Solving Problems on Generalized Convex Graphs via Mim-Width*

Flavia Bonomo-Braberman[1 [0000−0002−9872−7528]],
Nick Brettell[2 [0000−0002−1136−418X]],
Andrea Munaro[3 [0000−0003−1509−8832]], and
Daniël Paulusma[4 [0000−0001−5945−9287]]

[1] ICC (CONICET-UBA) and Departamento de Computación,
Universidad de Buenos Aires, Argentina, fbonomo@dc.uba.ar
[2] School of Mathematics and Statistics, Victoria University of Wellington,
New Zealand, nick.brettell@vuw.ac.nz
[3] School of Mathematics and Physics, Queen's University Belfast, UK,
a.munaro@qub.ac.uk
[4] Department of Computer Science, Durham University, UK,
daniel.paulusma@durham.ac.uk

Abstract. A bipartite graph $G = (A, B, E)$ is $\mathcal{H}$-convex, for some family of graphs $\mathcal{H}$, if there exists a graph $H \in \mathcal{H}$ with $V(H) = A$ such that the set of neighbours in A of each $b \in B$ induces a connected subgraph of H. Many NP-complete problems become polynomial-time solvable for $\mathcal{H}$-convex graphs when $\mathcal{H}$ is the set of paths. In this case, the class of $\mathcal{H}$-convex graphs is known as the class of convex graphs. The underlying reason is that this class has bounded mim-width. We extend the latter result to families of $\mathcal{H}$-convex graphs where (i) $\mathcal{H}$ is the set of cycles, or (ii) $\mathcal{H}$ is the set of trees with bounded maximum degree and a bounded number of vertices of degree at least 3. As a consequence, we can reprove and strengthen a large number of results on generalized convex graphs known in the literature. To complement result (ii), we show that the mim-width of $\mathcal{H}$-convex graphs is unbounded if $\mathcal{H}$ is the set of trees with arbitrarily large maximum degree or an arbitrarily large number of vertices of degree at least 3. In this way we are able to determine complexity dichotomies for the aforementioned graph problems. Afterwards we perform a more refined width-parameter analysis, which shows even more clearly which width parameters are bounded for classes of $\mathcal{H}$-convex graphs.

1 Introduction

Many computationally hard graph problems can be solved efficiently if we place constraints on the input. Instead of solving individual problems in an ad hoc

* Brettell and Paulusma received support from the Leverhulme Trust (RPG-2016-258). Bonomo received support from UBACyT (20020170100495BA and 20020160100095BA). Brettell also received support from a Rutherford Foundation Postdoctoral Fellowship, administered by the Royal Society Te Apārangi.

way we may try to decompose the vertex set of the input graph into large sets of "similarly behaving" vertices and to exploit this decomposition for an algorithmic speed up that works for many problems simultaneously. This requires some notion of an "optimal" vertex decomposition, which depends on the type of vertex decomposition used and which may relate to the minimum number of sets or the maximum size of a set in a vertex decomposition. An optimal vertex decomposition gives us the "width" of the graph. A graph class has *bounded width* if every graph in the class has width at most some constant c. Boundedness of width is often the underlying reason why a graph-class-specific algorithm runs efficiently: in such a case, the proof that the algorithm is efficient for some special graph class reduces to a proof showing that the width of the class is bounded by some constant. We will give examples, but also refer to the surveys [16,19,22,26,42] for further details and examples.

Width parameters differ in strength. A width parameter p *dominates* a width parameter q if there is a function f such that $p(G)$ is at most $f(q(G))$ for every graph G. If p dominates q but q does not dominate p, then we say that p is *more powerful* than q. If both p and q dominate each other, then p and q are *equivalent*. If neither p is more powerful than q nor q is more powerful than p, then p and q are *incomparable*. If p is more powerful than q, then the class of graphs for which p is bounded is larger than the class of graphs for which q is bounded and so efficient algorithms for bounded p have greater applicability with respect to the graphs under consideration. The *trade-off* is that fewer problems exhibit an efficient algorithm for the parameter p, compared to the parameter q.

The notion of powerfulness leads to a large hierarchy of width parameters, in which new width parameters continue to be defined. The well-known parameters boolean-width, clique-width, module-width and rank-width are equivalent to each other [10,34,38]. They are more powerful than the equivalent parameters branch-width and treewidth [14,39,42] but less powerful than mim-width [42], which is less powerful than sim-width [27]. To give another example, thinness is more powerful than path-width [33], but less powerful than mim-width and incomparable to clique-width or treewidth [4].

For each group of equivalent width parameters, a growing set of NP-complete problems is known to be tractable on graph classes of bounded width. However, there are still large families of graph classes for which boundedness of width is not known for many width parameters.

Our Focus. We consider the relatively new width parameter *mim-width*, which we define below. Recently, we showed in [7,8] that boundedness of mim-width is the underlying reason why some specific hereditary graph classes, characterized by two forbidden induced subgraphs, admit polynomial-time algorithms for a range of problems including k-Colouring and its generalization List k-Colouring (the algorithms are given in [13,15,20]). Here we prove that the same holds for certain *superclasses of convex graphs* known in the literature. Essentially all the known polynomial-time algorithms for such classes are obtained by reducing to the class of convex graphs. We show that our new approach via mim-width simplifies the analysis, unifies the sporadic approaches and explains the reductions to convex graphs.

Mim-width. A set of edges M in a graph G is a *matching* if no two edges of M share an endpoint. A matching M is *induced* if there is no edge in G between vertices of different edges of M. Let $(A, \overline{A})$ be a partition of the vertex set of a graph G. Then $G[A, \overline{A}]$ denotes the bipartite subgraph of G induced by the edges with one endpoint in A and the other in $\overline{A}$. Vatshelle [42] introduced the notion of *maximum induced matching width*, also called mim-width. Mim-width measures the extent to which it is possible to decompose a graph G along certain vertex partitions $(A, \overline{A})$ such that the size of a maximum induced matching in $G[A, \overline{A}]$ is small. The kind of vertex partitions permitted stem from classical branch decompositions. A *branch decomposition* for a graph G is a pair (T, δ), where T is a subcubic tree and δ is a bijection from $V(G)$ to the leaves of T. Every edge $e \in E(T)$ partitions the leaves of T into two classes, L_e and $\overline{L_e}$, depending on which component of $T - e$ they belong to. Hence, e induces a partition $(A_e, \overline{A_e})$ of $V(G)$, where $\delta(A_e) = L_e$ and $\delta(\overline{A_e}) = \overline{L_e}$. Let $\mathrm{cutmim}_G(A_e, \overline{A_e})$ be the size of a maximum induced matching in $G[A_e, \overline{A_e}]$. Then the *mim-width* $\mathrm{mimw}_G(T, \delta)$ of (T, δ) is the maximum value of $\mathrm{cutmim}_G(A_e, \overline{A_e})$ over all edges $e \in E(T)$. The *mim-width* $\mathrm{mimw}(G)$ of G is the minimum value of $\mathrm{mimw}_G(T, \delta)$ over all branch decompositions (T, δ) for G. We refer to Figure 1 for an example.

Computing the mim-width is NP-hard [40], and approximating the mim-width in polynomial time within a constant factor of the optimal is not possible unless $\mathsf{NP} = \mathsf{ZPP}$ [40]. It is not known how to compute in polynomial time a branch decomposition for a graph G whose mim-width is bounded by some function in the mim-width of G. However, for graph classes of bounded mim-width this might be possible. In that case, the mim-width of $\mathcal{G}$ is said to be *quickly computable*. One can then try to develop a polynomial-time algorithm for the graph problem under consideration via dynamic programming over the computed branch decomposition. We give examples of such problems later.

Convex Graphs and Generalizations. A bipartite graph $G = (A, B, E)$ is *convex* if there exists a path P with $V(P) = A$ such that the neighbours in A of each $b \in B$ induce a connected subpath of P. Convex graphs generalize bipartite permutation graphs (see, e.g., [5]) and form a well-studied graph class.

Belmonte and Vatshelle [1] proved that the mim-width of convex graphs is bounded and quickly computable. We consider superclasses of convex graphs and research to what extent mim-width can play a role in obtaining polynomial-time algorithms for problems on these classes.

Let $\mathcal{H}$ be a family of graphs. A bipartite graph $G = (A, B, E)$ is $\mathcal{H}$*-convex* if there exists a graph $H \in \mathcal{H}$ with $V(H) = A$ such that the set of neighbours in A of each $b \in B$ induces a connected subgraph of H. If $\mathcal{H}$ consists of all paths, we obtain the class of convex graphs. A *caterpillar* is a tree T that contains a path P, the *backbone* of T, such that every vertex not on P has a neighbour on P. A caterpillar with a backbone consisting of one vertex is a *star*. A *comb* is a caterpillar such that every backbone vertex has exactly one neighbour outside the backbone. The *subdivision* of an edge uv replaces uv by a new vertex w and edges uw and wu. A *triad* is a tree that can be obtained from a 4-vertex star after a sequence of subdivisions. For $t, \Delta \geq 0$, a (t,Δ)-*tree* is a tree with maximum degree at most Δ and containing at most t vertices of degree at least 3; note that, for example, a triad is a $(1,3)$-tree. If $\mathcal{H}$ consists of all cycles, all trees, all stars, all triads, all combs or all (t,Δ)-trees, then we obtain the class of *circular convex graphs*, *tree convex graphs*, *star convex graphs*, *triad convex graphs*, *comb convex graphs* or (t,Δ)-*tree convex graphs*, respectively. See Figure 1 for an example.

To show the relationships between the above graph classes we need some extra terminology. Let $\mathcal{C}_{t,\Delta}$ be the class of (t,Δ)-tree convex graphs. For fixed t or Δ, we have increasing sequences $\mathcal{C}_{t,0} \subseteq \mathcal{C}_{t,1} \subseteq \cdots$ and $\mathcal{C}_{0,\Delta} \subseteq \mathcal{C}_{1,\Delta} \subseteq \cdots$. For $t \in \mathbb{N}$, the class of (t,∞)-*tree convex graphs* is $\bigcup_{\Delta\in\mathbb{N}} \mathcal{C}_{t,\Delta}$, denoted by $\mathcal{C}_{t,\infty}$. Similarly, for $\Delta \in \mathbb{N}$, the class of (∞,Δ)-*tree convex graphs* is $\bigcup_{t\in\mathbb{N}} \mathcal{C}_{t,\Delta}$, denoted by $\mathcal{C}_{\infty,\Delta}$. Hence, $\mathcal{C}_{t,\infty}$ and $\mathcal{C}_{\infty,\Delta}$ are the set-theoretic limits of the increasing sequences $\{\mathcal{C}_{t,\Delta}\}_{\Delta\in\mathbb{N}}$ and $\{\mathcal{C}_{t,\Delta}\}_{t\in\mathbb{N}}$, respectively. The class of (∞,∞)-*tree convex graphs* is $\bigcup_{t,\Delta\in\mathbb{N}} \mathcal{C}_{t,\Delta}$, which coincides with the class of tree convex graphs. Notice that the class of convex graphs coincides with $\mathcal{C}_{t,2}$, for any $t \in \mathbb{N} \cup \{\infty\}$, and with $\mathcal{C}_{0,\Delta}$, for any $\Delta \in \mathbb{N} \cup \{\infty\}$. The class of star convex graphs coincides with $\mathcal{C}_{1,\infty}$. Moreover, each triad convex graph belongs to $\mathcal{C}_{1,3}$ and each comb convex graph belongs to $\mathcal{C}_{\infty,3}$. A bipartite graph is *chordal bipartite* if every induced cycle in it has exactly four vertices. Every convex graph is chordal bipartite (see, e.g., [5]) and every chordal bipartite graph is tree convex (see [24,29]). In Figure 2 we display these and other relationships, which directly follow from the definitions.

Brault-Baron et al. [6] proved that chordal bipartite graphs have unbounded mim-width. Hence, the result of [1] for convex graphs cannot be generalized to

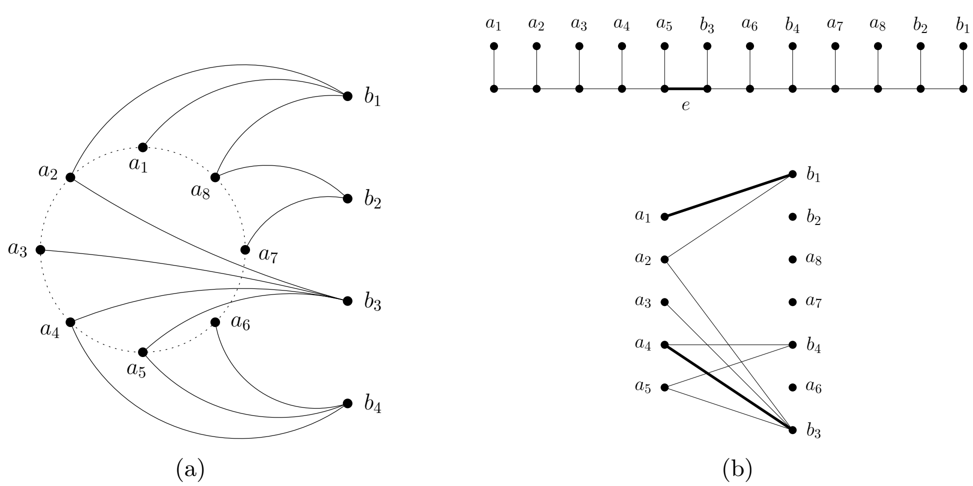

Fig. 1: (a) A circular convex graph $G = (A, B, E)$ with a circular ordering on A. (b) A branch decomposition (T, δ) for G, where T is a caterpillar with a specified edge e, together with the graph $G[A_e, \overline{A_e}]$. The bold edges in $G[A_e, \overline{A_e}]$ form an induced matching and it is easy to see that $\text{cutmim}_G(A_e, \overline{A_e}) = 2$.

chordal bipartite graphs. We determine the mim-width of the other classes in Figure 2 but first discuss known algorithmic results for these classes.

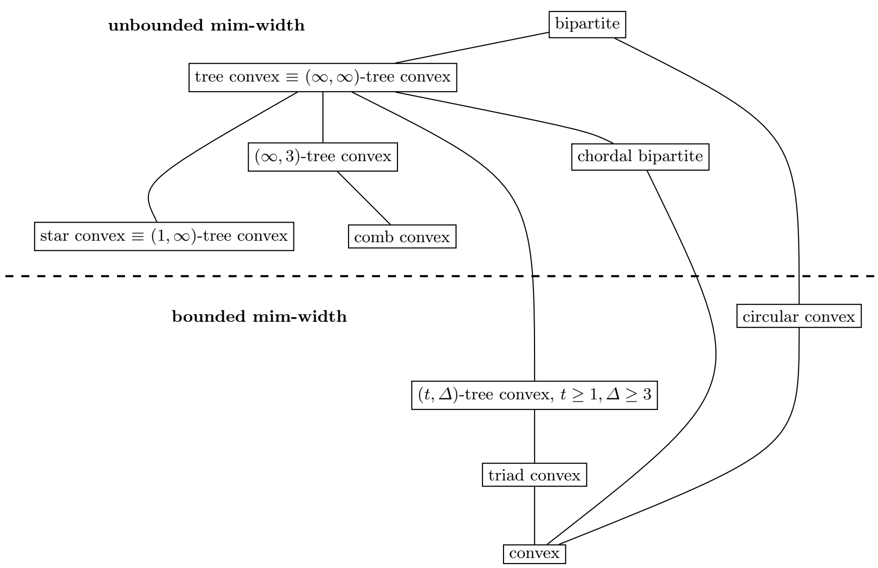

Fig. 2: The inclusion relations between the classes we consider. A line from a lower-level class to a higher one means the first class is contained in the second.

Known Results. Belmonte and Vatshelle [1] and Bui-Xuan et al. [11] proved that so-called Locally Checkable Vertex Subset and Vertex Partitioning (LC-VSVP) problems are polynomial-time solvable on graph classes whose mim-width is bounded and quickly computable. This result was extended by Bergougnoux and Kanté [2] to variants of such problems with additional constraints on connectivity or acyclicity. Each of the problems mentioned below is a special case of a Locally Checkable Vertex Subset (LCVS) problem possibly with one of the two extra constraints. Panda et al. [36] proved that Induced Matching is polynomial-time solvable for circular convex and triad convex graphs, but NP-complete for star convex and comb convex graphs. Pandey and Panda [37] proved that Dominating Set is polynomial-time solvable for circular convex, triad convex and $(1, \Delta)$-tree convex graphs for every $\Delta \geq 1$. Liu et al. [31] proved that Connected Dominating Set is polynomial-time solvable for circular convex and triad convex graphs. Chen et al. [12] showed that (Connected) Dominating Set and Total Dominating Set are NP-complete for star convex and comb convex graphs. Lu et al. [32] proved that Independent Dominating Set is polynomial-time solvable for circular convex and triad convex graphs. The latter result was shown already in [41] using a dynamic programming approach instead of a reduction to convex graphs [32]. Song et al. [41] showed in fact a stronger result, namely that Independent Dominating Set is polynomial-time solvable for (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$. They also showed in [41] that Independent Dominating Set is NP-complete for star convex and comb convex graphs. Hence, they obtained a dichotomy: Independent Dominating Set is polynomial-time solvable for (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$ but NP-complete for $(\infty, 3)$-tree convex graphs and $(1, \infty)$-tree convex graphs.

The same dichotomy (explicitly formulated in [44]) holds for Feedback Vertex Set and is obtained similarly. Namely, Jiang et al. [25] proved that this problem is polynomial-time solvable for triad convex graphs and mentioned that their algorithm can be generalized to (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$. Jiang et al. [24] proved that Feedback Vertex Set is NP-complete for star convex and comb convex graphs. In addition, Liu et al. [30] proved that Feedback Vertex Set is polynomial-time solvable for circular convex graphs, whereas Jiang et al. [24] proved that the Weighted Feedback Vertex Set problem is polynomial-time solvable for triad convex graphs.

It turns out that the above problems are polynomial-time solvable on circular convex graphs and subclasses of (t, Δ)-tree convex graphs, but NP-complete for star convex graphs and comb convex graphs. In contrast, Panda and Chaudhary [35] proved that Dominating Induced Matching is not only polynomial-time solvable on circular convex and triad convex graphs, but also on star convex graphs. Nevertheless, we notice *a common pattern*: many dominating set, induced matching and graph transversal type of problems are polynomial-time solvable for (t, Δ)-tree convex graphs, for every $t \geq 1$ and $\Delta \geq 3$, and NP-complete for comb convex graphs, and thus for $(\infty, 3)$-tree convex graphs, and star convex graphs, or equivalently, $(1, \infty)$-tree convex graphs. Moreover, essentially all the polynomial-time algorithms reduce the input to a convex graph.

Our Results. We simplify the analysis, unify the above approaches and explain the reductions to convex graphs, using mim-width. We prove three results that, together with the fact that chordal bipartite graphs have unbounded mim-width [6], explain the dotted line in Figure 2. The first two results generalize the result of [1] for convex graphs. The third result gives two new reasons why tree convex graphs (that is, (∞, ∞)-tree convex graphs) have unbounded mim-width.

Theorem 1. *Let G be a circular convex graph. Then* $\mathrm{mimw}(G) \leq 2$. *Moreover, we can construct in polynomial time a branch decomposition (T, δ) for G with* $\mathrm{mimw}_G(T, \delta) \leq 2$.

Theorem 2. *Let G be a (t, Δ)-tree convex graph with $t, \Delta \in \mathbb{N}$ and $t \geq 1$ and $\Delta \geq 3$. Let*

$$f(t, \Delta) = \max \left\{ 2 \left\lfloor \left(\frac{\Delta}{2} \right)^2 \right\rfloor, 2\Delta - 1 \right\} + t^2 \Delta.$$

Then $\mathrm{mimw}(G) \leq f(t, \Delta)$. *Moreover, we can construct in polynomial time a branch decomposition (T, δ) for G with* $\mathrm{mimw}_G(T, \delta) \leq f(t, \Delta)$.

Theorem 3. *The class of star convex graphs and the class of comb convex graphs each has unbounded mim-width.*

Hence, we obtain a structural dichotomy (recall that star convex graphs are the $(1, \infty)$-tree convex graphs and that comb convex graphs are $(\infty, 3)$-tree convex):

Corollary 1. *Let $t, \Delta \in \mathbb{N} \cup \{\infty\}$ with $t \geq 1$ and $\Delta \geq 3$. The class of (t, Δ)-tree convex graphs has bounded mim-width if and only if $\{t, \Delta\} \cap \{\infty\} = \varnothing$.*

Algorithmic Consequences. As discussed, the following six problems were shown to be NP-complete for star convex and comb convex graphs, and thus for $(1, \infty)$-tree convex graphs and $(\infty, 3)$-tree convex graphs: Feedback Vertex Set [2,24]; Dominating Set, Connected Dominating Set, Total Dominating Set [12]; Independent Dominating Set [41]; Induced Matching [36]. These problems are examples of LCVS problems, possibly with connectivity or acyclicity constraints. Hence, they are polynomial-time solvable for every graph class whose mim-width is bounded and quickly computable [1,2,11]. Recall that the same holds for Weighted Feedback Vertex Set [23] and (Weighted) Subset Feedback Vertex Set [3]; these three problems generalize Feedback Vertex Set and are thus NP-complete for star convex graphs and comb convex graphs. Combining these results with Corollary 1 yields the following complexity dichotomy.

Corollary 2. *Let $t, \Delta \in \mathbb{N} \cup \{\infty\}$ with $t \geq 1$, $\Delta \geq 3$ and Π be one of the nine problems mentioned above, restricted to (t, Δ)-tree convex graphs. If $\{t, \Delta\} \cap \{\infty\} = \varnothing$, then Π is polynomial-time solvable; otherwise, Π is NP-complete.*

It is worth noting that this complexity dichotomy does not hold for all LCVS problems; recall that Dominating Induced Matching is polynomial-time solvable on star convex graphs [35]. Theorems 1 and 2, combined with the result of [11], imply that this problem is also polynomial-time solvable on circular convex graphs and (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$.

Further Algorithmic Consequences. Theorems 1 and 2, combined with the result of [28], also generalize a result of Díaz et al. [17] for List k-Colouring on convex graphs to circular convex and (t, Δ)-tree convex graphs ($t \geq 1$, $\Delta \geq 3$).

Additional Structural Results. We prove Theorems 1–3 in Sections 2–4, respectively. In Section 5 we perform a more refined analysis. We consider a hierarchy of width parameters and determine exactly which of the generalized convex classes considered in the previous sections have bounded width for each of these parameters. This does not yet yield any new algorithmic results. In the same section we also give some other research directions.

Preliminaries. Let $G = (V, E)$ be a graph. For $v \in V$, the *neighbourhood* $N_G(v)$ is the set of vertices adjacent to v. The *degree* $d(v)$ of a vertex $v \in V$ is the size $|N_G(v)|$. A vertex of degree k is a *k-vertex*. A graph is *subcubic* if every vertex has degree at most 3. We let $\Delta(G) = \max\{d(v) : v \in V\}$. For disjoint $S, T \subseteq V$, we say that S is *complete to* T if every vertex of S is adjacent to every vertex of T. For $S \subseteq V$, $G[S] = (S, \{uv : u, v \in S, uv \in E\})$ is the subgraph of G *induced* by S. The *disjoint union* $G + H$ of graphs G and H has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. A graph is *r-partite*, for $r \geq 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. A 2-partite graph is also called *bipartite*. A graph G is a *support* for a hypergraph $H = (V, \mathcal{S})$ if the vertices of G correspond to the vertices of H and, for each hyperedge $S \in \mathcal{S}$, the subgraph of G induced by S is connected. When a bipartite graph $G = (A, B, E)$ is viewed as a hypergraph $H = (A, \{N(b) : b \in B\})$, then a support T for H with $T \in \mathcal{H}$ is a witness that G is $\mathcal{H}$-convex.

2 The Proof of Theorem 1

We need the following known lemma on recognizing circular convex graphs.

Lemma 1 (see, e.g., Buchin et al. [9]). *Circular convex graphs can be recognized and a cycle support computed, if it exists, in polynomial time.*

For an integer $\ell \geq 1$, an *ℓ-caterpillar* is a subcubic tree T on 2ℓ vertices with $V(T) = \{s_1, \ldots, s_\ell, t_1, \ldots, t_\ell\}$, such that $E(T) = \{s_i t_i \ : \ 1 \leq i \leq \ell\} \cup \{s_i s_{i+1} \ : \ 1 \leq i \leq \ell - 1\}$. Note that we label the leaves of an ℓ-caterpillar $t_1, t_2, \ldots, t_\ell$, in this order. Given a total ordering $\prec$ of length ℓ, we say that (T, δ) is *obtained from* $\prec$ if T is an ℓ-caterpillar and δ is the natural bijection from the ℓ ordered elements to the leaves of T. We are now ready to prove Theorem 1.

Theorem 1 (restated). *Let G be a circular convex graph. Then* $\mathrm{mimw}(G) \leq 2$. *Moreover, we can construct in polynomial time a branch decomposition (T, δ) for G with* $\mathrm{mimw}_G(T, \delta) \leq 2$.

Proof. Let $G = (A, B, E)$ be a circular convex graph with a circular ordering on A. By Lemma 1, we construct in polynomial time such an ordering $a_1, \ldots, a_n$, where $n = |A|$ (see Figure 1). Let $B_1 = N(a_n)$ and $B_2 = B \setminus B_1$. We obtain a total ordering $\prec$ on $V(G)$ by extending the ordering $a_1, \ldots, a_n$ as follows. Each $b \in B_1$ is inserted after a_n, breaking ties arbitrarily. Each $b \in B_2$ is inserted immediately after the largest element of A it is adjacent to (hence immediately after some a_i with $1 \leq i < n$), breaking ties arbitrarily.

Let T be the $|V(G)|$-caterpillar obtained from $\prec$. Below we will prove that $\mathrm{mimw}_G(T, \delta) \leq 2$. Let $e \in E(T)$. We may assume without loss of generality that e is not incident to a leaf of T. Let M be a maximum induced matching of $G[A_e, \overline{A_e}]$. As e is not incident to a leaf, we may assume without loss of generality that each vertex in $\overline{A_e}$ is larger than any vertex in A_e in the ordering $\prec$.

We first observe that at most one edge of M has one endpoint in B_2. Indeed, suppose there exist two edges $xy, x'y' \in M$, each with one endpoint in B_2, say without loss of generality $\{y, y'\} \subseteq B_2$. Since each vertex in B_2 is adjacent only to smaller vertices, $\{y, y'\} \subseteq \overline{A_e}$ and $\{x, x'\} \subseteq A_e$. Without loss of generality, $y \prec y'$. However, $N(y)$ and $N(y')$ are intervals of the ordering and so either $x \in N(y')$ or $x' \in N(y)$, contradicting the fact that M is induced.

We now show that at most two edges in M have an endpoint in B_1 and, if exactly two such edges are in M, then no edge with an endpoint in B_2 is.

First suppose that three edges of M have one endpoint in B_1 and let u_1, u_2, u_3 be these endpoints. Since $N(u_1)$, $N(u_2)$ and $N(u_3)$ are intervals of the circular ordering on A all containing a_n, one of these neighbourhoods is contained in the union of the other two, contradicting the fact that M is induced.

Finally suppose exactly two edges u_1v_1 and $u_2v_2 \in M$ have one endpoint in B_1 and thus their other endpoint in A. Let $\{u_1, u_2\} \subseteq \overline{A_e}$ and $\{v_1, v_2\} \subseteq A_e$. Then, as each vertex in $\overline{A_e}$ is larger than any vertex in A_e in $\prec$, we find that u_1 and u_2 belong to B_1 and thus $\{v_1, v_2\} \subseteq A$. Now if there is some edge $u_3v_3 \in M$ such that $u_3 \in B_2$, then $u_3 \in \overline{A_e}$. Recall that $N(u_1)$ and $N(u_2)$ are intervals of the circular ordering on A both containing a_n. Since M is induced, for each $i, j \in \{1, 2\}$, we have that $v_i \in N(u_j)$, if $i = j$, and $v_i \notin N(u_j)$, if $i \neq j$. This implies that one of v_1 and v_2 is larger than v_3 in $\prec$ and so it is contained in $N(u_3)$, contradicting the fact that M is induced. This concludes the proof. □

3 The Proof of Theorem 2

We need the following lemma on recognizing (t, Δ)-tree convex graphs[5].

Lemma 2. *For $t, \Delta \in \mathbb{N}$, (t, Δ)-tree convex graphs can be recognized and a (t, Δ)-tree support computed, if it exists, in $O(n^{t+3})$ time.*

Proof. Given a hypergraph $H = (V, \mathcal{S})$ together with degrees d_i for each $i \in V$, Buchin et al. [9] provided an $O(|V|^3 + |\mathcal{S}||V|^2)$ time algorithm that solves the following decision problem: Is there a tree support for H such that each vertex i of the tree has degree at most d_i? If it exists, the algorithm computes a tree support satisfying this property. Given as input a bipartite graph $G = (A, B, E)$, we consider the hypergraph $H = (A, \mathcal{S})$, where $\mathcal{S} = \{N(b) : b \in B\}$. For each of the $\binom{|A|}{t} = O(|A|^t)$ subsets $A' \subseteq A$ of size t we proceed as follows: we assign a degree Δ to each of its elements and a degree 2 to each element in $A \setminus A'$. We then apply the algorithm in [9] to the $O(|A|^t)$ instances thus constructed. If G is (t, Δ)-tree convex, then the algorithm returns a (t, Δ)-tree support for H. □

The proof of Theorem 2 heavily relies on the following result for mim-width.

Lemma 3 (Brettell et al. [8]). *Let G be a graph and $(X_1, \ldots, X_p)$ be a partition of $V(G)$ such that $\operatorname{cutmim}_G(X_i, X_j) \leq c$ for all distinct $i, j \in \{1, \ldots, p\}$, and $p \geq 2$. Let $h = \max\left\{c \left\lfloor \left(\frac{p}{2}\right)^2 \right\rfloor, \max_{i \in \{1,\ldots,p\}}\{\operatorname{mimw}(G[X_i])\} + c(p-1)\right\}$. Then $\operatorname{mimw}(G) \leq h$. Moreover, given a branch decomposition (T_i, δ_i) for $G[X_i]$ for each i, we can construct in $O(p)$ time a branch decomposition (T, δ) for G with $\operatorname{mimw}_G(T, \delta) \leq h$.*

[5] Jiang et al. [24] proved that Weighted Feedback Vertex Set is polynomial-time solvable for triad convex graphs if a triad support is given as input. They observed that an associated tree support can be constructed in linear time, but this does not imply that a triad support can be obtained. Lemma 2 shows that indeed a triad support can be obtained in polynomial time and need not be provided on input.

We also need the following lemma (proof omitted).

Lemma 4. *Let G be a $(1, \Delta)$-tree convex graph, for some $\Delta \geq 3$. Let $f(\Delta) = \max\left\{2\left\lfloor\left(\frac{\Delta}{2}\right)^2\right\rfloor, 2\Delta - 1\right\}$. Then $\operatorname{mimw}(G) \leq f(\Delta)$, and we can construct in polynomial time a branch decomposition (T, δ) for G with $\operatorname{mimw}_G(T, \delta) \leq f(\Delta)$.*

We are now ready to prove Theorem 2.

Theorem 2 (restated). *Let G be a (t, Δ)-tree convex graph with $t, \Delta \in \mathbb{N}$ and $t \geq 1$ and $\Delta \geq 3$. Let*

$$f(t, \Delta) = \max\left\{2\left\lfloor\left(\frac{\Delta}{2}\right)^2\right\rfloor, 2\Delta - 1\right\} + t^2\Delta.$$

Then $\operatorname{mimw}(G) \leq f(t, \Delta)$. Moreover, we can construct in polynomial time a branch decomposition (T, δ) for G with $\operatorname{mimw}_G(T, \delta) \leq f(t, \Delta)$.

Proof. We use induction on t. If $t = 1$, the result follows from Lemma 4. Let $t > 1$ and let $G = (A, B, E)$ be a (t, Δ)-tree convex graph. By Lemma 2, we can compute in polynomial time a (t, Δ)-tree T with $V(T) = A$ and such that, for each $v \in B$, $N_G(v)$ forms a subtree of T. Consider an edge $uv \in E(T)$ such that $T - uv$ is the disjoint union of a (t_1, Δ)-tree T_1 containing u and a (t_2, Δ)-tree T_2 containing v, where $\max\{t_1, t_2\} < t$ and $t_1, t_2 \geq 1$. Clearly such an edge can be found in linear time. For $i \in \{1, 2\}$, let $V(T_i) = A_i$. Clearly, $A = A_1 \cup A_2$. We now partition B into two classes as follows. The set B_1 contains all vertices in B with at least one neighbour in A_1, and $B_2 = B \setminus B_1$. In view of Lemma 3, we then consider the partition $(A_1 \cup B_1, A_2 \cup B_2)$ of $V(G)$. For $i \in \{1, 2\}$, $G[A_i \cup B_i]$ is a (t_i, Δ)-tree convex graph with $t_i < t$ and so, by the induction hypothesis, $\operatorname{mimw}(G[A_i \cup B_i]) \leq \max\left\{2\left\lfloor\left(\frac{\Delta}{2}\right)^2\right\rfloor, 2\Delta - 1\right\} + (t-1)^2\Delta$.

We now claim that $\operatorname{cutmim}_G(A_1 \cup B_1, A_2 \cup B_2) \leq \Delta(t-1)$. Let M be a maximum induced matching in $G[A_1 \cup B_1, A_2 \cup B_2]$. Since no vertex in B_2 has a neighbour in A_1, all edges in M have one endpoint in B_1 and the other in A_2. We now consider the (t_2, Δ)-tree T_2 as a tree rooted at v, so that the nodes of T_2 inherit a corresponding ancestor/descendant relation. Since T_2 has maximum degree at most Δ and contains at most t_2 vertices of degree at least 3, it has at most $\Delta t_2 \leq \Delta(t-1)$ leaves. Suppose, to the contrary, that $|M| > \Delta(t-1)$. We first claim that there exist $xy, x'y' \in M$ with $\{y, y'\} \subseteq A_2$ and such that y' is a descendant of y. Indeed, for each leaf z of T_2, consider the unique z, v-path in T_2. There are at most $\Delta(t-1)$ such paths and each vertex of T_2 is contained in one of them. By the pigeonhole principle, there exist two matching edges $xy, x'y' \in M$, with $\{y, y'\} \subseteq A_2$, such that y and y' belong to the same path; without loss of generality, y' is then a descendant of y, as claimed. Since $N_G(x')$ induces a subtree of T, the definition of $(A_1 \cup B_1, A_2 \cup B_2)$ implies that $N_G(x') \cap V(T_2)$ contains v and induces a subtree of T_2. But then this subtree contains y and so x' is adjacent to y as well, contradicting the fact that M is induced.

Combining the previous paragraphs and Lemma 3, we then obtain that

$$\begin{aligned}\mathrm{mimw}(G) &\leq \max\left\{\Delta(t-1), \max\left\{2\left\lfloor\left(\frac{\Delta}{2}\right)^2\right\rfloor, 2\Delta-1\right\} + (t-1)^2\Delta + \Delta(t-1)\right\}\\ &= \max\left\{2\left\lfloor\left(\frac{\Delta}{2}\right)^2\right\rfloor, 2\Delta-1\right\} + (t-1)^2\Delta + \Delta(t-1)\\ &\leq \max\left\{2\left\lfloor\left(\frac{\Delta}{2}\right)^2\right\rfloor, 2\Delta-1\right\} + t^2\Delta.\end{aligned}$$

Finally, we compute a branch decomposition of G. We do this recursively by using Lemmas 3 and 4. □

4 The Proof of Theorem 3

For proving Theorem 3, we need the following lemma.

Lemma 5 (see Wang et al. [43]). *Let $G = (A, B, E)$ be a bipartite graph and G' be the bipartite graph obtained from G by making k new vertices complete to B. If $k = 1$, then G' is star convex. If $k = |A|$, then G' is comb convex.*

Theorem 3 (restated). *The class of star convex graphs and the class of comb convex graphs each has unbounded mim-width.*

Proof. We show that, for every integer ℓ, there exist star convex graphs and comb convex graphs with mim-width larger than ℓ. Therefore, let $\ell \in \mathbb{N}$. There exists a bipartite graph $G = (A, B, E)$ such that $\mathrm{mimw}(G) > \ell$ (see, e.g., [7]). Let G' be the star convex graph obtained as in Lemma 5. Adding a vertex does not decrease the mim-width [42]. Then $\mathrm{mimw}(G') \geq \mathrm{mimw}(G) > \ell$. Let now G'' be the comb convex graph obtained as in Lemma 5. Then $\mathrm{mimw}(G'') \geq \mathrm{mimw}(G) > \ell$. □

5 A Refined Parameter Analysis and Final Remarks

We perform a more refined analysis on width parameters for the graph classes listed in Figure 2. We will consider the graph width parameters listed in Figure 3. Our results are summarized in Figure 4. We omit the proofs but note that we provide a *complete* picture with respect to the width parameters and graph classes considered.

We are not aware of any new algorithmic implications. In particular, it would be interesting to research if there are natural problems that are NP-complete for graphs of bounded mim-width but polynomial-time solvable for graphs of bounded thinness or bounded linear mim-width. In addition, it would also be interesting to obtain dichotomies for more graph problems solvable in polynomial time for graph classes whose mim-width is bounded and quickly computable. For example, what is the complexity of List k-Colouring ($k \geq 3$) for star convex and comb convex graphs? We leave this for future research.

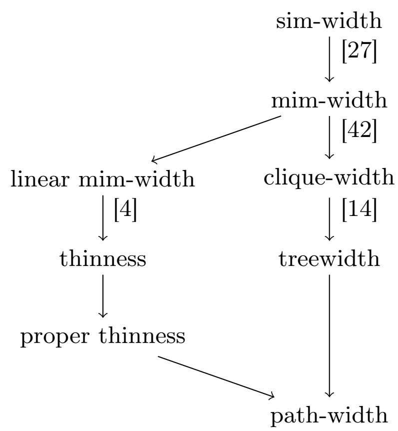

Fig. 3: The relationships between the different width parameters that we consider in Section 5. Parameter p is more powerful than parameter q if and only if there exists a directed path from p to q. To explain the incomparabilities, proper interval graphs have proper thinness 1 [33] and unbounded clique-width [18], whereas trees have tree-width 1 and unbounded linear mim-width [21]. Unreferenced arrows follow from the definitions of the width parameters involved except for the arrow from proper thinness to path-width whose proof we omitted.

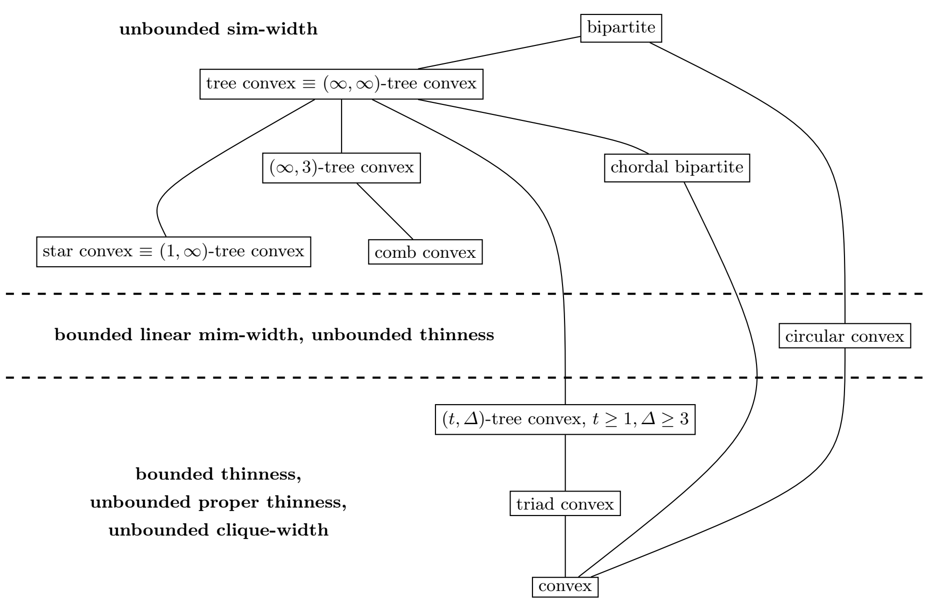

Fig. 4: The inclusion relations between the classes we consider. A line from a lower-level class to a higher one means the first class is contained in the second.

References

1. Belmonte, R., Vatshelle, M.: Graph classes with structured neighborhoods and algorithmic applications. Theoretical Computer Science **511**, 54–65 (2013)
2. Bergougnoux, B., Kanté, M.M.: More applications of the d-neighbor equivalence: Connectivity and acyclicity constraints. Proc. ESA 2019, LIPIcs **144**, 17:1–17:14 (2019)
3. Bergougnoux, B., Papadopoulos, C., Telle, J.A.: Node multiway cut and subset feedback vertex set on graphs of bounded mim-width. Proc. WG 2020, LNCS **12301**, 388–400 (2020)
4. Bonomo, F., de Estrada, D.: On the thinness and proper thinness of a graph. Discrete Applied Mathematics **261**, 78–92 (2019)
5. Brandstädt, A., Le, V.B., Spinrad, J.P.: Graph Classes: A Survey. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, PA (1999)
6. Brault-Baron, J., Capelli, F., Mengel, S.: Understanding model counting for beta-acyclic CNF-formulas. Proc. STACS 2015, LIPIcs **30**, 143–156 (2015)
7. Brettell, N., Horsfield, J., Munaro, A., Paesani, G., Paulusma, D.: Bounding the mim-width of hereditary graph classes. Proc. IPEC 2020, LIPIcs **180**, 6:1–6:18 (2020)
8. Brettell, N., Horsfield, J., Munaro, A., Paulusma, D.: List k-colouring P_t-free graphs: a mim-width perspective. CoRR, abs/2008.01590 (2020)
9. Buchin, K., van Kreveld, M., Meijer, H., Speckmann, B., Verbeek, K.: On planar supports for hypergraphs. Journal of Graph Algorithms and Applications **15**, 533–549 (2011)
10. Bui-Xuan, B.M., Telle, J.A., Vatshelle, M.: Boolean-width of graphs. Theoretical Computer Science **412**, 5187–5204 (2011)
11. Bui-Xuan, B.M., Telle, J.A., Vatshelle, M.: Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. Theoretical Computer Science **511**, 66–76 (2013)
12. Chen, H., Lei, Z., Liu, T., Tang, Z., Wang, C., Xu, K.: Complexity of domination, hamiltonicity and treewidth for tree convex bipartite graphs. Journal of Combinatorial Optimization **32**, 95–110 (2016)
13. Chudnovsky, M., Spirkl, S., Zhong, M.: List 3-coloring P_t-free graphs with no induced 1-subdivision of $K_{1,s}$. Discrete Mathematics **343**, 112086 (2020)
14. Courcelle, B., Olariu, S.: Upper bounds to the clique width of graphs. Discrete Applied Mathematics **101**, 77–114 (2000)
15. Couturier, J.F., Golovach, P.A., Kratsch, D., Paulusma, D.: List coloring in the absence of a linear forest. Algorithmica **71**, 21–35 (2015)
16. Dabrowski, K.K., Johnson, M., Paulusma, D.: Clique-width for hereditary graph classes. London Mathematical Society Lecture Note Series **456**, 1–56 (2019)
17. Díaz, J., Diner, Ö.Y., Serna, M.J., Serra, O.: On list k-coloring convex bipartite graphs. Proc. CTW 2020, AIRO Springer Series **5**, 15–26 (2020)
18. Golumbic, M.C., Rotics, U.: On the clique-width of some perfect graph classes. International Journal of Foundations of Computer Science **11**, 423–443 (2000)
19. Hliněný, P., Oum, S., Seese, D., Gottlob, G.: Width parameters beyond tree-width and their applications. The Computer Journal **51**, 326–362 (2008)
20. Hoàng, C.T., Kamiński, M., Lozin, V.V., Sawada, J., Shu, X.: Deciding k-colorability of P_5-free graphs in polynomial time. Algorithmica **57**, 74–81 (2010)
21. Høgemo, S., Telle, J.A., Vågset, E.R.: Linear mim-width of trees. Proc. WG 2019, LNCS **11789**, 218–231 (2019)

22. Jaffke, L.: Bounded Width Graph Classes in Parameterized Algorithms. Ph.D. thesis, University of Bergen (2020)
23. Jaffke, L., Kwon, O., Telle, J.A.: Mim-width II. The Feedback Vertex Set problem. Algorithmica **82**, 118–145 (2020)
24. Jiang, W., Liu, T., Wang, C., Xu, K.: Feedback vertex sets on restricted bipartite graphs. Theoretical Computer Science **507**, 41–51 (2013)
25. Jiang, W., Liu, T., Xu, K.: Tractable feedback vertex sets in restricted bipartite graphs. Proc. COCOA 2011, LNCS **6831**, 424–434 (2011)
26. Kamiński, M., Lozin, V.V., Milanič, M.: Recent developments on graphs of bounded clique-width. Discrete Applied Mathematics **157**, 2747–2761 (2009)
27. Kang, D.Y., Kwon, O., Strømme, T.J.F., Telle, J.A.: A width parameter useful for chordal and co-comparability graphs. Theoretical Computer Science **704**, 1–17 (2017)
28. Kwon, O.: Personal communication (2020)
29. Liu, T.: Restricted bipartite graphs: Comparison and hardness results. Proc. AAIM 2014, LNCS **8546**, 241–252 (2014)
30. Liu, T., Lu, M., Lu, Z., Xu, K.: Circular convex bipartite graphs: Feedback vertex sets. Theoretical Computer Science **556**, 55–62 (2014)
31. Liu, T., Lu, Z., Xu, K.: Tractable connected domination for restricted bipartite graphs. Journal of Combinatorial Optimization **29**, 247–256 (2015)
32. Lu, M., Liu, T., Xu, K.: Independent domination: Reductions from circular- and triad-convex bipartite graphs to convex bipartite graphs. Proc. FAW-AAIM 2013, LNCS **7924**, 142–152 (2013)
33. Mannino, C., Oriolo, G., Ricci, F., Chandran, S.: The stable set problem and the thinness of a graph. Operations Research Letters **35**, 1–9 (2007)
34. Oum, S., Seymour, P.D.: Approximating clique-width and branch-width. Journal of Combinatorial Theory, Series B **96**, 514–528 (2006)
35. Panda, B.S., Chaudhary, J.: Dominating induced matching in some subclasses of bipartite graphs. Proc. CALDAM 2019, LNCS **11394**, 138–149 (2019)
36. Panda, B.S., Pandey, A., Chaudhary, J., Dane, P., Kashyap, M.: Maximum weight induced matching in some subclasses of bipartite graphs. Journal of Combinatorial Optimization **40**, 713–732 (2020)
37. Pandey, A., Panda, B.: Domination in some subclasses of bipartite graphs. Discrete Applied Mathematics **252**, 51–66 (2019)
38. Rao, M.: Clique-width of graphs defined by one-vertex extensions. Discrete Mathematics **308**, 6157–6165 (2008)
39. Robertson, N., Seymour, P.D.: Graph minors. X. Obstructions to tree-decomposition. Journal of Combinatorial Theory, Series B **52**, 153–190 (1991)
40. Sæther, S.H., Vatshelle, M.: Hardness of computing width parameters based on branch decompositions over the vertex set. Theoretical Computer Science **615**, 120–125 (2016)
41. Song, Y., Liu, T., Xu, K.: Independent domination on tree convex bipartite graphs. Proc. FAW-AAIM 2012, LNCS **7285**, 129–138 (2012)
42. Vatshelle, M.: New Width Parameters of Graphs. Ph.D. thesis, University of Bergen (2012)
43. Wang, C., Chen, H., Lei, Z., Tang, Z., Liu, T., Xu, K.: Tree convex bipartite graphs: NP-complete domination, hamiltonicity and treewidth. Proc. FAW 2014, LNCS **8497**, 252–263 (2014)
44. Wang, C., Liu, T., Jiang, W., Xu, K.: Feedback vertex sets on tree convex bipartite graphs. Proc. COCOA 2012, LNCS **7402**, 95–102 (2012)