = 1/\log r$ for $r \ge \mathrm{e}$ (see [10, pp. 509–510]) and $\mathbb{P}(\xi > r) = \exp(-(\log r)^\beta)$ for $r \ge 1$ with $\beta \in (0, 1/2)$ (see [28, p. 895]).

(b) The intuition behind Theorem 6.1 is as follows. The condition (6.1) means that the biggest jump so far is a.s. on a bigger scale than all the other jumps combined, and so the projection on the sphere is determined by the Q corresponding to the current biggest jump. As times goes on, one sees an i.i.d. subsequence of the Qs associated with the biggest jumps, and so the walk explores the sphere over the set A.

(c) Theorem 6.1 can be compared to the construction of random walks with desired limit properties of [9–11, 19].

Proof of Theorem 6.1. Write $X_i = Q_i\xi_i$ where the Q_i are i.i.d. copies of Q and the ξ_i are i.i.d. copies of ξ. Let $T_n = \sum_{i=1}^n \xi_i$, $M_n = \max_{1\le i\le n} \xi_i$, and $B_n = T_n - M_n$; then (6.1) is equivalent to $B_n/M_n \to 0$, a.s. Also set $k(1) := 1$ and, for $n \in \mathbb{N}$,

$$k(n+1) := \begin{cases} k(n) & \text{if } \xi_{n+1} \le M_n, \\ n+1 & \text{if } \xi_{n+1} > M_n. \end{cases}$$

Then $M_n = \xi_{k(n)}$. Define $R_n := S_n - M_n Q_{k(n)}$. Since $\|Q_{k(n)}\| = 1$, repeated application of the triangle inequality yields

$$\begin{aligned} \|\hat{S}_n - Q_{k(n)}\| &= \left\| \frac{M_n Q_{k(n)} + R_n - \|S_n\| Q_{k(n)}}{\|S_n\|} \right\| \\ &\le \frac{|M_n - \|S_n\||}{\|S_n\|} + \frac{\|R_n\|}{\|S_n\|} \\ &\le \frac{2\|R_n\|}{M_n - \|R_n\|}. \end{aligned}$$

But $\|R_n\| = \| \sum_{i \in \{1,\dots,n\} \setminus \{k(n)\}} X_i \| \le B_n$ where $B_n = T_n - M_n$, so

$$\|\hat{S}_n - Q_{k(n)}\| \le \frac{2(B_n/M_n)}{1 - (B_n/M_n)} \to 0, \text{ a.s.,}$$

by (6.1).

Since M_n is a non-decreasing sequence in $\mathbb{R}_+$ with $M_n \to \infty$ a.s. (as easily follows from (6.1) and the fact that $\mathbb{P}(\xi > 0) > 0$) the sequence $k(1), k(2), \dots$ is a non-decreasing subsequence of $\mathbb{Z}_+$ with $k(n) \to \infty$ a.s., and since the Q_i are independent of the ξ_i, the sequence $k(1), k(2), \dots$ is independent of the sequence $Q_1, Q_2, \dots$. Let $\ell_1 = 1$ and for $n \in \mathbb{N}$ define $\ell_{n+1} = \min\{m > \ell_n : k(m) > k(\ell_n)\}$, so that $1 = k(\ell_1) < k(\ell_2) < k(\ell_3) < \cdots$. Then the sequence $Q_{k(\ell_1)}, Q_{k(\ell_2)}, \dots$ has the same law as a sequence of i.i.d. copies of Q. Hence if $\mathbf{u} \in A$ we have

$$\liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| \le \lim_{n\to\infty} \|\hat{S}_n - Q_{k(\ell_n)}\| + \liminf_{n\to\infty} \|Q_{k(\ell_n)} - \mathbf{u}\| = 0, \text{ a.s.}$$

Thus $\mathbf{u} \in \mathcal{D}$. This shows that $A \subseteq \mathcal{D}$.

On the other hand, if $\mathbf{u} \notin A$ we have that since $\mathbb{S}^{d-1} \setminus A$ is open in $\mathbb{S}^{d-1}$ there is some $r > 0$ such that $\mathbb{P}(Q \in B_s(\mathbf{u}; r)) = 0$, and

$$\liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| \ge \liminf_{n\to\infty} \|Q_{k(\ell_n)} - \mathbf{u}\| - \lim_{n\to\infty} \|\hat{S}_n - Q_{k(\ell_n)}\| \ge r, \text{ a.s.,}$$

so that $\mathbf{u} \notin \mathcal{D}$. Thus $\mathcal{D} \subseteq A$ and the proof is complete. □

7 Convexity and an upper bound

We start this section with a straightforward result (Theorem 7.1) that is sometimes useful for giving an upper bound on $\mathcal{D}$ in terms of the support of $\hat{S}_n$. We then present (in Proposition 7.3 below) a simpler description of the upper bound in terms of the distribution of X alone, rather than its convolutions. To do so, we need an appropriate notion of convexity, which will also be useful in Sections 8 and 9 below when we look at one-dimensional projections and the convex hull of the walk.

Let $\mathcal{X}_n = (\operatorname{supp} \hat{S}_n) \setminus \{\mathbf{0}\}$, and let $\mathcal{X}^\star = \operatorname{cl}(\cup_{n\ge1} \mathcal{X}_n)$. Here is the upper bound.

Theorem 7.1. *We have that $\mathcal{D} \subseteq \mathcal{X}^\star$.*

Proof. Suppose that $\mathbf{u} \in \mathbb{S}^{d-1} \setminus \mathcal{X}^\star$. Since $\mathcal{X}^\star$ is closed, there exists $r > 0$ such that $B_s(\mathbf{u}; r) \cap \mathcal{X}_n = \emptyset$ for all $n \in \mathbb{N}$, and so $\mathbb{P}(\hat{S}_n \in B_s(\mathbf{u}; r)) = 0$ for all $n \in \mathbb{N}$. Then the Borel–Cantelli lemma shows that $\mathbb{P}(A(\mathbf{u}; r)) = \mathbb{P}(\hat{S}_n \in B_s(\mathbf{u}; r) \text{ i.o.}) = 0$. Hence, by Proposition 2.5, we have $\mathbf{u} \notin \mathcal{D}$. Hence $\mathcal{D} \subseteq \mathcal{X}^\star$. □

For $\mathbf{u}, \mathbf{v} \in \mathbb{S}^{d-1}$ and $\alpha \in [0, 1]$, let

$$I_\alpha(\mathbf{u}, \mathbf{v}) := \frac{\alpha \mathbf{u} + (1-\alpha)\mathbf{v}}{\|\alpha \mathbf{u} + (1-\alpha)\mathbf{v}\|},$$

unless $\mathbf{u} = -\mathbf{v}$ and $\alpha = 1/2$, in which case we set $I_{1/2}(\mathbf{u}, -\mathbf{u}) := \mathbf{0}$. If $\mathbf{u} \neq -\mathbf{v}$, set $I(\mathbf{u}, \mathbf{v}) := \{I_\alpha(\mathbf{u}, \mathbf{v}) : \alpha \in [0, 1]\}$, and set $I(\mathbf{u}, -\mathbf{u}) := \{\mathbf{u}, -\mathbf{u}\}$ (i.e., ignore $\alpha = 1/2$).

Definition 7.2. Say that $A \subseteq \mathbb{S}^{d-1}$ is *s-convex* if for every $\mathbf{u}, \mathbf{v} \in A$, one has $I(\mathbf{u}, \mathbf{v}) \subseteq A$.

Note that we only need to check the condition in Definition 7.2 for $\mathbf{v} \neq -\mathbf{u}$. In words, $A \subseteq \mathbb{S}^{d-1}$ is s-convex if for any $\mathbf{u}, \mathbf{v} \in A$, the radial projection onto $\mathbb{S}^{d-1}$ of the straight line segment from $\mathbf{u}$ to $\mathbf{v}$ in $\mathbb{R}^d$ lies in A. See also Lemma 7.5 below.

Denote by $\operatorname{hull} A$ the convex hull of $A \subseteq \mathbb{R}^d$. For $A \subseteq \mathbb{S}^{d-1}$, define

$$\text{s-hull}\, A := \{\hat{\mathbf{x}} : \mathbf{x} \in \operatorname{hull} A,\ \mathbf{x} \neq \mathbf{0}\}.$$

We will show (see Lemma 7.7) that s-hull A is s-convex. Let $\mathcal{X} := (\operatorname{supp} \hat{X}) \setminus \{\mathbf{0}\}$.

Proposition 7.3. *We have that $\mathcal{X}^\star = \operatorname{cl} \text{s-hull}\, \mathcal{X}$, and $\mathcal{X}^\star$ is s-convex.*

We work towards a proof of Proposition 7.3. Let $\mathcal{X}' := \{\hat{\mathbf{x}} : \mathbf{x} \in \operatorname{supp} X\}$.

Lemma 7.4. *For $X \in \mathbb{R}^d$ any random variable, we have that $\mathcal{X} = (\operatorname{cl} \mathcal{X}') \setminus \{\mathbf{0}\}$.*

Proof. Recall that $\operatorname{supp} X$ is the smallest closed $A \subseteq \mathbb{R}^d$ such that $\mathbb{P}(X \in A) = 1$, or, equivalently, $\operatorname{supp} X = \{\mathbf{x} \in \mathbb{R}^d : \mathbb{P}(X \in B(\mathbf{x}; r)) > 0 \text{ for all } r > 0\}$. Since $\operatorname{supp} \hat{X}$ is a closed subset of $\mathbb{S}^{d-1} \cup \{\mathbf{0}\}$, it follows that $\mathcal{X}$ is a closed subset of $\mathbb{S}^{d-1}$.

Suppose that $\mathbf{u} \in \mathcal{X}'$ with $\mathbf{u} \neq \mathbf{0}$. Then $\mathbf{u}r \in \operatorname{supp} X$ for some $r > 0$. This means that $\mathbb{P}(X \in B(\mathbf{u}r; s)) > 0$ for all $s \in (0, r/2)$, say; but, for any $\mathbf{x} \in B(\mathbf{u}r; s)$,

$$\begin{aligned} \|\hat{\mathbf{x}} - \mathbf{u}\| &= \|\mathbf{x}\|^{-1} \left(\|\mathbf{x} - \|\mathbf{x}\|\mathbf{u}\|\right) \\ &\leq \|\mathbf{x}\|^{-1} \left(\|\mathbf{x} - r\mathbf{u}\| + |r - \|\mathbf{x}\||\right) \leq 4s/r, \end{aligned}$$

so $\mathbb{P}(\hat{X} \in B(\mathbf{u}; 4s/r)) \geq \mathbb{P}(X \in B(\mathbf{u}r; s)) > 0$ for all $s \in (0, r/2)$. Hence $\mathbf{u} \in \operatorname{supp} \hat{X}$. Thus $\mathcal{X}' \subseteq \mathcal{X} \cup \{\mathbf{0}\}$, and since $\mathcal{X} \cup \{\mathbf{0}\}$ is closed we get $\operatorname{cl} \mathcal{X}' \subseteq \mathcal{X} \cup \{\mathbf{0}\}$.

On the other hand suppose that $\mathbf{u} \in \mathcal{X}$. Let $r_n > 0$ be such that $r_n \to 0$. Then $\mathbb{P}(X \in C(\mathbf{u}; r_n)) = \mathbb{P}(\hat{X} \in B(\mathbf{u}; r_n)) > 0$ for all n, which means that $C(\mathbf{u}; r_n) \cap \operatorname{supp} X \neq \emptyset$, i.e., for every n there exists $\mathbf{x}_n \in \operatorname{supp} X$ with $\|\hat{\mathbf{x}}_n - \mathbf{u}\| \leq r_n$. Hence $\hat{\mathbf{x}}_n \in \mathcal{X}'$ with $\hat{\mathbf{x}}_n \to \mathbf{u}$, so $\mathbf{u} \in \operatorname{cl} \mathcal{X}'$, and we get $\mathcal{X} \subseteq \operatorname{cl} \mathcal{X}'$. □

The next result characterizes a set as s-convex if and only if all normalized conical combinations are contained within the set.

Lemma 7.5. *The set $A \subseteq \mathbb{S}^{d-1}$ is s-convex if and only if for all $n \in \mathbb{N}$, all $\mathbf{u}_1, \dots, \mathbf{u}_n \in A$, and all $\beta_1, \dots, \beta_n \in (0, \infty)$,*

$$\frac{\sum_{i=1}^n \beta_i \mathbf{u}_i}{\|\sum_{i=1}^n \beta_i \mathbf{u}_i\|} \in A, \text{ whenever } \sum_{i=1}^n \beta_i \mathbf{u}_i \neq \mathbf{0}. \tag{7.1}$$

Proof. The 'if' half follows immediately (take $n = 2$ and $\beta_1 + \beta_2 = 1$). Suppose that A is s-convex. We proceed by an induction on n. Then (7.1) holds for $n = 2$, since

$$\frac{\beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2}{\|\beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2\|} = \frac{\frac{\beta_1}{\beta_1+\beta_2}\mathbf{u}_1 + \frac{\beta_2}{\beta_1+\beta_2}\mathbf{u}_2}{\left\|\frac{\beta_1}{\beta_1+\beta_2}\mathbf{u}_1 + \frac{\beta_2}{\beta_1+\beta_2}\mathbf{u}_2\right\|}.$$

Suppose that (7.1) holds for all $n \in \{1, \dots, m\}$ with $m \geq 2$, and consider $\mathbf{u}_1, \dots, \mathbf{u}_{m+1} \in A$ and $\beta_1, \dots, \beta_{m+1} \in (0, \infty)$ with $\sum_{i=1}^{m+1} \beta_i \mathbf{u}_i \neq \mathbf{0}$. We may also suppose that $\beta_m \mathbf{u}_m + \beta_{m+1} \mathbf{u}_{m+1} \neq \mathbf{0}$, or else the inductive hypothesis would apply directly. Set $\mathbf{u}'_i = \mathbf{u}_i$ for $1 \leq i \leq m-1$ and

$$\mathbf{u}'_m = \frac{\frac{\beta_m}{\beta_m+\beta_{m+1}}\mathbf{u}_m + \frac{\beta_{m+1}}{\beta_m+\beta_{m+1}}\mathbf{u}_{m+1}}{\left\|\frac{\beta_m}{\beta_m+\beta_{m+1}}\mathbf{u}_m + \frac{\beta_{m+1}}{\beta_m+\beta_{m+1}}\mathbf{u}_{m+1}\right\|}.$$

Then since A is s-convex, $\mathbf{u}'_m \in A$, and

$$\frac{\sum_{i=1}^{m+1} \beta_i \mathbf{u}_i}{\|\sum_{i=1}^{m+1} \beta_i \mathbf{u}_i\|} = \frac{\sum_{i=1}^{m} \beta'_i \mathbf{u}'_i}{\|\sum_{i=1}^{m} \beta'_i \mathbf{u}'_i\|},$$

where $\beta'_i = \beta_i$ for $1 \leq i \leq m-1$ and $\beta'_m = \|\beta_m \mathbf{u}_m + \beta_{m+1} \mathbf{u}_{m+1}\|$. By inductive hypothesis, the expression in the last display is thus in A. This completes the inductive step. $\square$

Corollary 7.6. *Suppose that $A \subseteq \mathbb{S}^{d-1}$ is s-convex. Then $A = \mathbb{S}^{d-1} \cap \operatorname{hull} A$.*

Proof. It is clear that $A \subseteq \mathbb{S}^{d-1} \cap \operatorname{hull} A$. So suppose that $\mathbf{u} \in \mathbb{S}^{d-1} \cap \operatorname{hull} A$. Then (see e.g. Lemma 3.1 of [14, p. 42]) there exist $n \in \mathbb{N}$, $\mathbf{v}_1, \dots, \mathbf{v}_n \in A$, and $\lambda_1, \dots, \lambda_n \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$, for which $\mathbf{u} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$. But, since A is s-convex and $\|\mathbf{u}\| = 1$, Lemma 7.5 shows that $\sum_{i=1}^n \lambda_i \mathbf{v}_i \in A$. So $\mathbb{S}^{d-1} \cap \operatorname{hull} A \subseteq A$. $\square$

The next result shows that s-hull A has a similar characterization to the usual hull A.

Lemma 7.7. *For $A \subseteq \mathbb{S}^{d-1}$,* s-hull A *is the smallest s-convex $B \subseteq \mathbb{S}^{d-1}$ with $A \subseteq B$.*

Proof. Let $\mathbf{u}, \mathbf{v} \in$ s-hull A with $\mathbf{v} \neq -\mathbf{u}$, and $\alpha \in (0, 1)$. Then $\mathbf{u} = \hat{\mathbf{x}}$ and $\mathbf{v} = \hat{\mathbf{y}}$ for some $\mathbf{x}, \mathbf{y} \in \operatorname{hull} A$ with $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. Choose $\beta \in (0, 1)$ given by

$$\beta = \frac{\alpha \|\mathbf{y}\|}{\alpha \|\mathbf{y}\| + (1 - \alpha)\|\mathbf{x}\|}.$$

Consider $\mathbf{w} = \beta \mathbf{x} + (1 - \beta)\mathbf{y}$. Then, since hull A is convex, $\mathbf{w} \in \operatorname{hull} A$, and $\mathbf{w} \neq \mathbf{0}$ since $\hat{\mathbf{x}} \neq -\hat{\mathbf{y}}$, so $\hat{\mathbf{w}} \in$ s-hull A. But

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{\alpha \hat{\mathbf{x}} + (1 - \alpha)\hat{\mathbf{y}}}{\|\alpha \hat{\mathbf{x}} + (1 - \alpha)\hat{\mathbf{y}}\|}$$

is thus in s-hull A for all $\alpha \in (0, 1)$, verifying that s-hull A is s-convex.

Next we claim that if $A \subseteq \mathbb{S}^{d-1}$ is s-convex, then s-hull $A = A$. Clearly $A \subseteq$ s-hull A. So suppose that A is s-convex, and consider $\mathbf{u} \in$ s-hull A. Then $\mathbf{u} = \hat{\mathbf{x}}$ for some $\mathbf{x} \in \operatorname{hull} A$, $\mathbf{x} \neq \mathbf{0}$, and thus (see e.g. Lemma 3.1 of [14, p. 42]) there exist $n \in \mathbb{N}$, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in A$, and $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$, for which $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$. Then Lemma 7.5 shows that $\hat{\mathbf{x}} \in A$. In other words, s-hull $A \subseteq A$, as required.

Suppose B is s-convex with $A \subseteq B$; then the preceding paragraph shows that s-hull $A \subseteq$ s-hull $B = B$, which completes the proof of the lemma. □

Lemma 7.8. *Let $A \subseteq \mathbb{S}^{d-1}$ be s-convex. Then* $\operatorname{cl} A$ *is also s-convex.*

Proof. It suffices to suppose $\mathbf{u}, \mathbf{v} \in \operatorname{cl} A$ with $\mathbf{u} \neq -\mathbf{v}$. Then there exist $\mathbf{u}_1, \mathbf{u}_2, \ldots \in A$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots \in A$ with $\mathbf{u}_n \to \mathbf{u}$ and $\mathbf{v}_n \to \mathbf{v}$, and there exists $n_0 \in \mathbb{N}$ such that $\mathbf{u}_n \neq -\mathbf{v}_n$ for all $n \geq n_0$. Since A is s-convex, $I_\alpha(\mathbf{u}_n, \mathbf{v}_n) \in A$ for all $n \geq n_0$ and all $\alpha \in [0,1]$. By continuity of the function $\mathbf{x} \mapsto \hat{\mathbf{x}}$ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$, it follows that $I_\alpha(\mathbf{u}, \mathbf{v}) = \lim_{n\to\infty} I_\alpha(\mathbf{u}_n, \mathbf{v}_n) \in \operatorname{cl} A$ for all $\alpha \in [0,1]$. Hence $\operatorname{cl} A$ is s-convex. □

Proof of Proposition 7.3. First we use induction to show that $\mathcal{X}_n \subseteq \operatorname{cl}$ s-hull $\mathcal{X}$ for all $n \in \mathbb{N}$. Clearly this is true for $n = 1$. So suppose, for the inductive hypothesis, that $\mathcal{X}_m \subseteq \operatorname{cl}$ s-hull $\mathcal{X}$ for all $m \in \{1, \ldots, n\}$. Now, provided that $S_{n+1} \neq \mathbf{0}$, we have

$$\hat{S}_{n+1} = \frac{\alpha_n \hat{S}_n + (1-\alpha_n)\hat{X}_{n+1}}{\|\alpha_n \hat{S}_n + (1-\alpha_n)\hat{X}_{n+1}\|}, \text{ where } \alpha_n = \frac{\|S_n\|}{\|S_n\| + \|X_{n+1}\|}.$$

In particular, since $\mathbb{P}(\hat{S}_n \in \mathcal{X}_n \cup \{\mathbf{0}\}) = 1$ and $\mathbb{P}(\hat{X}_{n+1} \in \mathcal{X} \cup \{\mathbf{0}\}) = 1$, we have

$$\mathbb{P}\left(\hat{S}_{n+1} \in (\cup\{I(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in \operatorname{cl} \text{s-hull}\, \mathcal{X}\}) \cup \{\mathbf{0}\}\right) = 1,$$

by the inductive hypothesis. But cl s-hull $\mathcal{X}$ is s-convex, by Lemmas 7.7 and 7.8, so $\mathbb{P}(\hat{S}_{n+1} \in (\operatorname{cl} \text{s-hull}\, \mathcal{X}) \cup \{\mathbf{0}\}) = 1$, which means that $\mathcal{X}_{n+1} \subseteq \operatorname{cl}$ s-hull $\mathcal{X}$, completing the induction. Thus we conclude that $\mathcal{X}^\star \subseteq \operatorname{cl}$ s-hull $\mathcal{X}$.

Next we show that $\mathcal{X}^\star$ is s-convex. It suffices to suppose that $\mathbf{u}, \mathbf{v} \in \mathcal{X}^\star$ with $\mathbf{u} \neq -\mathbf{v}$. Then there exist sequences $\mathbf{u}_{n_k} \in \mathcal{X}_{n_k}$ and $\mathbf{v}_{m_k} \in \mathcal{X}_{m_k}$ with $\mathbf{u}_{n_k} \to \mathbf{u}$ and $\mathbf{v}_{m_k} \to \mathbf{v}$. Lemma 7.4 shows that, correspondingly, there exist sequences $\mathbf{x}_{n_{k,1}}, \mathbf{x}_{n_{k,2}}, \ldots \in \operatorname{supp} S_{n_k}$ and $\mathbf{y}_{m_{k,1}}, \mathbf{y}_{m_{k,2}}, \ldots \in \operatorname{supp} S_{m_k}$ with $\lim_{i\to\infty} \hat{\mathbf{x}}_{n_{k,i}} = \mathbf{u}_{n_k}$ and $\lim_{j\to\infty} \hat{\mathbf{y}}_{m_{k,j}} = \mathbf{v}_{m_k}$, and, for all k sufficiently large and all i, j sufficiently large, $\hat{\mathbf{x}}_{n_{k,i}} \neq -\hat{\mathbf{y}}_{m_{k,j}}$. Now for $s, t \in \mathbb{Z}_+$, $s\mathbf{x}_{n_{k,i}} + t\mathbf{y}_{m_{k,j}} \in \operatorname{supp} S_{sn_k + tm_k}$. Applying Lemma 7.4 with $X = S_{sn_k+tm_k}$ we see that $\mathbf{w} \in \mathcal{X}_{sn_k+tm_k} \subseteq \mathcal{X}^\star$, where

$$\mathbf{w} = \frac{s\mathbf{x}_{n_{k,i}} + t\mathbf{y}_{m_{k,j}}}{\|s\mathbf{x}_{n_{k,i}} + t\mathbf{y}_{m_{k,j}}\|} = I_{\alpha_{s,t,i,j}}(\hat{\mathbf{x}}_{n_{k,i}}, \hat{\mathbf{y}}_{m_{k,j}}),$$

with

$$\alpha_{s,t,i,j} = \frac{s\|\mathbf{x}_{n_{k,i}}\|}{s\|\mathbf{x}_{n_{k,i}}\| + t\|\mathbf{y}_{m_{k,j}}\|}.$$

For fixed k, i, j and $\alpha \in [0,1]$, we may choose $s, t \to \infty$ such that $\alpha_{s,t,i,j} \to \alpha$, and since for $\mathbf{u} \neq -\mathbf{v}$, $\alpha \mapsto I_\alpha(\mathbf{u}, \mathbf{v})$ is continuous over $\alpha \in [0,1]$, and $\mathcal{X}^\star$ is closed, we get

$$I_\alpha(\hat{\mathbf{x}}_{n_{k,i}}, \hat{\mathbf{y}}_{m_{k,j}}) = \lim_{s,t\to\infty} I_{\alpha_{s,t,i,j}}(\hat{\mathbf{x}}_{n_{k,i}}, \hat{\mathbf{y}}_{m_{k,j}}) \in \mathcal{X}^\star, \text{ for all } \alpha \in [0,1].$$

Then by continuity of $(\mathbf{u}, \mathbf{v}) \mapsto I_\alpha(\mathbf{u}, \mathbf{v})$ away from $\mathbf{u} = -\mathbf{v}$ we get

$$I_\alpha(\mathbf{u}, \mathbf{v}) = \lim_{k\to\infty} I_\alpha(\mathbf{u}_{n_k}, \mathbf{v}_{m_k}) = \lim_{k\to\infty} \lim_{i,j\to\infty} I_\alpha(\hat{\mathbf{x}}_{n_{k,i}}, \hat{\mathbf{y}}_{m_{k,j}}) \in \mathcal{X}^\star,$$

for all $\alpha \in [0, 1]$. Hence $\mathcal{X}^\star$ is s-convex, and $\mathcal{X} \subseteq \mathcal{X}^\star$, so, by Lemma 7.7, we have $\operatorname{s-hull} \mathcal{X} \subseteq \mathcal{X}^\star$, and since $\mathcal{X}^\star$ is closed, we get $\operatorname{cl} \operatorname{s-hull} \mathcal{X} \subseteq \mathcal{X}^\star$.

Thus we conclude that $\mathcal{X}^\star = \operatorname{cl} \operatorname{s-hull} \mathcal{X}$, and the latter is s-convex by Lemmas 7.7 and 7.8. □

We finish this section with a result on the boundary of an s-convex set, which will be useful in Section 8 below. For $A \subseteq \mathbb{S}^{d-1}$, denote by $\operatorname{s-int} A$ the interior of A relative to $\mathbb{S}^{d-1}$, i.e., $\mathbf{u} \in \operatorname{s-int} A$ if and only if $B_s(\mathbf{u}; \delta) \subseteq A$ for some $\delta > 0$. Also, for $A \subseteq \mathbb{S}^{d-1}$, we write $\partial_s A$ for the boundary of A relative to $\mathbb{S}^{d-1}$, i.e., $\partial_s A := (\operatorname{cl} A) \setminus (\operatorname{s-int} A)$.

Lemma 7.9. *If $A \subseteq \mathbb{S}^{d-1}$ is s-convex, then (i)* $\operatorname{s-int} A = \operatorname{s-int} \operatorname{cl} A$*; and (ii)* $\partial_s A = \partial_s \operatorname{cl} A$.

Proof. Suppose that $\mathbf{u} \in \operatorname{s-int} \operatorname{cl} A$. Then there exist $m \in \mathbb{N}$ and $\mathbf{u}_1, \dots, \mathbf{u}_m \in \operatorname{cl} A$ such that $\mathbf{u} \in \operatorname{s-int} P_s(\mathbf{u}_1, \dots, \mathbf{u}_m)$, where $P_s(\mathbf{u}_1, \dots, \mathbf{u}_m) := \operatorname{s-hull}\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$. Let

$$R_s(\mathbf{v}_1, \dots, \mathbf{v}_m; \mathbf{u}) = \inf\{\|\mathbf{v} - \mathbf{u}\| : \mathbf{v} \in \mathbb{S}^{d-1} \setminus P_s(\mathbf{v}_1, \dots, \mathbf{v}_m)\},$$

which is zero unless $\mathbf{u}$ lies in the interior of $P_s(\mathbf{v}_1, \dots, \mathbf{v}_m)$, when it is equal to the shortest distance from $\mathbf{u}$ to the boundary of $P_s(\mathbf{v}_1, \dots, \mathbf{v}_m)$. In particular, note that $R_s(\mathbf{u}_1, \dots, \mathbf{u}_m; \mathbf{u}) = \delta_0 > 0$. For $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{S}^{d-1}$, the map $(\mathbf{v}_1, \dots, \mathbf{v}_m) \mapsto P_s(\mathbf{v}_1, \dots, \mathbf{v}_m)$, as a function from $(\mathbb{S}^{d-1})^m$ to compact subsets of $\mathbb{R}^d$ with the Hausdorff metric, is continuous. So the map from $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ to $R_s(\mathbf{v}_1, \dots, \mathbf{v}_m; \mathbf{u})$ is also continuous. Hence for any $\delta \in (0, \delta_0)$, we can find $\varepsilon > 0$ sufficiently small such that $B_s(\mathbf{u}; \delta)$ is contained in $P_s(\mathbf{v}_1, \dots, \mathbf{v}_m)$ for all $\mathbf{v}_i \in \mathbb{S}^{d-1}$ with $\|\mathbf{v}_i - \mathbf{u}_i\| < \varepsilon$. Since $\mathbf{u}_i \in \operatorname{cl} A$, we can find $\mathbf{v}_i \in A$ with $\|\mathbf{v}_i - \mathbf{u}_i\| < \varepsilon$, which means that $B_s(\mathbf{u}; \delta) \subseteq P_s(\mathbf{v}_1, \dots, \mathbf{v}_m) \subseteq A$, since A is s-convex. Hence $\mathbf{u} \in \operatorname{s-int} A$. This establishes (i). Then (ii) follows since $\partial_s \operatorname{cl} A = \operatorname{cl} A \setminus \operatorname{s-int} \operatorname{cl} A = \operatorname{cl} A \setminus \operatorname{s-int} A = \partial_s A$. □

8 Projection asymptotics

In Section 9 we study the way in which the random walk fills space via the convex hull of the trajectory. Pertinent for this is the behaviour of one-dimensional projections of the walk, so we turn to this first. For fixed $\mathbf{u} \in \mathbb{S}^{d-1}$, the projection $S_n \cdot \mathbf{u}$ defines a random walk on $\mathbb{R}$, with increment distribution $X \cdot \mathbf{u}$, which either tends to $+\infty$, to $-\infty$, or oscillates (see Lemma 8.1 below). However, this, by itself, does not exclude that there might exist (random) $\mathbf{u} \in \mathbb{S}^{d-1}$ for which $S_n \cdot \mathbf{u}$ does something out of the ordinary, such as having a finite lim sup. While not central for what follows, we show that such *exceptional projections* do not exist, at least for $d \leq 2$.

Define the random sets

$$\mathcal{P}_+ := \left\{\mathbf{u} \in \mathbb{S}^{d-1} : \lim_{n\to\infty} (S_n \cdot \mathbf{u}) = +\infty\right\}, \quad \mathcal{P}_- := \left\{\mathbf{u} \in \mathbb{S}^{d-1} : \lim_{n\to\infty} (S_n \cdot \mathbf{u}) = -\infty\right\},$$
$$\mathcal{P}_\pm := \left\{\mathbf{u} \in \mathbb{S}^{d-1} : -\infty = \liminf_{n\to\infty} (S_n \cdot \mathbf{u}) < \limsup_{n\to\infty} (S_n \cdot \mathbf{u}) = +\infty\right\},$$

and their non-random counterparts

$$\mathcal{D}_+ := \left\{\mathbf{u} \in \mathbb{S}^{d-1} : \lim_{n\to\infty} (S_n \cdot \mathbf{u}) = +\infty, \text{ a.s.}\right\},$$

$$\mathcal{D}_- := \{\mathbf{u} \in \mathbb{S}^{d-1} : \lim_{n\to\infty}(S_n \cdot \mathbf{u}) = -\infty, \text{ a.s.}\},$$
$$\mathcal{D}_\pm := \{\mathbf{u} \in \mathbb{S}^{d-1} : -\infty = \liminf_{n\to\infty}(S_n \cdot \mathbf{u}) < \limsup_{n\to\infty}(S_n \cdot \mathbf{u}) = +\infty, \text{ a.s.}\},$$

Then $\mathcal{P}_+ = -\mathcal{P}_-$, $\mathcal{P}_\pm = -\mathcal{P}_\pm$, and similarly for the non-random versions.

Lemma 8.1. *The sets $\mathcal{D}_+, \mathcal{D}_-, \mathcal{D}_\pm$ partition $\mathbb{S}^{d-1}$.*

Proof. Let $\mathbf{u} \in \mathbb{S}^{d-1}$. Then (see e.g. [8, Theorem 4.1.2]) exactly one of the following holds: (i) $\mathbf{u} \in \mathcal{D}_+$, (ii) $\mathbf{u} \in \mathcal{D}_-$, (iii) $\mathbf{u} \in \mathcal{D}_\pm$, or (iv) $\mathbb{P}(X \cdot \mathbf{u} = 0) = 1$. Case (iv) is ruled out by our assumption that the walk is genuinely d-dimensional. □

It is not immediately obvious that $\mathcal{P}_+, \mathcal{P}_-, \mathcal{P}_\pm$ also partition $\mathbb{S}^{d-1}$. We define

$$\mathcal{E}_+ := \{\mathbf{u} \in \mathbb{S}^{d-1} : \limsup_{n\to\infty}(S_n \cdot \mathbf{u}) \in \mathbb{R}\}, \quad \mathcal{E}_- := \{\mathbf{u} \in \mathbb{S}^{d-1} : \liminf_{n\to\infty}(S_n \cdot \mathbf{u}) \in \mathbb{R}\}.$$

We call $\mathbf{u} \in \mathcal{E} := \mathcal{E}_+ \cup \mathcal{E}_-$ an *exceptional projection* of the walk. Since $\mathcal{E}_- = -\mathcal{E}_+$, we have $\mathcal{E} = -\mathcal{E}$. Lemma 8.1 means that $\mathbb{P}(\mathbf{u} \in \mathcal{E}) = 0$ for all fixed $\mathbf{u} \in \mathbb{S}^{d-1}$. Recall the definition of s-convexity from Definition 7.2.

Lemma 8.2. *The sets $\mathcal{P}_+$, $\mathcal{P}_-$, $\mathcal{P}_+ \cup \mathcal{E}_-$, $\mathcal{P}_- \cup \mathcal{E}_+$, $\mathcal{D}_+$, and $\mathcal{D}_-$ are s-convex.*

Proof. Suppose that $\mathbf{u}, \mathbf{v} \in \mathcal{P}_+$ with $\mathbf{v} \neq -\mathbf{u}$. Then

$$S_n \cdot (\alpha\mathbf{u} + (1-\alpha)\mathbf{v}) = \alpha S_n \cdot \mathbf{u} + (1-\alpha) S_n \cdot \mathbf{v},$$

and both $S_n \cdot \mathbf{u}$ and $S_n \cdot \mathbf{v}$ tend to infinity, so $I_\alpha(\mathbf{u}, \mathbf{v}) \in \mathcal{P}_+$ for all $\alpha \in [0, 1]$. Hence $\mathcal{P}_+$ is s-convex, and so is $\mathcal{P}_- = -\mathcal{P}_+$ as well. The argument for $\mathcal{D}_+$, $\mathcal{D}_-$ is essentially the same. Note that $\mathbf{u} \in \mathcal{P}_+ \cup \mathcal{E}_-$ if and only if $\liminf_{n\to\infty}(S_n \cdot \mathbf{u}) > -\infty$. Hence if $\mathbf{u}, \mathbf{v} \in \mathcal{P}_+ \cup \mathcal{E}_-$,

$$\liminf_{n\to\infty}(S_n \cdot (\alpha\mathbf{u} + (1-\alpha)\mathbf{v})) \geq \alpha \liminf_{n\to\infty}(S_n \cdot \mathbf{u}) + (1-\alpha) \liminf_{n\to\infty}(S_n \cdot \mathbf{v}) > -\infty,$$

so $\mathcal{P}_+ \cup \mathcal{E}_-$ is s-convex; similarly for $\mathcal{P}_- \cup \mathcal{E}_+$. □

The following result shows that random set $\mathcal{P}_+$ can differ from the non-random set $\mathcal{D}_+$ in a rather limited way. In particular, since $\mathcal{P}_+$ and $\mathcal{D}_+$ are s-convex (by Lemma 8.2), Proposition 8.3(i) with Lemma 7.9 shows that $\mathbb{P}(\partial_s \mathcal{P}_+ = \partial_s \mathcal{D}_+) = 1$. Similarly for $\mathcal{P}_-$ and $\mathcal{D}_-$.

Proposition 8.3. *(i) We have*

$$\mathbb{P}(\operatorname{cl} \mathcal{P}_+ = \operatorname{cl} \mathcal{P}_+ \cup \operatorname{cl} \mathcal{E}_- = \operatorname{cl} \mathcal{D}_+) = 1, \text{ and } \mathbb{P}(\operatorname{cl} \mathcal{P}_- = \operatorname{cl} \mathcal{P}_- \cup \operatorname{cl} \mathcal{E}_+ = \operatorname{cl} \mathcal{D}_-) = 1.$$

(ii) Moreover, $\mathbb{P}(\operatorname{cl} \mathcal{E}_+ \subseteq \partial_s \mathcal{D}_-) = \mathbb{P}(\operatorname{cl} \mathcal{E}_- \subseteq \partial_s \mathcal{D}_+) = 1$.

Proof. For part (i), it suffices to prove the first statement. For ease of notation, write $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{E}_-$. Since, by Lemma 8.2, $\mathcal{P}$ is s-convex, so is $\operatorname{cl} \mathcal{P}$, by Lemma 7.8. Thus, by Corollary 7.6, $\operatorname{cl} \mathcal{P} = \mathbb{S}^{d-1} \cap \operatorname{hull} \operatorname{cl} \mathcal{P}$. Since $\operatorname{cl} \mathcal{P}$ is bounded, $A = \operatorname{hull} \operatorname{cl} \mathcal{P} = \operatorname{cl} \operatorname{hull} \mathcal{P}$ [14, p. 45]. The set A is convex and compact, and so it is uniquely determined by its support function $h_A : \mathbb{R}^d \to \mathbb{R}$ given by $h_A(\mathbf{x}) = \sup\{\mathbf{x} \cdot \mathbf{y} : \mathbf{y} \in A\}$, which is continuous [14, p. 56]. Since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, h_A is determined by $\{h_A(\mathbf{x}) : \mathbf{x} \in \mathbb{Q}^d\}$. By the Hewitt–Savage theorem, each member of this countable collection of random variables is a.s. constant, so h_A is a.s. constant. Thus the set A is non-random, and then $\mathbb{P}(\operatorname{cl}\mathcal{P} = S) = 1$ for the non-random closed, s-convex set $S = \mathbb{S}^{d-1} \cap A$. Note that

$$\mathbb{P}(\mathbf{u} \in \operatorname{cl}\mathcal{P}) = \begin{cases} 1 & \text{if } \mathbf{u} \in S, \\ 0 & \text{if } \mathbf{u} \notin S. \end{cases}$$

Since every $\mathbf{u} \in \mathcal{D}_+$ has $\mathbb{P}(\mathbf{u} \in \mathcal{P}_+ \subseteq \mathcal{P}) = 1$, we have $\mathcal{D}_+ \subseteq S$, and since S is closed, $\operatorname{cl}\mathcal{D}_+ \subseteq S$. On the other hand, if $S \backslash \operatorname{cl}\mathcal{D}_+ \neq \emptyset$, there is some $\mathbf{u} \in S \backslash \operatorname{cl}\mathcal{D}_+$ and some $\varepsilon > 0$ such that $S \cap B_s(\mathbf{u}; \varepsilon)$ does not intersect $\operatorname{cl}\mathcal{D}_+$. The compact set S contains a countable dense subset, Q, say, and every $\mathbf{v} \in Q \cap B_s(\mathbf{u}; \varepsilon)$ has $\mathbf{v} \notin \mathcal{D}_+$, so $\mathbb{P}(\mathbf{v} \in \mathcal{P}_+) = 0$. Also, $\mathbb{P}(\mathbf{v} \in \mathcal{E}_-) = 0$. Thus no member of $Q \cap B_s(\mathbf{u}; \varepsilon)$ is in $\mathcal{P}$. Since $\mathcal{P}$ is s-convex with closure S, this implies that there is a neighbourhood of $\mathbf{u}$ in S that does not intersect $\mathcal{P}$. Hence $\mathbf{u} \in S \backslash \operatorname{cl}\mathcal{P}$. But $\mathbb{P}(S \backslash \operatorname{cl}\mathcal{P} = \emptyset) = 1$. Thus $\mathbb{P}(\operatorname{cl}\mathcal{P}_+ \cup \operatorname{cl}\mathcal{E}_- = \operatorname{cl}\mathcal{D}_+) = 1$. Repeating the preceding argument, but taking $\mathcal{P} = \mathcal{P}_+$ throughout, gives $\mathbb{P}(\operatorname{cl}\mathcal{P}_+ = \operatorname{cl}\mathcal{D}_+) = 1$ too.

For part (ii), we have from (i) that $\mathbb{P}(\operatorname{cl}\mathcal{E}_- \subseteq \operatorname{cl}\mathcal{P}_+) = 1$. Moreover, we must have $\mathbb{P}(\operatorname{cl}\mathcal{E}_- \cap \operatorname{s-int}\mathcal{P}_+ = \emptyset) = 1$, or else we would have $\mathcal{E}_- \cap \mathcal{P}_+ \neq \emptyset$. Thus $\mathbb{P}(\operatorname{cl}\mathcal{E}_- \subseteq \partial_s \mathcal{P}_+) = 1$. But since $\mathcal{P}_+$ and $\mathcal{D}_+$ are s-convex and a.s. have the same closure, Lemma 7.9 shows that $\mathbb{P}(\partial_s \mathcal{P}_+ = \partial_s \mathcal{D}_+) = 1$. This gives (ii). □

Corollary 8.4. *If $\mathcal{D}_\pm = \mathbb{S}^{d-1}$, then $\mathbb{P}(\mathcal{P}_\pm = \mathbb{S}^{d-1}) = 1$.*

Proof. If $\mathcal{D}_\pm = \mathbb{S}^{d-1}$, then $\mathcal{D}_+ = \mathcal{D}_- = \emptyset$, by Lemma 8.1, and Proposition 8.3 shows that $\mathbb{P}(\operatorname{cl}\mathcal{P}_+ \cup \operatorname{cl}\mathcal{P}_- \cup \operatorname{cl}\mathcal{E} = \emptyset) = 1$. □

We turn briefly to the question of whether $\mathcal{E}$ is in fact empty.

Lemma 8.5. *With probability 1, $\operatorname{cl}\mathcal{E}$ is a perfect set.*

Proof. For a measurable $B \subseteq \mathbb{S}^{d-1}$ and let $N(B) = \#(B \cap \operatorname{cl}\mathcal{E})$, the number of points of $\operatorname{cl}\mathcal{E}$ in B. We claim that, for any B that is open in $\mathbb{S}^{d-1}$,

$$\mathbb{P}(N(B) = 0) = 1 \text{ or } \mathbb{P}(N(B) = \infty) = 1. \tag{8.1}$$

Indeed, the $\mathbb{Z}_+ \cup \{\infty\}$-valued random variable $N(B)$ is a.s. constant, by the Hewitt–Savage theorem: $\mathbb{P}(N(B) = K) = 1$ for some (non-random) K. If $1 \leq K < \infty$, we may label the elements of $B \cap \operatorname{cl}\mathcal{E} = B \cap \mathcal{E}$ in an arbitrary order as $\mathbf{u}_1, \ldots, \mathbf{u}_K$, and each is a.s. constant, by the Hewitt–Savage theorem again, so there exist constant $\mathbf{u}_1, \ldots, \mathbf{u}_K \in B$ with $\mathbb{P}(\mathbf{u}_j \in \mathcal{E}) = 1$ for each j. But $\mathbb{P}(\mathbf{u} \in \mathcal{E}) = 0$ for all $\mathbf{u}$. Hence $K \in \{0, \infty\}$. This establishes (8.1).

Recall that $\mathcal{R}$ denotes the (countable) set of all $B_s(\mathbf{u}; r)$ with $\mathbf{u} \in \mathbb{Q}^d \cap \mathbb{S}^{d-1}$ and $r \in \mathbb{Q} \cap (0, \infty)$. From (8.1) we have that $\mathbb{P}(N(B) \in \{0, \infty\}$ for all $B \in \mathcal{R}) = 1$, which means that $\operatorname{cl}\mathcal{E}$ contains no isolated points. □

Corollary 8.6. *Suppose that $d \in \{1, 2\}$. Then $\mathbb{P}(\operatorname{cl}\mathcal{E} = \emptyset) = 1$.*

Proof. For $d = 1$ this is evident, so suppose that $d = 2$. By Proposition 8.3, $\mathbb{P}(\operatorname{cl}\mathcal{E}_- \subseteq \partial_s \mathcal{D}_+) = 1$, while Lemma 8.2 shows that $\mathcal{D}_+$ is s-convex, so $\partial_s \mathcal{D}_+$ contains at most two points. Similarly for $\operatorname{cl}\mathcal{E}_+$. Thus $\operatorname{cl}\mathcal{E}$ has at most four points. Lemma 8.5 then shows that $\mathbb{P}(\operatorname{cl}\mathcal{E} = \emptyset) = 1$. □

9 The convex hull

For $n \in \mathbb{Z}_+$ let $\mathcal{H}_n := \text{hull}\{S_0, S_1, \ldots, S_n\}$ (a convex polytope). Set $\mathcal{H}_\infty := \cup_{n \geq 0} \mathcal{H}_n$. If $x, y \in \mathcal{H}_\infty$ then $x, y \in \mathcal{H}_n$ for some n, and since $\mathcal{H}_n$ is convex, $\theta x + (1-\theta) y \in \mathcal{H}_n \subseteq \mathcal{H}_\infty$ for all $\theta \in [0,1]$. Thus $\mathcal{H}_\infty$ is convex, and hence so is $\text{cl}\, \mathcal{H}_\infty$ [14, p. 44]. Define

$$\mathcal{S}_\infty := \{S_0, S_1, \ldots\}. \tag{9.1}$$

If S_n is transient, then $\mathcal{S}_\infty$ has no finite limit points. Since $\mathcal{H}_n \subseteq \text{hull}\, \mathcal{S}_\infty$, we have $\mathcal{H}_\infty \subseteq \text{hull}\, \mathcal{S}_\infty$, while $\mathcal{H}_\infty$ is a convex set containing $\mathcal{S}_\infty$, so $\text{hull}\, \mathcal{S}_\infty \subseteq \mathcal{H}_\infty$. That is,

$$\mathcal{H}_\infty = \text{hull}\, \mathcal{S}_\infty = \text{hull}\{S_0, S_1, S_2, \ldots\}.$$

Also define

$$r_n := \inf\{\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^d \setminus \mathcal{H}_n\}.$$

Note that r_n is non-decreasing, so $r_\infty := \lim_{n\to\infty} r_n$ exists in $[0, \infty]$. In [24] it is shown that if $\mathbb{P}(r_\infty = \infty) = 1$, then there is a zero–one law for random variables that are tail-measurable for the sequence $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$: see [24, §3].

Lemma 9.1. *We have* $\mathbb{P}(r_\infty = \infty) = \mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) \in \{0, 1\}$.

Proof. By definition of r_n, we have $B(\mathbf{0}; r_n) \subseteq \mathcal{H}_n \subseteq \mathcal{H}_\infty$. Thus if $r_\infty = \infty$, we have $\mathcal{H}_\infty = \mathbb{R}^d$. On the other hand, if $\mathcal{H}_\infty = \mathbb{R}^d$, then for any $r \in (0, \infty)$ there exists some $n \in \mathbb{N}$ for which $B(\mathbf{0}; r) \subseteq \mathcal{H}_n$. (If not, there is some r and $\mathbf{x} \in B(\mathbf{0}; r)$ with $\mathbf{x} \notin \mathcal{H}_\infty$.) Then $r_n \geq r$, so $r_\infty \geq r$. Since r was arbitrary, we get $r_\infty = \infty$. Thus $\mathbb{P}(r_\infty = \infty) = \mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d)$, and the proof is completed by the Hewitt–Savage theorem. □

A consequence of a theorem of Carathéodory is that if $A \subseteq \mathbb{R}^d$ is compact, then $\text{hull}\, A$ is also compact (see e.g. Corollary 3.1 of [14, p. 44]). Thus $\text{hull}\, \mathcal{D}$ is compact, by Theorem 2.1. The following result relates several concepts from earlier to the question of whether the convex hull eventually fills all of space. Here 'int' denotes interior.

Theorem 9.2. *Consider the following statements.*

(i) $\mathbf{0} \in \text{int}\, \text{hull}\, \mathcal{D}$.

(ii) $\mathbb{P}(r_\infty = \infty) = 1$.

(iii) $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$.

(iv) $\mathcal{D}_\pm = \mathbb{S}^{d-1}$.

(v) $\mathbf{0} \in \text{hull}\, \mathcal{D}$.

Then the following logical relationships apply: (i) ⇒ (ii) ⇔ (iii) ⇔ (iv) ⇒ (v).

Remarks 9.3. (a) If the random walk is recurrent, then $\mathcal{D} = \mathbb{S}^{d-1}$ (Proposition 2.6) and so (i) and hence (iv) hold, so that $\mathcal{D}_+ = \emptyset$. In other words, if $\mathcal{D}_+ \neq \emptyset$, then $\mathcal{D} \neq \mathbb{S}^{d-1}$, and the random walk is transient.

(b) Examples 10.1 and 10.2 below show that (i) is not necessary for (iii), and (v) is not sufficient for (iii).

In [24], it was shown that sufficient for $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$ is that the random walk is recurrent; this follows from Theorem 9.2 and the fact that recurrence implies that $\mathcal{D} = \mathbb{S}^{d-1}$ (Proposition 2.6). Here are some further sufficient conditions.

Corollary 9.4. *Suppose that either (i)* $X \stackrel{d}{=} -X$*, or (ii)* $\mathbb{E}\|X\| < \infty$ *and* $\mu = \mathbf{0}$*. Then* $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$*.*

Proof. By Theorem 9.2, it suffices to show that $\mathcal{D}_\pm = \mathbb{S}^{d-1}$. But under either hypotheses (i) or (ii), the non-degenerate one-dimensional random walk with increment distribution $X \cdot \mathbf{u}$ oscillates. □

Proof of Theorem 9.2. First suppose that (i) holds. If $\mathbf{0} \in \operatorname{int}\operatorname{hull}\mathcal{D}$ then there exist $m \in \mathbb{N}$ and $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathcal{D}$ such that $\mathbf{0}$ is also in the interior of the convex polytope $P(\mathbf{u}_1, \ldots, \mathbf{u}_m) := \operatorname{hull}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. Let

$$R(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \inf\{\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^d \setminus P(\mathbf{v}_1, \ldots, \mathbf{v}_m)\},$$

which is zero unless $\mathbf{0}$ lies in the interior of $P(\mathbf{v}_1, \ldots, \mathbf{v}_m)$, when it is equal to the shortest distance from $\mathbf{0}$ to the boundary of $P(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. In particular, note that $R(\mathbf{u}_1, \ldots, \mathbf{u}_m) = \delta_0 > 0$.

For $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^d$, the map $(\mathbf{v}_1, \ldots, \mathbf{v}_m) \mapsto P(\mathbf{v}_1, \ldots, \mathbf{v}_m)$, as a function from $\mathbb{R}^{md}$ to convex, compact subsets of $\mathbb{R}^d$ with the Hausdorff metric, is continuous. So the map from $(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ to $R(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is also continuous. Hence for any $\delta \in (0, \delta_0)$, we can find $\varepsilon > 0$ sufficiently small such that $B(\mathbf{0}; \delta)$ is contained in $P(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ for all $\mathbf{v}_i$ with $\|\mathbf{v}_i - \mathbf{u}_i\| < \varepsilon$. For such an $\varepsilon > 0$, let

$$C_i(r, \varepsilon) = \{\mathbf{x} \in \mathbb{R}^d : \|\hat{\mathbf{x}} - \mathbf{u}_i\| < \varepsilon, \ \|\mathbf{x}\| \geq r\}.$$

Then for any $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $\mathbf{x}_i \in C_i(r, \varepsilon)$, we have that $\operatorname{hull}\{\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_m\}$ contains the ball $B(\mathbf{0}; \delta)$. Thus, since $\|\mathbf{x}_i\| \geq r$,

$$B(\mathbf{0}; r\delta) \subseteq \operatorname{hull}\{r\hat{\mathbf{x}}_1, \ldots, r\hat{\mathbf{x}}_m\} \subseteq \operatorname{hull}\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}.$$

Since $\mathbf{u}_i \in \mathcal{D} = \mathcal{D}_\infty$ (by Theorem 2.8), we have $S_n \in C_i(r, \varepsilon)$ i.o., a.s. Thus $B(\mathbf{0}; r\delta) \subseteq \mathcal{H}_n$ for all but finitely many n. That is $\liminf_{n\to\infty} r_n \geq r\delta$, a.s. Since $r > 0$ was arbitrary, we get $r_\infty = \infty$, a.s. Thus (i) implies (ii), and (ii) is equivalent to (iii) by Lemma 9.1.

Suppose that $\mathbf{u} \in \mathcal{D}_+$, so that $\mathbb{P}(\mathbf{u} \in \mathcal{L}_+) = 1$. Then $S_n \cdot \mathbf{u} \to \infty$, so that $\inf_{n\geq 0}(S_n \cdot \mathbf{u}) = c$ for some $c > -\infty$. It follows that $S_0, S_1, S_2, \ldots$ are contained in the half-space $H_+(\mathbf{u}) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{u} \geq c\}$. Thus $\mathcal{H}_n \subseteq H_+(\mathbf{u})$ for all n, and hence $\mathcal{H}_\infty \subseteq H_+(\mathbf{u})$. Thus $\mathcal{H}_\infty = \mathbb{R}^d$ implies $\mathcal{D}_+ = \mathcal{D}_- = \emptyset$, and so, by Lemma 8.1, (iii) implies (iv).

To show that (iv) implies (iii), we prove the contrapositive. By Lemma 9.1, it suffices to suppose that $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 0$. Since $\operatorname{cl}\mathcal{H}_\infty$ is closed and convex, it can be written as an intersection of hyperplanes (see e.g. Corollary 4.1 of [14, p. 55]); in particular, if $\operatorname{cl}\mathcal{H}_\infty$ is not the whole of $\mathbb{R}^d$, it is contained in a half-space $H_-(\mathbf{u}) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{u} \leq c\}$ for some $\mathbf{u} \in \mathbb{S}^{d-1}$ and $c \in \mathbb{R}$. Thus $\sup_{n\geq 0}(S_n \cdot \mathbf{u}) < \infty$. In particular, $\mathcal{P}_\pm$ is not the whole of $\mathbb{S}^{d-1}$. By Corollary 8.4, this implies that $\mathcal{D}_\pm \neq \mathbb{S}^{d-1}$.

Finally, we show that (iv) implies (v). Suppose that $\mathbf{0} \notin \operatorname{hull}\mathcal{D}$. Since $\operatorname{hull}\mathcal{D}$ is closed, this means that there is a hyperplane that separates $\mathbf{0}$ from $\operatorname{hull}\mathcal{D}$, so there is a $\mathbf{u} \in \mathbb{S}^{d-1}$ and $c < 0$ such that $S(\mathbf{u}) = \{\mathbf{x} \in \mathbb{S}^{d-1} : \mathbf{x} \cdot \mathbf{u} \geq c\}$ contains no point of $\mathcal{D}$. Since $S(\mathbf{u})$ is compact, it must thus contain only finitely many of $\hat{S}_0, \hat{S}_1, \ldots$. That is $\limsup_{n\to\infty}(\hat{S}_n \cdot \mathbf{u}) \leq c$, and hence $\limsup_{n\to\infty}(S_n \cdot \mathbf{u}) \leq 0$. In particular, $\mathcal{P}_\pm$ is not the whole of $\mathbb{S}^{d-1}$, and Corollary 8.4 shows that $\mathcal{D}_\pm \neq \mathbb{S}^{d-1}$. □

10 Some examples

Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard orthonormal basis vectors of $\mathbb{R}^d$. For convenience we locate all our random walks on the integer lattice $\mathbb{Z}^d$, but this is not essential. We write $\xi \sim \text{Rad}$ to mean that $\mathbb{P}(\xi = +1) = \mathbb{P}(\xi = -1) = 1/2$ (a Rademacher distribution), and, for $\alpha > 0$, write $\zeta \sim S(\alpha)$ to mean that $\zeta \in \mathbb{Z}$ has $\mathbb{P}(\zeta \geq r) = \mathbb{P}(\zeta \leq -r) = \frac{1}{2} r^{-\alpha}$ for $r \in \mathbb{N}$. Our examples are constructed mostly from components that are copies of $\xi \sim \text{Rad}$ or $\zeta \sim S(\alpha)$.

If $\xi_1, \xi_2, \ldots$ are independent copies of $\xi \sim \text{Rad}$, then we write $W_n = \sum_{i=1}^n \xi_i$ for the associated simple symmetric random walk (SSRW) on $\mathbb{Z}$. If $\zeta_1, \zeta_2, \ldots$ are independent copies of $\zeta \sim S(\alpha)$, then we write $Y_n = \sum_{i=1}^n \zeta_i$.

We recall some well-known facts about W_n and Y_n. The local limit theorem for SSRW on $\mathbb{Z}$ (see e.g. [8, pp. 141–143]) says that, with ϕ the standard Gaussian density function,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}} \left| \frac{n^{1/2}}{2} \mathbb{P}\left(W_n = 2x - n\right) - \phi\left(\frac{2x-n}{\sqrt{n}}\right) \right| = 0. \tag{10.1}$$

If $\alpha \in (0, 1)$, then Y_n is transient and oscillates: $|Y_n| \to \infty$ and Y_n takes both signs i.o., and, moreover (see e.g. Theorem 3.5 of [13])

$$\text{if } \alpha \in (0,1), \text{ then } \liminf_{n\to\infty} n^{-1} |Y_n| = \infty, \text{ a.s.} \tag{10.2}$$

If $\alpha \in (0, 2)$, $\alpha \neq 1$, then $n^{-1/\alpha} Y_n$ converges in distribution to (a constant multiple of) a symmetric α-stable random variable, since ζ is in the corresponding domain of normal attraction, with no centering (see e.g. Theorem 2.6.7 of [16] and [12, p. 580]). If g is the density of this limiting random variable, then Gnedenko's local limit theorem (see Theorem 4.2.1 of [16]) says that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}} \left| n^{1/\alpha} \mathbb{P}\left(Y_n = x\right) - g(n^{-1/\alpha} x) \right| = 0. \tag{10.3}$$

Note that g is uniformly bounded: this follows from the inversion formula for densities and the fact that the characteristic function of a symmetric stable random variable is of the form $\mathrm{e}^{-c|t|^\alpha}$, for some $c > 0$ (see e.g. [12, p. 570]).

Example 10.1. Suppose that $d = 2$. Let $X = \mathbf{e}_1 + \mathbf{e}_2 \zeta$ where $\zeta \sim S(\alpha)$ for $\alpha > 0$.

If $\alpha > 1$ then $\mathbb{E}\,\|X\| < \infty$ and $\mathbb{E}\,X = \mathbf{e}_1$, so the SLLN implies that S_n is transient with limiting direction $\mathbf{e}_1$, and Proposition 4.1 shows that $\mathcal{D} = \{\mathbf{e}_1\}$.

If $\alpha \in (0, 1)$, then $\|S_n\| \geq |S_n \cdot \mathbf{e}_1| = n$ so the walk is again transient. Write $X_i = \mathbf{e}_1 + \mathbf{e}_2 \zeta_i$ where the ζ_i are independent copies of ζ. Let $Y_n = \sum_{i=1}^n \zeta_i$. For $j = \pm 1$,

$$\|\hat{S}_n - j\mathbf{e}_2\| \leq \frac{n}{\|S_n\|} + \left| \frac{Y_n}{\|S_n\|} - j \right|.$$

By (10.2) we have that $n/\|S_n\| \leq n/|Y_n| \to 0$, a.s., and so $\|S_n\|/|Y_n| \to 1$, a.s., and hence

$$\left| \frac{Y_n}{\|S_n\|} - \operatorname{sgn}(Y_n) \right| \leq \left| \frac{|Y_n|}{\|S_n\|} - 1 \right| \to 0, \text{ a.s.}$$

It follows that, for $j = \pm 1$,

$$\liminf_{n\to\infty} \|\hat{S}_n - j\mathbf{e}_2\| = \liminf_{n\to\infty} |\operatorname{sgn}(Y_n) - j| = 0, \text{ a.s.}$$

Hence $\{\pm\mathbf{e}_2\} \subseteq \mathcal{D}$. On the other hand, if $\mathbf{u} \in \mathbb{S}^1 \setminus \{\pm\mathbf{e}_2\}$, we have $u_1 := \mathbf{u} \cdot \mathbf{e}_1 \neq 0$, and

$$\liminf_{n\to\infty} \|\hat{S}_n - \mathbf{u}\| \geq \liminf_{n\to\infty} \left| \frac{n}{\|S_n\|} - u_1 \right| = |u_1| > 0,$$

so $\mathbf{u} \notin \mathcal{D}$. Thus $\mathcal{D} = \{\pm\mathbf{e}_2\}$.

Finally, note that this example obviously has $\mathcal{H}_\infty \neq \mathbb{R}^2$ (since $S_n \geq 0$ for all n) while $\mathbf{0} \in \operatorname{hull}\mathcal{D}$, but $\mathbf{0} \notin \operatorname{int}\operatorname{hull}\mathcal{D}$. This shows that (iii) and (v) of Theorem 9.2 are not equivalent. △

Example 10.2. Suppose that $d = 2$. Let $X = \mathbf{e}_1\xi + \mathbf{e}_2\zeta$ where ξ and ζ are independent, $\xi \sim \text{Rad}$, and $\zeta \sim S(\alpha)$ for $\alpha > 0$.

First suppose that $\alpha > 2$. Here $\mathbb{E}(\|X\|^2) < \infty$ and $\mathbb{E}\, X = \mathbf{0}$, so the central limit theorem applies, and Corollary 2.10 shows that $\mathcal{D} = \mathbb{S}^1$. Alternatively, note that the walk in this case is recurrent (see e.g. [8, Theorem 4.2.8]) and apply Proposition 2.6.

Next suppose that $\alpha \in (1, 2)$. In this case $\mathbb{E}\, X = \mathbf{0}$ but $\mathbb{E}(\|X\|^2) = \infty$. Here the walk is transient, as follows from the Borel–Cantelli lemma and the local limit theorems (10.1) and (10.3), which together show that $\mathbb{P}(S_n = \mathbf{0}) = \mathbb{P}(W_n = 0)\mathbb{P}(Y_n = 0) = O(n^{-(1/2)-(1/\alpha)})$. By construction, $X \overset{d}{=} -X$, so Proposition 5.2 shows that $\mathcal{D} = \mathbb{S}^1$.

Finally, suppose that $\alpha \in (0, 1)$. Since $|S_n \cdot \mathbf{e}_1| \leq n$, a similar argument to that in Example 10.1 shows that $\mathcal{D} = \{\pm\mathbf{e}_2\}$. Note that this walk is transient, by Corollary 2.7, and, by Corollary 9.4, $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$. This example has $\mathbf{0} \in \operatorname{hull}\mathcal{D}$, $\mathbf{0} \notin \operatorname{int}\operatorname{hull}\mathcal{D}$, and $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$, showing that (i) and (iii) of Theorem 9.2 are not equivalent. △

Example 10.3. Suppose that $d \geq 4$. Let $X = \sum_{k=1}^{d-1} \mathbf{e}_k \zeta^{(k)} + \mathbf{e}_d \xi$ where $\xi, \zeta^{(1)}, \ldots, \zeta^{(d-1)}$ are independent, $\xi \sim \text{Rad}$, and $\zeta^{(k)} \sim S(\alpha)$ for $\alpha \in (1, 2)$. This random walk has $X \overset{d}{=} -X$, $\mu = \mathbf{0}$, and is transient. Let $E_d := \{\mathbf{u} \in \mathbb{S}^{d-1} : \mathbf{u} \cdot \mathbf{e}_d = 0\}$, a copy of $\mathbb{S}^{d-2}$.

Recall that $C(\mathbf{u}; r) = \{\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\} : \|\hat{\mathbf{x}} - \mathbf{u}\| < r\}$. Fix $\varepsilon > 0$, and set

$$B_n := \{(x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : |x_d| \leq n^{(1/2)+\varepsilon}\}.$$

Then we have the estimate

$$\mathbb{P}(S_n \in C(\mathbf{u}; r)) \leq \mathbb{P}(|S_n \cdot \mathbf{e}_d| > n^{(1/2)+\varepsilon}) + \sum_{\mathbf{x} \in B_n \cap C(\mathbf{u}; r)} \mathbb{P}(S_n = \mathbf{x}).$$

Here we have from the local limit theorems (10.1) and (10.3) that, for some $C < \infty$,

$$\mathbb{P}(S_n = \mathbf{x}) = \mathbb{P}(W_n = x_d) \prod_{i=1}^{d-1} \mathbb{P}(Y_n = x_i) \leq C n^{-(d-1)/\alpha} \cdot n^{-1/2},$$

for all $\mathbf{x} \in \mathbb{Z}^d$. Standard binomial tail bounds show that for SSRW $\mathbb{P}(|W_n| > n^{(1/2)+\varepsilon}) \leq C \exp(-cn^{2\varepsilon})$ for constants $c > 0$ and $C < \infty$. Thus we get

$$\mathbb{P}(S_n \in C(\mathbf{u}; r)) \leq C \exp(-cn^{2\varepsilon}) + C \sum_{\mathbf{x} \in B_n \cap C(\mathbf{u}; r)} n^{-(d-1)/\alpha} \cdot n^{-1/2}. \tag{10.4}$$

Fix $\mathbf{u} \notin E_d$, and take $0 < r < |\mathbf{u} \cdot \mathbf{e}_d|$. Then any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in C(\mathbf{u}; r)$ has

$$\big| x_d - \|\mathbf{x}\| \mathbf{u} \cdot \mathbf{e}_d \big| \leq \big\| \mathbf{x} - \|\mathbf{x}\| \mathbf{u} \big\| < r \|\mathbf{x}\|.$$

Thus $(|\mathbf{u}\cdot\mathbf{e}_d|-r)\|\mathbf{x}\| < |x_d| < (|\mathbf{u}\cdot\mathbf{e}_d|+r)\|\mathbf{x}\|$. It follows that there is a constant $C < \infty$ such that $|x_i| < C|x_d|$ for all $1 \le i \le d-1$ and all $\mathbf{x} \in C(\mathbf{u}; r)$. Hence the number of $\mathbf{x} \in B_n \cap C(\mathbf{u}; r)$ is at most $O(n^{(d/2)+d\varepsilon})$. Thus we obtain from (10.4) that

$$\mathbb{P}(S_n \in C(\mathbf{u}; r)) \le C \exp(-cn^{2\varepsilon}) + Cn^{d\varepsilon} n^{-(d-1)(2-\alpha)/(2\alpha)},$$

where $C < \infty$ depends on $\mathbf{u}$ and r, but not ε. Thus for any α satisfying

$$1 < \alpha < \frac{2(d-1)}{1+d} \tag{10.5}$$

we can choose $\varepsilon > 0$ small enough to ensure that $\sum_{n\ge1} \mathbb{P}(S_n \in C(\mathbf{u}; r)) < \infty$. We can find α satisfying (10.5) provided $d > 3$.

Thus if we have $d \ge 4$ and α satisfying (10.5), the Borel–Cantelli lemma shows that $\mathbf{u} \notin \mathcal{D}$ for any $\mathbf{u} \notin E_d$, i.e., $\mathcal{D} \subseteq E_d$. On the other hand, we have $n^{-1/\alpha} S_n$ converges in distribution to $Z = (Z_1, \dots, Z_{d-1}, 0)$, where the Z_i are independent α-stable random variables with $\operatorname{supp} Z_i = \mathbb{R}$. It follows that $\operatorname{supp} \hat{Z} = E_d$, and so, by Proposition 2.9, we conclude that $\mathcal{D} = E_d$. △

We write $\zeta \sim S_+(\alpha)$ to mean that $\zeta \in \mathbb{Z}_+$ has $\mathbb{P}(\zeta \ge r) = r^{-\alpha}$ for $r \in \mathbb{N}$.

Example 10.4. Let $d \in \mathbb{N}$ and $\alpha \in (0, 1)$. Let $X = \sum_{j=1}^k \mathbf{u}_j \zeta^{(j)}$ where $k \in \mathbb{N}$, the $\zeta^{(j)} \sim S_+(\alpha)$ are independent, and $\mathbf{u}_1, \dots, \mathbf{u}_k$ are fixed vectors in $\mathbb{R}^d$. For $\mathbf{z} = (z_1, \dots, z_k) \in \mathbb{R}^k$, set $\Lambda(\mathbf{z}) := \sum_{j=1}^k z_j \mathbf{u}_j$.

Write $X_i = \sum_{j=1}^k \mathbf{u}_j \zeta_i^{(j)}$, where the $\zeta_i^{(j)}$ are independent copies of $\zeta^{(j)}$, and let $Y_n^{(j)} = \sum_{i=1}^n \zeta_i^{(j)}$. Then $n^{-1/\alpha}(Y_n^{(1)}, \dots, Y_n^{(k)})$ converges in distribution to $(Z_1, \dots, Z_k)$, where $Z_1, \dots, Z_k$ are independent, positive α-stable random variables supported on $\mathbb{R}_+$. By the continuous mapping theorem, $n^{-1/\alpha} S_n$ converges in distribution to $\sum_{j=1}^k \mathbf{u}_j Z_j =: V$. Since V is continuous, $\mathbb{P}(V = \mathbf{0}) = 0$, and so $\mathbb{P}(\hat{V} \in \mathbb{S}^{d-1}) = 1$. Thus

$$\operatorname{supp} \hat{V} = C := C(\mathbf{u}_1, \dots, \mathbf{u}_k) := \operatorname{cl}\left\{ \frac{\Lambda(\mathbf{z})}{\|\Lambda(\mathbf{z})\|} : \mathbf{z} \in \mathbb{R}^k,\ z_1, \dots, z_k > 0,\ \|\Lambda(\mathbf{z})\| > 0 \right\}.$$

Hence by Proposition 2.9(ii) we have that $C \subseteq \mathcal{D}$.

To get an inclusion in the other direction, we use the notation of Section 7. We have $\operatorname{supp} X = \operatorname{cl}\{\Lambda(\mathbf{z}) : \mathbf{z} \in \mathbb{N}^k\}$, and for any $\mathbf{x} \in \operatorname{supp} X$, either $\hat{\mathbf{x}} = \mathbf{0}$ (if $\mathbf{x} = \mathbf{0}$) or else $\hat{\mathbf{x}} = \lim_{n\to\infty} \hat{\mathbf{x}}_n \in \mathbb{S}^{d-1}$ with $\mathbf{x}_n = \Lambda(\mathbf{z}_n)$ and $\mathbf{z}_n \in \mathbb{N}^k$. It follows that

$$\left\{ \frac{\Lambda(\mathbf{z})}{\|\Lambda(\mathbf{z})\|} : \mathbf{z} \in \mathbb{N}^k,\ \|\Lambda(\mathbf{z})\| > 0 \right\} \subseteq \mathcal{X}' \subseteq \{\mathbf{0}\} \cup \operatorname{cl}\left\{ \frac{\Lambda(\mathbf{z})}{\|\Lambda(\mathbf{z})\|} : \mathbf{z} \in \mathbb{N}^k,\ \|\Lambda(\mathbf{z})\| > 0 \right\}.$$

Lemma 7.4 then shows that

$$\mathcal{X} = \operatorname{cl}\left\{ \frac{\Lambda(\mathbf{z})}{\|\Lambda(\mathbf{z})\|} : \mathbf{z} \in \mathbb{N}^k,\ \|\Lambda(\mathbf{z})\| > 0 \right\} = \operatorname{cl}\left\{ \frac{\Lambda(\mathbf{z})}{\|\Lambda(\mathbf{z})\|} : \mathbf{z} \in \lambda\mathbb{N}^k,\ \|\Lambda(\mathbf{z})\| > 0 \right\},$$

for any $\lambda > 0$, by scale invariance. It follows that

$$\mathcal{X} = \operatorname{cl}\left\{ \frac{\Lambda(\mathbf{z})}{\|\Lambda(\mathbf{z})\|} : \mathbf{z} \in \mathbb{Q}^k,\ z_1, \dots, z_k > 0,\ \|\Lambda(\mathbf{z})\| > 0 \right\}.$$

Since $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$, we get $\mathcal{X} = C$. Moreover, C is the closure of an s-convex set, and hence itself s-convex, by Lemma 7.8, and hence $\operatorname{cl}\operatorname{s-hull} \mathcal{X} = \operatorname{s-hull} \mathcal{X} = C$, by Lemma 7.7. Then Theorem 7.1 confirms that $\mathcal{D} = C$. △

11 Concluding remarks

The Borel–Cantelli lemma shows that if for some $\varepsilon > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(\|\hat{S}_n - \mathbf{u}\| < \varepsilon) < \infty$, then $\mathbb{P}(S_n \in C(\mathbf{u};\varepsilon) \text{ i.o.}) = 0$, and so $\mathbf{u} \notin \mathcal{D}$, by Proposition 2.5. This is not sharp, however, as is already shown by the case of $d = 1$, when, for example, $+1 \in \mathcal{D}$ if and only if $\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(S_n > 0) = \infty$ [12, p. 415].

Problem 11.1. *Is there a criterion for* $\mathbf{u} \in \mathcal{D}$ *in terms of* $\mathbb{P}(S_n \in \cdot\,)$*?*

We do not necessarily expect a simple answer to Problem 11.1: in $d = 1$, Kesten (Corollary 1 of [19, p. 1177]) gives a criterion for $\mathbf{x} \in \mathcal{A}^\alpha$ where $\mathcal{A}^\alpha$ is as defined at (3.1).

Proposition 5.2 leaves the following question.

Problem 11.2. *Suppose that* $d = 2$, $\mathbb{E}\,\|X\| < \infty$*, and* $\mu = \mathbf{0}$*. Is* $\mathcal{D}$ *always equal to* $\mathbb{S}^1$*?*

A The recurrent case

For most of the questions in the present paper, the main interest is the transient case, because, loosely speaking, any recurrent random walk explores all of space and hence all directions at all distances. Proposition A.1 is a precise version of this statement. Recall [8, p. 190] that S_n is *recurrent* if there is a non-empty set $\mathcal{R}$ of points $\mathbf{x} \in \mathbb{R}^d$ (the recurrent values) such that, for any $\varepsilon > 0$, $\|S_n - \mathbf{x}\| < \varepsilon$ i.o., a.s.

Proposition A.1. *If* S_n *is recurrent, then there exists* $h > 0$ *such that a.s., for any* $\mathbf{x} \in \mathbb{R}^d$*,* $S_n \in B(\mathbf{x}; h)$ *i.o.*

Proof. Since S_n is recurrent, the set $\mathcal{R}$ of recurrent values is a closed subgroup of $\mathbb{R}^d$ and coincides with the set of *possible values* for the walk: see [8, p. 190]. Since S_n is genuinely d-dimensional, it follows from e.g. Theorem 21.2 of [1, p. 225] that $\mathcal{R}$ contains a further closed subgroup $\mathcal{R}'$ of the form $H\mathbb{Z}^d$ where H is a non-singular d by d matrix. Hence there exists $h > 0$ such that for every $\mathbf{x} \in \mathbb{R}^d$ there exists $\mathbf{y} \in \mathcal{R}'$ with $\|\mathbf{x} - \mathbf{y}\| < h/2$, and since $\mathcal{R}'$ is a countable set of recurrent values for the walk, we have that, a.s., for any $\mathbf{x} \in \mathbb{R}^d$, $S_n \in B(\mathbf{x}; h)$ i.o. □

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