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Continuum Strong Coupling Expansion of Yang-Mills Theory: Quark Confinement and Infra-Red Slavery

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Abstract

We solve Schrödinger's equation for the ground-state of *four*-dimensional Yang-Mills theory as an expansion in inverse powers of the coupling. Expectation values computed with the leading order approximation are reduced to a calculation in *two*-dimensional Yang-Mills theory which is known to confine. Consequently the Wilson loop in the four-dimensional theory obeys an area law to leading order and the coupling becomes infinite as the mass-scale goes to zero.

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1. Introduction

Despite considerable success in describing high energy phenomena it remains to be shown that Quantum Chromodynamics accounts for the known and conjectured properties of the Strong Interactions at low energies such as quark confinement and chiral symmetry breakdown. The strong coupling expansion on the lattice [1], which leads to an area law for Wilson loops, provides a framework for the understanding of confinement but only in a calculational regime where the continuous nature of space-time has been spoilt and the connection with continuum field theory has only been established numerically. The difficulty with attempting a strong-coupling expansion directly in the continuum is that our conventional understanding of quantum field theory is based on a different kind of expansion, namely the semi-classical, or loop, expansion. Indeed, the particle interpretation of field theory derives from analysing the free part of the action in terms of harmonic oscillators. This is encoded into the LSZ formalism which underpins the use of field theory in Particle Physics. In Yang-Mills theory asymptotic freedom [2] implies that the LSZ realisation of field theory is applicable at high energies. As the energy scale is reduced the running coupling grows but it is not known if it continues to grow beyond the perturbative regime, although it is usually thought to do so, giving rise to infra-red slavery. In ordinary Quantum Mechanics the Born approximation, in contrast to the semi-classical approximation, provides an expansion in decreasing powers of $\hbar^2$. In this paper we develop such a scheme for continuum Yang-Mills theory in four dimensions. We will solve Schrödinger's equation for the wave-functional of the ground-state as an expansion in inverse powers of the coupling. Our strategy will be based on an *ansatz* in which we assume that the logarithm of the wave-functional is the integral of a local function of the gauge-potential. This enables us to relate ground-state expectation values in the four-dimensional theory to calculations in two-dimensional Yang-Mills theory. It is well known that Yang-Mills theory in two dimensions confines, and as a consequence, our leading order approximation for the Wilson loop in the four-dimensional theory has area-law behaviour. Furthermore the coupling depends on an arbitrary mass-scale, μ, growing stronger as μ is reduced. The relationship between two-dimensional Yang-Mills theory and certain string theories has received attention recently [3], so our result provides a link between this and the four-dimensional theory.

The Yang-Mills action can be written with the coupling factored out

$$S_{YM} = -\frac{1}{2g^2}\int d^3x\, dt\, tr\left(\mathbf{E}^2 - \mathbf{B}^2\right),$$

$$\mathbf{E} = -\dot{\mathbf{A}} + \nabla A_0 + [\mathbf{A}, A_0], \quad \mathbf{B} = \nabla \wedge \mathbf{A} + \mathbf{A} \wedge \mathbf{A}. \tag{1.1}$$

(The gauge potentials are elements of the Lie algebra of the gauge group so $\mathbf{A} = \mathbf{A}^C T^C$, and $(T^C)^\dagger = -T^C$, $tr\,(T^A T^B) = -\delta^{AB}$, $[T^A, T^B] = f^{ABC}T^C$.) g^2 plays the role of Planck's constant since the action occurs in the path-integral as $exp\, iS/\hbar$. Thus the LSZ

realisation of Yang-Mills theory is appropriate to the semi-classical regime where physical quantities are expandable as power series in g^2. If we want to understand the strong coupling limit of the theory we must expand in powers of $1/g^2$, in other words we need to study the ultra-quantum mechanical limit where the (essentially semi-classical) notion of particles may not be relevant. In this paper we will solve Schrödinger's equation using such an expansion for the ground-state. Of course, either kind of expansion should be viewed as being of a formal nature since renormalisation leads to the introduction of a length scale in terms of which the running coupling is fixed as a function of a mass scale. Rather, the two formal power series should be viewed as different ways of organising the calculation.

To simplify things let's consider Schrödinger's equation for a quantum mechanical particle of mass m moving in a potential $V(x)$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right)\Psi=\mathcal{E}\Psi. \tag{1.2}$$

If we write the wave-function as $\Psi = exp\ -W(x)$ then this becomes

$$-\frac{\hbar^2}{2m}\left((W')^2-W''\right)+V=\mathcal{E}. \tag{1.3}$$

In the semi-classical approximation, which corresponds to the loop expansion in the usual perturbative approach to field theory, we expand

$$W=\sum_{n=0}^{\infty}\hbar^{n-1}W_n, \tag{1.4}$$

so that to leading order we can neglect W'' and obtain the Hamilton-Jacobi equation

$$-\frac{1}{2m}(W_0')^2+V=\mathcal{E}, \tag{1.5}$$

which identifies W_0 as the classical action. When the potential is sufficently weak we can use instead the Born approximation. Assuming that $V\to 0$ as $|x|\to\infty$ and that there are no bound states implies that the lowest value of $\mathcal{E}$ is zero corresponding to a (non-square-integrable) wave-function that is constant at spatial infinity. If we take $\Psi = 1-\psi$, with ψ a small perturbation, then re-arranging (1.2) gives

$$\psi=\left(-\frac{1}{2m}\frac{d^2}{dx^2}+\frac{V}{\hbar^2}\right)^{-1}\frac{V}{\hbar^2} \tag{1.6}$$

which can be expanded in powers of $V/\hbar^2$.

Another way of organising the same expansion is to write W as a power series in $1/\hbar^2$, $W=\sum_{n=1}^{\infty}\hbar^{-2n}\tilde{W}_n$ and expand (1.3). To leading order and next to leading order this gives (with $\mathcal{E}=0$)

$$\frac{1}{2m}\tilde{W}_1''+V=0,\quad \tilde{W}_2''=(\tilde{W}_1')^2. \tag{1.7}$$

It is easy to check that this gives the same results to this order as expanding (1.6). This is the approach we will take to the Schrödinger equation for Yang-Mills theory.

We will follow the formulation of the Schrödinger equation for Yang-Mills given by Jackiw in [4], which has the merits of being both simple and closely related to ordinary quantum mechanics. (This reference also explains the modification to the wave-functional required when there is a non-zero θ-angle.) We work in the Weyl gauge ($A_0 = 0$) so that the canonical coordinates are $\mathbf{A}^C(\mathbf{x})$ and their conjugate momenta are $-g^{-2}\mathbf{E}^C(\mathbf{x})$. The Hamiltonian is

$$H[\mathbf{A}, \mathbf{E}] = -\frac{1}{2g^2}\int d^3x\, tr\left(\mathbf{E}^2 + \mathbf{B}^2\right). \tag{1.8}$$

Since $\dot{A}_0(\mathbf{x})$ does not appear in the action the Euler-Lagrange equation obtained by varying $A_0(\mathbf{x})$,

$$\nabla \cdot \mathbf{E} + \mathbf{A} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{A} = 0, \tag{1.9}$$

is a constraint rather than a Hamiltonian equation of motion. In the Schrödinger representation with $\hbar = 1$ $\mathbf{E}^C$ is represented by the operator

$$\mathbf{E}^C(\mathbf{x}) = ig^2 \frac{\delta}{\delta \mathbf{A}^C(\mathbf{x})}, \tag{1.10}$$

(showing explicitly that g^2 plays the role of Planck's constant). Schrödinger's equation is

$$\left(-\frac{1}{2}g^2\Delta + g^{-2}\mathcal{B}\right)\Psi[\mathbf{A}] = \mathcal{E}\Psi[\mathbf{A}],$$

$$\Delta \equiv \int d^3x \frac{\delta}{\delta \mathbf{A}^C(\mathbf{x})} \cdot \frac{\delta}{\delta \mathbf{A}^C(\mathbf{x})}, \quad \mathcal{B} = -\frac{1}{2}\int d^3x\, tr\, \mathbf{B}^2, \tag{1.11}$$

and the constraint is imposed as

$$\Gamma\,\Psi[\mathbf{A}] = 0, \quad \Gamma \equiv \nabla \cdot \frac{\delta}{\delta \mathbf{A}^C(\mathbf{x})} + f^{RSC}\mathbf{A}^R \cdot \frac{\delta}{\delta \mathbf{A}^S(\mathbf{x})}. \tag{1.12}$$

Now Γ is the generator of infinitesimal gauge transformations acting on functionals of $\mathbf{A}$ so this constraint simply implies that the wave-functional is gauge invariant. The Schrödinger equation stands in need of regularisation because the two functional derivatives act at the same point in space. In the next section we shall construct a regulated operator, Δ_ϕ, that preserves the gauge invariance and underlying geometry of the Hamiltonian operator. It acts on integrals of local functions of $\mathbf{A}$ to produce other local integrals with finite coefficents, unlike Δ which produces coefficents that are ill-defined. This means that we have a Hamiltonian operator that has a finite action on the space of sums of products of integrals of local functions of $\mathbf{A}$ with finite coefficents. The price for being able to do this is, as always, the introduction of an arbitrariness in the renormalisation procedure that, in this case, is encoded into a function, ϕ. This is needed to separate the finite part from quantities that diverge as we remove the cut-off. If the theory is renormalisable, as we

hope, we will be able to absorb this arbitrariness into the coupling constant so that physical quantities will not depend on ϕ. Thus the finite version of the Schrödinger equation will be

$$\left(-\frac{1}{2}g^2\Delta_\phi + \frac{1}{g^2}\mathcal{B}\right)\Psi[\mathbf{A}] = \mathcal{E}\Psi[\mathbf{A}] \tag{1.13}$$

where g is now a functional of ϕ. If we attempt a strong-coupling expansion of the ground-state wave-functional

$$\Psi[\mathbf{A}] \equiv exp - S[\mathbf{A}] = exp\left(-\sum_{n=1}^{\infty} g^{-4n} S_n[\mathbf{A}]\right), \quad \mathcal{E} = \sum_{n=1}^{\infty} g^{2-4n}\mathcal{E}_n \tag{1.14}$$

the Schrödinger equation becomes

$$\frac{1}{2}g^2\Delta_\phi S - \frac{1}{2}g^2\left(\int d^3x \, \frac{\delta S}{\delta \mathbf{A}^C}\cdot\frac{\delta S}{\delta \mathbf{A}^C}\right) + \frac{1}{g^2}\mathcal{B} = \mathcal{E} \tag{1.15}$$

so that to lowest order

$$\Delta_\phi S_1 = 2\left(\mathcal{E}_1 - \mathcal{B}\right). \tag{1.16}$$

The right-hand side is the integral of a gauge invariant function that is local in the sense of being constructed from fields at the point $\mathbf{x}$ and a finite number of derivatives. By construction, Δ_ϕ will act on integrals of local gauge-invariant functions to produce other integrals of local gauge-invariant fuctions with finite coefficents. So we can look for a solution for S_1 that is itself the integral of a local function. In the next section we will show that

$$\Delta_\phi\mathcal{B} = \beta_1\left(\int d^3x\right) + \beta_2\mathcal{B}, \tag{1.17}$$

where, for the gauge group $SU(N)$ the coefficents are $\beta_1 = \phi^{(5)}(0)\,(N^2-1)/320\sqrt{\pi}^3$, and $\beta_2 = -11\phi'(0)\,N/12\sqrt{\pi}^3$. So if we take

$$S_1 = -\frac{2\mathcal{B}}{\beta_2}, \quad \mathcal{E}_1 = -\frac{\beta_1}{\beta_2}\int d^3x, \tag{1.18}$$

then equation (1.16) is satisfied. The infra-red divergent integral over space in $\mathcal{E}_1$ is what we would expect for the energy of the ground-state, since by translational invariance the energy density must be constant. More generally we adopt an *ansatz* of locality as a strategy for solving the Schrödinger equation in a strong coupling expansion. We will assume that $S[\mathbf{A}]$ is a sum of integrals of local functions of $\mathbf{A}$. Then, by construction $\Delta_\phi S$ will also be a local integral, and so too will the other terms entering Schrödinger's equation in the form (1.15). Taking S to be gauge-invariant satisfies the constraint (1.12), and taking the integrals to be invariant under spatial translations and rotations acting on

A ensures that we are studying the ground-state which can carry neither momentum nor angular momentum. We also require that the ground-state wave-functional be Lorentz invariant. This gives a very different wave-functional from that given by the semi-classical expansion which would contain propagators, and so be non-local, but then we should not expect the strong-coupling limit to resemble the semi-classical one. To leading order the ground-state wave-functional is

$$\Psi[\mathbf{A}] = exp\,\frac{1}{4\gamma^2}\int d^3x\, tr\mathbf{B}^2, \quad \gamma^2 = \frac{11g^4\phi'(0)N}{48\sqrt{\pi}^3} \equiv g^4\kappa. \tag{1.19}$$

Now that we have an approximation for Ψ we can compute expectation values of operators $\Omega[\mathbf{A},\mathbf{E}]$. We will be principally interested in the Wilson loop, in which case Ω can be taken independent of $\mathbf{E}$, so $\Omega = \omega[\mathbf{A}]$, say.(More generally we use the Schrödinger representation for $\mathbf{E}$, (1.10) to reduce the action of Ω on Ψ to multiplication by some functional dependent on Ψ.)

$$\langle\Omega\rangle = \frac{\int \mathcal{D}\mathbf{A}\,\Psi^*[\mathbf{A}]\,\omega[\mathbf{A}]\,\Psi[\mathbf{A}]}{\int \mathcal{D}\mathbf{A}\,\Psi^*[\mathbf{A}]\,\Psi[\mathbf{A}]}, \tag{1.20}$$

and to leading order this is

$$\langle\Omega\rangle \simeq \frac{\int \mathcal{D}\mathbf{A}\,\omega[\mathbf{A}]\,exp\left(\frac{1}{2\gamma^2}\int d^3x\, tr\mathbf{B}^2\right)}{\int \mathcal{D}\mathbf{A}\,exp\left(\frac{1}{2\gamma^2}\int d^3x\, tr\mathbf{B}^2\right)}, \tag{1.21}$$

But this is the functional integral for the vacuum expectation value of ω in three dimensional Yang-Mills theory, Wick rotated to Euclidean space, and with coupling γ. We are back to the problem we started with but in one dimension lower. Feynman has discussed the qualitative structure of the wave-functional for three-dimensional Yang-Mills theory, arguing that there is a mass gap [5]. We want to calculate (1.21) for large g, this means that we can repeat the above analysis. We begin by undoing the Wick rotation and working in three-dimensional Minkowski space with time coordinate x^3, and quantisation surface $x^3 = constant$. We will denote vectors in the 12-plane by a bar. Working in the Weyl gauge $A_3 = 0$ the wave-functional of the vacuum is a functional of the two components of the gauge potential A_1, A_2, and in the Schrödinger representation

$$\nabla_3\bar{A}^C(\bar{x}) = -i\kappa g^4\frac{\delta}{\delta\bar{A}^C(\bar{x})} \tag{1.22}$$

The Schrödinger equation is

$$-\frac{1}{2}\int d^2x\left(\kappa g^4\frac{\delta}{\delta\bar{A}^C(\bar{x})}\cdot\frac{\delta}{\delta\bar{A}^C(\bar{x})} + \frac{1}{\kappa g^4}tr\,B^2\right)\Phi[\bar{A}] = \tilde{\mathcal{E}}\Phi[\bar{A}],$$
$$B = \epsilon^{ij}\left(\nabla_i A_j + A_i A_j\right) \tag{1.23}$$

as well as a constraint that demands that Φ be gauge-invariant.

$$\tilde{\Gamma}\,\Phi[\bar{A}]=0,\quad \tilde{\Gamma}\equiv\bar{\nabla}\cdot\frac{\delta}{\delta\bar{A}^C(\bar{x})}+f^{RSC}\bar{A}^R\cdot\frac{\delta}{\delta\bar{A}^S(\bar{x})} \tag{1.24}$$

We again construct a Hamiltonian operator that has a finite action on a space of sums of products of local integrals with finite coefficents. Consequently the regulated Laplacian depends on a new arbitrary function, $\tilde{\phi}$, and with $\tilde{\mathcal{B}}=-\frac{1}{2}\int d^2x\,tr\,B^2$, Schrödinger's equation becomes

$$\left(-\frac{1}{2}\gamma^2\tilde{\Delta}_{\tilde{\phi}}+\frac{1}{\gamma^2}\tilde{\mathcal{B}}\right)\Phi[\bar{A}]=\tilde{\mathcal{E}}\Phi[\bar{A}]. \tag{1.25}$$

Expanding in a power series in g^{-8}

$$\Phi[\bar{A}]\equiv exp-\tilde{S}[\bar{A}]=exp\left(-\sum_{n=1}^{\infty}g^{-8n}\tilde{S}_n[\bar{A}]\right),\quad \tilde{\mathcal{E}}=\sum_{n=1}^{\infty}g^{4-8n}\tilde{\mathcal{E}}_n \tag{1.26}$$

we obtain to leading order

$$\kappa\tilde{\Delta}_{\tilde{\phi}}\tilde{S}_1=2(\tilde{\mathcal{E}}_1-\tilde{\mathcal{B}}/\kappa). \tag{1.27}$$

Again $\tilde{\Delta}_{\tilde{\phi}}$ acts on integrals of local gauge-invariant functions to generate further integrals of local gauge-invariant functions, and in the next section we compute

$$\tilde{\Delta}_{\tilde{\phi}}\tilde{\mathcal{B}}=\tilde{\beta}_1\left(\int d^2x\right)+\tilde{\beta}_2\tilde{\mathcal{B}} \tag{1.28}$$

With $\tilde{\beta}_1=(N^2-1)\tilde{\phi}^{(2)}(0)/8\pi$, $\tilde{\beta}_2=-23N/6\pi$ for $SU(N)$. Taking

$$\tilde{S}_1=-\frac{2}{\tilde{\beta}_2\kappa^2}\tilde{\mathcal{B}},\quad \tilde{\mathcal{E}}_1=-\frac{\tilde{\beta}_1}{\tilde{\beta}_2\kappa}\left(\int d^2x\right) \tag{1.29}$$

yields a solution to (1.27). The leading order expression for $\Phi[\bar{A}]$ is thus

$$\Phi[\bar{A}]=exp\frac{1}{4\gamma_*^2}\int d^2x\,tr\,B^2,\quad \gamma_*^2=\frac{23\gamma^4N}{24\pi} \tag{1.30}$$

If we restrict our attention to those $\omega[\mathbf{A}]$ that can be expressed as functionals of only $\bar{A}$ on the quantisation surface i.e. $\omega[\mathbf{A}]=\tilde{\omega}[\bar{A}]$ then

$$\langle\Omega\rangle=\frac{\langle\Psi|\,\Omega\,|\Psi\rangle}{\langle\Psi|\Psi\rangle}\simeq\frac{\langle\Phi|\,\tilde{\omega}\,|\Phi\rangle}{\langle\Phi|\Phi\rangle} \tag{1.31}$$

so to leading order in the strong-coupling expansion

$$\langle\Omega\rangle\simeq\frac{\int\mathcal{D}\bar{A}\,\Phi^*[\bar{A}]\,\tilde{\omega}[\bar{A}]\,\Phi[\bar{A}]}{\int\mathcal{D}\bar{A}\,\Phi^*[\bar{A}]\,\Phi[\bar{A}]}\simeq\frac{\int\mathcal{D}\bar{A}\,\tilde{\omega}[\bar{A}]\,exp\left(\frac{1}{2\gamma_*^2}\int d^2x\,trB^2\right)}{\int\mathcal{D}\bar{A}\,exp\left(\frac{1}{2\gamma_*^2}\int d^2x\,trB^2\right)}, \tag{1.32}$$

This is a functional integral for Yang-Mills theory in two Euclidean dimensions so we are back to where we started, but in two dimensions lower. We will now stop this process of reducing the dimension, because the gauge-invariant two-dimensional theory is free in the gauge $A_2 = 0$ and we can evaluate (1.32) directly.

It is well known that Yang-Mills theory confines in two dimensions in the sense that the Wilson loop has an area law behaviour. The Wilson loop is the vacuum expectation value of the trace of the path-ordered exponential of the gauge potential A taken around a closed curve [1]. If the curve is a rectangle of sides R and T then in the Euclidean formulation of the theory this has the interpretation of being $exp - E(R)T$, where E is the energy of a pair of massive 'test-quarks' created for a time T and held a distance R apart. With an area-law behaviour $E(R)T = \sigma RT$ so the potential energy of the two static quarks is σR, requiring an infinite energy to completely separate them. We will now show how the Wilson loop for four dimensional Yang-Mills theory shows area-law behaviour in the strong-coupling limit. In the Euclidean formulation of the theory we want to compute

$$W[C] = \frac{\int \mathcal{D}A_\mu \, \left(tr\, P\, exp\left(-\oint_C dx \cdot A\right)\right) exp\left(\frac{1}{4g^2}\int d^4x\, tr\, F_{\mu\nu}F_{\mu\nu}\right)}{\int \mathcal{D}A_\mu \, exp\left(\frac{1}{4g^2}\int d^4x\, tr\, F_{\mu\nu}F_{\mu\nu}\right)} \tag{1.33}$$

P denotes path-ordering of the Lie algebra generators around the closed curve C. If C is planar we can use rotational invariance in four-space to place it in the 12-plane. Then $W[C]$ is given by (1.20) with $\omega = tr\, P\, exp - \oint_C d\mathbf{x} \cdot \mathbf{A}$ and to leading order by (1.32) with $\tilde{\omega} = tr\, P\, exp - \oint_C d\bar{x} \cdot \bar{A}$. Evaluating this in the gauge $A_2 = 0$ gives

$$\begin{aligned} W[C] &\simeq \frac{\int \mathcal{D}A_1 \, \left(tr\, P\, exp\left(-\oint_C dx^1 A_1\right)\right) \, exp\left(\frac{1}{2\gamma_*^2}\int d^2x\, tr\, (\partial_2 A_1)^2\right)}{\int \mathcal{D}A_1 \, exp\left(\frac{1}{2\gamma_*^2}\int d^2x\, tr\, (\partial_2 A_1)^2\right)} \\ &= tr\, P\, exp\left(-\frac{\gamma_*^2}{2}\oint_C \oint_C dx^1 dy^1 \mathcal{G}(\bar{x},\bar{y}) T^A T^A\right), \end{aligned} \tag{1.34}$$

where $\mathcal{G}$ is the Green's function for ∂_2^2, i.e.

$$\partial_2^2 \mathcal{G}(\bar{x},\bar{y}) = \delta(\bar{x},\bar{y}). \tag{1.35}$$

Bose symmetry requires that $\mathcal{G}$ be symmetric in its arguments, so we take

$$\mathcal{G}(\bar{x},\bar{y}) = \frac{1}{2}|x^2 - y^2|\, \delta(x^1 - y^1). \tag{1.36}$$

For the Lie algebra $su(N)$ we have $T^A T^A = -1(N^2-1)/N$. Because this is proportional to the identity and because of the $\delta(x^1 - y^1)$ in the Green's function, the path-ordering is redundant. Furthermore the area enclosed by $\mathcal{C}$ can be written as

$$\mathcal{A}[C] = -\oint \oint dx^1\, dy^1 \frac{1}{2}|x^2 - y^2|\, \delta(x^1 - y^1). \tag{1.37}$$

Putting all this together gives the leading order approximation for the Wilson loop as

$$W[C] \simeq N\, exp\, (-\sigma \mathcal{A}[C])\,, \quad \sigma = \frac{(N^2-1)\gamma_*^2}{2N} = \lambda N^2(N^2-1)(\phi'(0))^2 g^8. \tag{1.38}$$

Where λ is a numerical constant, approximately equal to 2.6×10^{-4}. Now σ is a physical quantity which must not depend on the arbitrary function ϕ, so the coupling must actually be a functional of ϕ. ϕ is dimensionless, but its argument has the dimension of [length], so $\phi'(0) \equiv \mu$ has the dimension of [mass]. Since our expansion is in increasing powers of $1/g^4$ we shall define a β-function

$$\beta \equiv \mu \frac{\partial}{\partial \mu}\left(\frac{1}{g^4}\right) \tag{1.39}$$

where the differentiation is taken by holding physical quantities such as σ fixed. Applying this to the above expression for σ gives

$$\beta = \frac{1}{g^4}, \tag{1.40}$$

which tells us that $1/g^4(\mu)$ decreases, as the mass scale is reduced (which we can also see directly from (1.38)). We should emphasise at this point that our expansion in powers of $1/g^4$ is of the ground-state wave-functional, and not the physical quantity σ. Thus if we were to compute the first correction to (1.38) and use this to calculate the Wilson loop it would give corrections to σ of higher order in $1/g^4$, as well as, possibly, a contribution of the same order in g as our leading approximation, (1.38). It is inevitable in any quantum theory with only a single dimensionless coupling, where dimensionful physical quantities acquire those dimensions through dimensional transmutation that those quantities cannot be obtained from perturbation series in powers of the coupling, or powers of any function of the coupling, $f(g)$ say. All physical masses, for example, must depend on f and the arbitrary mass scale in the same way, i.e. as $\mu\, exp - \int^f df'/\beta_f(f')$, with $\beta_f = \mu \frac{\partial f}{\partial \mu}$, in order to be independent of μ. This would mean that if we could expand two physical masses in power series in f we would only need to compute the first terms in order to know the ratio of the two masses to all orders. Put another way if $f \to 0$ as $\mu \to \mu_*$ say, then the μ-independent ratio could be computed in this limit from only the first terms in the series. Such a situation could not take into account the full non-linearity of the theory. That our 'strong-coupling' expansion of the ground-state wave-functional gives a non-zero result for σ tells us that necessarily this will not be the first term in a power series expansion in the coupling. We will not be despondent about this, as the alternative is an approximation scheme in which we cannot compute σ at all. However this may make the renormalisation of the expansion difficult.

The wave-functional (1.19) is Lorentz invariant. Under a Lorentz boost the gauge condition $A_0 = 0$ must be preserved, so the change in $\mathbf{A}$ under an infinitesimal transformation has two pieces, the first being the usual one appropriate to the spatial components of a four-vector, whilst the second is a gauge transformation that preserves the Weyl gauge.

The generator is thus $L(\alpha) = i\alpha \cdot \int d^3x\, \mathbf{x}\mathcal{H}$, with $\mathcal{H}$ the Hamiltonian density. This can be regulated in the same way as H in the next section. Now, by construction, $H\Psi = \mathcal{E}\Psi$, so $\mathcal{H}\Psi = (\nabla\cdot\mathbf{J} + const.)\Psi$, for some $\mathbf{J}$. If we take $\int d^3x\, \mathbf{x} = 0$, then $L(\alpha)\Psi = -(\int d^3x\, \alpha\cdot\mathbf{J})\Psi$. $\mathbf{J}$ has dimensions of [length]$^{-3}$, and to the order we are working this can only come from $\mathcal{H}$ acting on $\mathcal{B}/\mu$, so $\mathbf{J}$ must be μ^{-1} multiplied by a vector of dimension [length]$^{-4}$, but we cannot construct a local gauge-invariant vector with this dimension out of $\mathbf{A}$ and positive powers of derivatives of ϕ, so $\mathbf{J}$ must vanish, as can be seen by explicit calculation, so $L(\alpha)\Psi = 0$.

2. Heat-Kernel Regulator

We will now address the crucial question of how to regulate Schrödinger's equation (1.11). The problem is that it contains a product of two functional derivatives at the same spatial point. So, for example Δ acting on $-tr\,\mathbf{A}^2(\mathbf{x})$ is proportional to $\delta(0)$ which is meaningless. More generally, if Σ denotes the space of sums of products of three-dimensional integrals of local functions of $\mathbf{A}$ with finite coefficents, then Δ acting on a typical element of Σ does not yield another element of Σ. We will replace Δ in Schrödinger's equation by a regulated operator, Δ_ϕ, which does have a finite action on elements of Σ and maps Σ to itself. We will insist that for those $\sigma \in \Sigma$ for which $\Delta\sigma \in \Sigma$ the action of Δ and Δ_ϕ coincide. In addition we require that Δ_ϕ is invariant under gauge transformations, as well as more general transformations of coordinates on configuration space, consistent with the interpretation of Δ as a Laplacian. This regularisation and renormalisation procedure introduces an arbitrary function ϕ on which Δ_ϕ depends. This is an inevitable consequence of isolating the finite part of a divergent quantity, and has the effect of introducing an arbitrary scale into the problem. Ensuring that physical quantities are independent of this arbitrariness, e.g. by absorbing it all into the coupling, is the basic problem of renormalisation and has the effect of making the coupling a functional of ϕ. To construct Δ_ϕ we begin by 'point-splitting' the functional derivatives. Consider the operator

$$T_t \equiv \int d^3x\, d^3y \frac{\delta}{\delta\mathbf{A}^R(\mathbf{x})} \cdot \mathbf{K}^{RS}(\mathbf{x},\mathbf{y};t) \cdot \frac{\delta}{\delta\mathbf{A}^S(\mathbf{y})} \tag{2.1}$$

Where the kernel, $\mathbf{K}$, is a second rank tensor in three-space, satisfying some generalised heat equation ((2.16) below) so that it depends smoothly on the proper time t, with the initial condition

$$Limit \quad t \downarrow 0 \quad \mathbf{K}^{RS}(\mathbf{x},\mathbf{y};t) = \delta(\mathbf{x}-\mathbf{y})\,\delta^{RS}\mathbf{1} \tag{2.2}$$

it has the interpretation of being the 'temperature' at the point $\mathbf{x}$ at time t due to a delta-function source placed at $\mathbf{y}$ at the initial time. $\mathbf{1}$ is the identity tensor (dyadic) in three-space. Taking t small but non-zero, $t = \epsilon$ say, gives a regulated operator that acts on elements of Σ to produce other elements of Σ. T_ϵ acting on $\sigma \in \Sigma$ produces local integrals because in the small time ϵ heat cannot flow vary far from its original source and so $\mathbf{K}(\mathbf{x},\mathbf{y},\epsilon)$ will depend only on $\mathbf{A}(\mathbf{x})$ for $\mathbf{x}$ in a neighbourhood of $\mathbf{y}$ of volume of order $\sqrt{\epsilon}^3$. However, the coefficents of the integrals resulting from the action of T_ϵ on $\sigma \in \Sigma$ will

contain powers of ϵ some of which diverge as $\epsilon \to 0$, so we cannot simply replace $\Delta\sigma$ by the *Limit* $\epsilon \to 0$ $T_\epsilon\sigma$. These powers of ϵ may be determined from dimensional analysis. We will see that the proper-time ϵ has dimensions of [length]^2, whilst T has dimensions of [length]$^{-1}$. We will choose $\mathbf{K}$ to preserve gauge and rotational invariance, so, given the local character of T_ϵ, we have, for example

$$T_\epsilon\mathcal{B} =$$

$$\int d^3x \left(\alpha_1\epsilon^{-5/2} + \alpha_2\epsilon^{-1/2} tr\mathbf{B}^2 + \alpha_3\epsilon^{1/2} tr\left((\nabla\wedge\mathbf{B} + \mathbf{A}\wedge\mathbf{B} + \mathbf{B}\wedge\mathbf{A})^2\right) + ...\right), \quad (2.3)$$

where the α_i are numerical constants. We can extract a finite result for $T_\epsilon\sigma$ when $\epsilon = 0$ using analytic continuation in an analogous fashion to zeta-function regularisation [6]. Suppose $\phi(p)$ is a differentiable function equal to one at the origin, and that $f(p)$ is also a differentiable function which is finite at the origin and such that $\phi(p)f(p)$ vanishes at infinity. Then on integrating by parts

$$I(s) \equiv \int_0^\infty dp\, p^{s-1}\phi(p)\, f(p) = \left[\frac{p^s}{s}\phi(p)\, f(p)\right]_0^\infty - \int_0^\infty dp\, \frac{p^s}{s}\frac{d}{dp}(\phi f). \quad (2.4)$$

For $s > 0$ the first term vanishes, so

$$Limit \quad s \downarrow 0 \qquad sI(s) = -\int_0^\infty dp\, \frac{d}{dp}(\phi f) = f(0). \quad (2.5)$$

If f diverges at the origin like p^{-n}, with n an integer, then $f(0)$ is meaningless, but we can give a meaning to the left-hand side of (2.5) and use this as our definition of $f(0)$. The integral $I(s)$ exists (provided ϕf has no other divergences) for $s > n$, so if we analytically continue $sI(s)$ to small values of s we can take the limit as $s \downarrow 0$ to obtain a finite result. Thus, repeated integration by parts for $s > n$ gives

$$sI(s) = -\int_0^\infty \frac{p^s}{(n-s)(n-s-1)..(1-s)}\frac{d^{n+1}}{dp^{n+1}}\left(\phi(p)\, f(p)\, p^n\right). \quad (2.6)$$

This expression allows an analytic continuation to $s < n$ so we can take the limit to obtain

$$Limit \quad s \downarrow 0 \qquad sI(s) = \frac{1}{n!}\frac{d^n}{dp^n}\left(\phi f p^n\right)|_{p=0}. \quad (2.7)$$

Clearly this result depends on the arbitrary function ϕ, as we should expect. Extracting a finite quantity from a divergent one requires the use of an arbitrary renormalisation procedure in general and the introduction of an arbitrary scale in particular. Physical quantities will be independent of ϕ if the theory is renormalisable. With the use of this tool we take the action of Δ_ϕ on an arbitrary element, $\sigma \in \Sigma$ to be

$$\Delta_\phi\sigma = \quad Limit \quad s \downarrow 0 \quad s\int_0^\infty dp\, p^{s-1}\phi(p)\, T_{p^2}\sigma, \quad (2.8)$$

where the integral on the right-hand side is defined by analytic continuation in s as in the example above. We take T at proper-time p^2 rather than p because of the square-roots

appearing in (2.3) which are generic to a theory in an odd number of dimensions, (by which we mean here the three spatial dimensions). For example

$$\Delta_\phi \mathcal{B} = \int d^3x \left(\frac{\alpha_1}{5!} \phi^{(5)}(0) + \alpha_2 \phi'(0)\, tr\, \mathbf{B}^2 \right). \tag{2.9}$$

More generally, Δ_ϕ acting on a local integral of dimension $[\text{length}]^{-D}$ results in other local integrals of dimensions $[\text{length}]^{-D'}$ with $D' \leq D$. We have done more than regulate Δ by constructing an operator whose action on elements of Σ does not depend on a cut-off. We have actually done part of the job of renormalisation. The rest, ensuring that physical quantities are independent of ϕ, remains to be done.

We now turn to the choice of heat-kernel. Under an infinitesimal gauge transformation parametrised by ω

$$\mathbf{A}^C(\mathbf{x}) \to \mathbf{A}^C(\mathbf{x}) + \nabla \omega^C(\mathbf{x}) - f^{RSC} \omega^R(\mathbf{x}) \mathbf{A}^S(\mathbf{x}), \tag{2.10}$$

the functional derivative transforms homogeneously as

$$\frac{\delta}{\delta \bar{A}^C(\bar{x})} \to \frac{\delta}{\delta \bar{A}^C(\bar{x})} - f^{RSC} \omega^R(\mathbf{x}) \frac{\delta}{\delta \bar{A}^S(\bar{x})}, \tag{2.11}$$

so that Δ is invariant. We will demand that Δ_ϕ is also gauge-invariant, which in turn requires that T_t be gauge-invariant, so we insist that the kernel transform under (2.10) as

$$\mathbf{K}^{RS}(\mathbf{x}, \mathbf{y}; t) \to \mathbf{K}^{RS}(\mathbf{x}, \mathbf{y}; t) - f^{ABR} \omega^A(\mathbf{x}) \mathbf{K}^{BS}(\mathbf{x}, \mathbf{y}; t) - \mathbf{K}^{RB}(\mathbf{x}, \mathbf{y}; t) f^{ABS} \omega^A(\mathbf{y}). \tag{2.12}$$

Or in matrix notation with $\omega = (\omega^{RS}) \equiv (\omega^Q f^{QRS})$,

$$\mathbf{K}(\mathbf{x}, \mathbf{y}; t) \to \mathbf{K}(\mathbf{x}, \mathbf{y}; t) + \omega(\mathbf{x})\, \mathbf{K}(\mathbf{x}, \mathbf{y}; t) - \mathbf{K}(\mathbf{x}, \mathbf{y}; t)\, \omega(\mathbf{y}). \tag{2.13}$$

It is easy to construct a kernel that satisfies these conditions. If $\mathbf{\Lambda}$ is a local operator that acts on variations of the gauge-potential as

$$(\Lambda \circ \delta \mathbf{A})^C(\mathbf{x}) = \int d^3y\, \mathbf{\Lambda}^{CD}(\mathbf{x}, \mathbf{y}) \cdot \delta \mathbf{A}^D(\mathbf{y}), \tag{2.14}$$

then we can view $\mathbf{\Lambda}$ as a matrix with indices $\mathbf{x}, \mathbf{y}, CD$ as well as the three-space tensor indices, and $\circ$ as matrix multiplication. If, in addition, $\mathbf{\Lambda}$ transforms under (2.10) as

$$\mathbf{\Lambda}(\mathbf{x}, \mathbf{y}) \to \mathbf{\Lambda}(\mathbf{x}, \mathbf{y}) + \omega(\mathbf{x})\, \mathbf{\Lambda}(\mathbf{x}, \mathbf{y}) - \mathbf{\Lambda}(\mathbf{x}, \mathbf{y})\, \omega(\mathbf{y}), \tag{2.15}$$

then we can take the kernel to be $exp\, -t\mathbf{\Lambda}\circ$ acting on the identity matrix. For example, if $\mathcal{D}$ is the covariant derivative acting in the adjoint representation , i.e. $\mathcal{D}_i \delta A_j = \nabla_i \delta A_j + [A_i, \delta A_j]$ then we could (but we won't) take $\mathbf{\Lambda} \circ \delta \mathbf{A} = -\mathcal{D}_i \mathcal{D}_i \delta \mathbf{A}$, in which case $\mathbf{\Lambda}(\mathbf{x}, \mathbf{y}) = \mathcal{D}_i \mathcal{D}_i \delta(\mathbf{x}, \mathbf{y})$ satisfies (2.15). We can then compute $\mathbf{K}$ by solving the heat equation

$$-\mathbf{\Lambda} \circ \mathbf{K} = \frac{\partial}{\partial t} \mathbf{K}, \tag{2.16}$$

subject to the initial condition (2.2).

There are many choices of operator $\mathbf{\Lambda}$ that satisfy the above conditions. We need a further criterion to pick the right one. This will emerge from considering the geometry that underlies Schrödinger's equation (1.11). If we think of $\mathbf{A}(\mathbf{x})$ as being a set of coordinates on the space of configurations, $\mathcal{A}$, then this space has a natural (flat) metric,

$$g(\delta\mathbf{A}, \delta\mathbf{A}) = -\int d^3x\, tr\, \delta\mathbf{A}(\mathbf{x}) \cdot \delta\mathbf{A}(\mathbf{x}), \tag{2.17}$$

which is invariant under (2.10). We can use this metric to define the volume element $\mathcal{D}\mathbf{A}$ in (1.20). Furthermore the operation $\circ$ is just the scalar product constructed using this metric. The operator Δ is then the Laplacian, expressed in terms of the coordinates $\mathbf{A}$, acting on functions on $\mathcal{A}$ such as Ψ. i.e. if $D/D\mathbf{A}$ is the Levi-Civita connection on $\mathcal{A}$ (indistinguishable from $\delta/\delta\mathbf{A}$ for the metric (2.11)) then

$$\Delta = g^{-1}\left(\frac{D}{D\mathbf{A}}, \frac{D}{D\mathbf{A}}\right) = \frac{D}{D\mathbf{A}} \circ \frac{D}{D\mathbf{A}}, \tag{2.18}$$

and

$$T = \frac{D}{D\mathbf{A}} \circ \mathbf{K} \circ \frac{D}{D\mathbf{A}}. \tag{2.19}$$

Physics should be independent of how we choose coordinates on $\mathcal{A}$, so Schrödinger's equation must transform covariantly under a change of coordinates $\mathbf{A}(\mathbf{x})$. Ψ transforms as a scalar, and Δ is the scalar Laplacian so $\mathcal{B} \equiv -\frac{1}{2}\int d^3x\, tr\, \mathbf{B}^2$ must also transform as a scalar. Now we want the regulated operator T to act on scalars to produce scalars, and this will be the case if $\mathbf{K}$ is a second rank tensor under transformations of $\mathcal{A}$. This in turn will be guaranteed if $\mathbf{\Lambda}$ is also a second rank tensor. So we need to find a local operator in three-space enabling us to construct the heat equation (2.10), which is also a tensor under coordinate transformations of $\mathcal{A}$. We can construct tensors by differentiating scalars using the Levi-Civita connection. The only scalar we have identified so far is $\mathcal{B}$, so

$$\frac{D}{D\mathbf{A}}\mathcal{B} = \frac{\delta}{\delta\mathbf{A}}\mathcal{B} = \nabla \wedge \mathbf{B} + \mathbf{A} \wedge \mathbf{B} + \mathbf{B} \wedge \mathbf{A} \equiv \mathcal{D} \wedge \mathbf{B} \tag{2.20}$$

is a covariant vector on $\mathcal{A}$, whilst a suitable candidate for $\mathbf{\Lambda}$ is

$$\mathbf{\Lambda} = \frac{D}{D\mathbf{A}} \otimes \frac{D}{D\mathbf{A}}\,\mathcal{B}. \tag{2.21}$$

With the flat metric (2.11) this gives

$$\mathbf{\Lambda} \circ \delta\mathbf{A} = -\mathcal{D} \cdot \mathcal{D}\delta\mathbf{A} + \mathcal{D}\mathcal{D} \cdot \delta\mathbf{A} + 2\mathbf{B} \wedge \delta\mathbf{A} + 2\delta\mathbf{A} \wedge \mathbf{B}$$

$$\equiv -\mathcal{D} \cdot \mathcal{D}\delta\mathbf{A} + \mathcal{D}\mathcal{D} \cdot \delta\mathbf{A} - 2\mathbf{F} \cdot \delta\mathbf{A} \tag{2.22}$$

This is just the operator that appears in the semi-classical expansion of three-dimensional Yang-Mills. $\mathbf{F}$ is the field-strength tensor in the adjoint representation.

We need to compute the coefficents α_1, α_2 that occur in the expansion (16). Now

$$T\mathcal{B} = \frac{\delta}{\delta \mathbf{A}} \left(\circ \mathbf{K} \circ \left(\frac{\delta}{\delta \mathbf{A}} \mathcal{B} \right) \right) = \left(\frac{\delta}{\delta \mathbf{A}} \circ \mathbf{K} \right) \circ \frac{\delta}{\delta \mathbf{A}} \mathcal{B} + Tr \left(\mathbf{K} \circ \frac{\delta}{\delta \mathbf{A}} \otimes \left(\frac{\delta}{\delta \mathbf{A}} \mathcal{B} \right) \right). \quad (2.23)$$

Tr denotes a trace over all the matrix indices. The first term on the right-hand side is proportional to $\mathcal{D} \wedge \mathbf{B}$ and so contributes to α_3 but not α_1 and α_2. We are left with

$$Tr \left(\mathbf{K} \circ \frac{\delta}{\delta \mathbf{A}} \otimes \left(\frac{\delta}{\delta \mathbf{A}} \mathcal{B} \right) \right) = Tr \left(\mathbf{K} \circ \mathbf{\Lambda} \right) = -\frac{\partial}{\partial t} Tr \left(\mathbf{K} \right) \quad (2.24)$$

Evaluated at $t = \epsilon$. So we need to compute the diagonal elements of $\mathbf{K}$. We could attempt to do this for a general $\mathbf{A}$, but since we only need to extract the numerical coefficents α_1, α_2 we can do the calculation for a background satisfying the classical equation of motion $\mathcal{D} \wedge \mathbf{B} = 0$. We will show that in this case

$$exp\ (-t\mathbf{\Lambda}\circ) = exp\ (t(\mathcal{D} \cdot \mathcal{D}\mathbf{1} + 2\mathbf{F})\cdot) - \int_0^t d\tau\, \mathcal{D}\, exp\ (\tau \mathcal{D} \cdot \mathcal{D})\, \mathcal{D}\cdot \quad (2.25)$$

($\mathbf{K}$ can be found for a general configuration by perturbing about this result in powers of $\mathcal{D} \wedge \mathbf{B}$.) For an arbitrary $\mathbf{A}$

$$[\, \mathcal{D} \cdot \mathcal{D}, \mathcal{D}\,] = -2\mathbf{F} \cdot \mathcal{D} + (\mathcal{D} \cdot \mathbf{F}), \quad (2.26)$$

so that

$$\mathbf{\Lambda} \circ \mathcal{D}\omega = -(\mathcal{D} \cdot \mathbf{F})\, \omega, \quad \mathcal{D} \cdot \mathbf{\Lambda} \circ \delta\mathbf{A} = (\mathcal{D} \cdot \mathbf{F}) \cdot \delta\mathbf{A}. \quad (2.27)$$

From now on we will assume that $\mathbf{A}$ satisfies $\mathcal{D} \wedge \mathbf{B} = 0$ which implies that $\mathcal{D} \cdot \mathbf{F}$ vanishes so then $\mathcal{D}\omega$ is a zero-mode of $\mathbf{\Lambda}\circ$ and $\mathcal{D} \cdot (\mathbf{\Lambda} \circ \delta\mathbf{A}) = 0$. Let $\mathcal{P}$ denote the projector onto the kernel of $\mathcal{D}\cdot$ so that

$$\mathcal{P} \circ \delta\mathbf{A} = \delta\mathbf{A} - \mathcal{D}\, (\mathcal{D} \cdot \mathcal{D})^{-1}\, \mathcal{D} \cdot \delta\mathbf{A}, \quad (2.28)$$

Now $\mathbf{\Lambda} \circ \mathcal{P} = \mathbf{\Lambda}$ so

$$exp\ (-t\mathbf{\Lambda}\circ)\, \mathcal{P} \circ \delta\mathbf{A} = exp\ (-t\mathbf{\Lambda}\circ)\, \delta\mathbf{A} - \mathcal{D}\, (\mathcal{D} \cdot \mathcal{D})^{-1}\, \mathcal{D} \cdot \delta\mathbf{A}, \quad (2.29)$$

If $\mathbf{u}$ is in the kernel of $\mathcal{D}\cdot$ then

$$\mathbf{\Lambda} \circ \mathbf{u} = -\mathcal{D} \cdot \mathcal{D}\mathbf{u} - 2\mathbf{F} \cdot \mathbf{u} \quad (2.30)$$

which will also be in the kernel of $\mathcal{D}\cdot$ so we can write

$$exp\ (-t\mathbf{\Lambda}\circ)\, \mathcal{P} = exp\ (t(\mathcal{D} \cdot \mathcal{D}\mathbf{1} + 2\mathbf{F})\cdot)\, \mathcal{P} \quad (2.31)$$

From (2.26)

$$(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot\mathcal{D}=\mathcal{D}\,\mathcal{D}\cdot\mathcal{D}, \tag{2.32}$$

so

$$exp\,(t(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot)\,\mathcal{D}=\mathcal{D}exp\,(t\mathcal{D}\cdot\mathcal{D})\,, \tag{2.33}$$

(as can be seen by expanding both exponentials in power series, and using (2.32) to move $\mathcal{D}$ to the left.) This enables us to simplify (2.31) to

$$exp\,(-t\mathbf{\Lambda}\circ)\,\mathcal{P}\circ=exp\,(t(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot)-\mathcal{D}exp\,(t\mathcal{D}\cdot\mathcal{D})\,(\mathcal{D}\cdot\mathcal{D})^{-1}\,\mathcal{D}\cdot \tag{2.34}$$

Finally we can write (2.29) as

$$exp\,(-t\mathbf{\Lambda}\circ)=exp\,(t(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot)-\mathcal{D}\,exp\,(t\mathcal{D}\cdot\mathcal{D})\,(\mathcal{D}\cdot\mathcal{D})^{-1}\,\mathcal{D}\cdot+\mathcal{D}\,(\mathcal{D}\cdot\mathcal{D})^{-1}\,\mathcal{D}\cdot \tag{2.35}$$

which is equivalent to (2.25). Using cyclicity of Tr we obtain

$$\frac{\partial}{\partial t}Tr\,\mathbf{K}=\frac{\partial}{\partial t}\left(Tr\,exp\,(t(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot)-Tr\,exp\,(t\mathcal{D}\cdot\mathcal{D})\right). \tag{2.36}$$

We can make this more explicit by introducing the heat-kernels

$$\begin{aligned}H^{CD}(\mathbf{x},\mathbf{y},t)&\equiv\ exp\,(t\mathcal{D}\cdot\mathcal{D})^{CD}\,\delta(\mathbf{x}-\mathbf{y}),\\ \mathbf{G}^{CD}(\mathbf{x},\mathbf{y},t)&\equiv\ exp\,(t(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot)^{CD}\,\delta(\mathbf{x}-\mathbf{y})\,\mathbf{1},\end{aligned} \tag{2.37}$$

which satisfy the generalised heat equations

$$\frac{\partial}{\partial t}H=\mathcal{D}\cdot\mathcal{D}H,\quad \frac{\partial}{\partial t}\mathbf{G}=(\mathcal{D}\cdot\mathcal{D}\mathbf{1}+2\mathbf{F})\cdot\mathbf{G}, \tag{2.38}$$

subject to the initial conditions

$$\begin{aligned}&Limit\quad t\downarrow 0\quad H^{BC}(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})\delta^{BC},\\ &Limit\quad t\downarrow 0\quad \mathbf{G}^{BC}(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})\delta^{BC}\mathbf{1}.\end{aligned} \tag{2.39}$$

In terms of these functions the right-hand side of (2.36) becomes

$$\frac{\partial}{\partial t}\left(\int d^3x\,G^{CC}_{ii}(\mathbf{x},\mathbf{x},t)-\int d^3x\,H^{CC}(\mathbf{x},\mathbf{x},t)\right), \tag{2.40}$$

so we need to know the diagonal matrix elements of $\mathbf{G}$ and H for short times. These may be computed using a standard technique [7] that assumes the existence of expansions of the form

$$H^{BC}(\mathbf{x},\mathbf{y},t) = \frac{e^{-(\mathbf{x}-\mathbf{y})^2/4t}}{\sqrt{4\pi t}^3}\sum_{n=0}^{\infty} t^n a_n^{BC}(\mathbf{x},\mathbf{y}),$$

$$\mathbf{G}^{BC}(\mathbf{x},\mathbf{y},t) = \frac{e^{-(\mathbf{x}-\mathbf{y})^2/4t}}{\sqrt{4\pi t}^3}\sum_{n=0}^{\infty} t^n \mathbf{b}_n^{BC}(\mathbf{x},\mathbf{y}). \tag{2.41}$$

The prefactor solves the ordinary heat equation and collapses to a delta-function at small times, consequently the initial conditions are satisfied by taking

$$a_0^{BC}(\mathbf{x},\mathbf{x}) = \delta^{BC}, \quad \mathbf{b}_0^{BC}(\mathbf{x},\mathbf{x}) = \delta^{BC}\mathbf{1}. \tag{2.42}$$

Substituting the expansions into the heat-equations (2.38) and equating terms of the same order in t leads to recurrence relations

$$(\mathbf{x}-\mathbf{y})\cdot\mathcal{D}\mathbf{b}_0(\mathbf{x},\mathbf{y}) = 0$$

$$(\mathcal{D}^2\mathbf{1} + 2\mathbf{F})\cdot\mathbf{b}_{n-1} - (\mathbf{x}-\mathbf{y})\cdot\mathcal{D}\mathbf{b}_n(\mathbf{x},\mathbf{y}) = n\mathbf{b}_n(\mathbf{x},\mathbf{y}), \quad n>0$$

$$(\mathbf{x}-\mathbf{y})\cdot\mathcal{D}a_0(\mathbf{x},\mathbf{y}) = 0,$$

$$\mathcal{D}^2 a_{n-1} - (\mathbf{x}-\mathbf{y})\cdot\mathcal{D}a_n(\mathbf{x},\mathbf{y}) = n a_n(\mathbf{x},\mathbf{y}), \quad n>0. \tag{2.43}$$

Repeated applications of $\mathcal{D}$ to these equations, followed by setting $\mathbf{x}=\mathbf{y}$ enables them to be solved for $\mathbf{b}_n(\mathbf{x},\mathbf{x}), a_n(\mathbf{x},\mathbf{x})$ and their derivatives. For example

$$a_1(\mathbf{x},\mathbf{x}) = 0, \quad a_2(\mathbf{x},\mathbf{x}) = \frac{1}{12}F_{ij}F_{ij},$$

$$\mathbf{b}_1(\mathbf{x},\mathbf{x}) = 2\mathbf{F}, \quad \mathbf{b}_2(\mathbf{x},\mathbf{x}) = 2\mathbf{F}\cdot\mathbf{F} + \frac{1}{12}\mathbf{1}\,F_{ij}F_{ij} + \frac{1}{3}\mathcal{D}\cdot\mathcal{D}\mathbf{F} \tag{2.44}$$

Using these in (2.40) and (2.23) gives eventually

$$\alpha_1 = \frac{3n_g}{\sqrt{4\pi}^3}, \quad \alpha_2 = \frac{11C}{6\sqrt{4\pi}^3} \tag{2.45}$$

where n_g is the number of Lie algebra generators and C is given by $f^{RPQ}f^{SQP} = -\delta^{RS}C$. For $SU(N)$ we have $n_g = N^2-1$ and $C = 2N$. Note that if we had performed this calculation for an Abelian theory, such as QED, then Λ would not depend on $\mathbf{A}$ and consequently the right-hand side of (2.9) would just be independent of $\mathbf{A}$. We would then be unable to solve (1.16) with our locality *ansatz*.

To construct the Schrödinger equation for three-dimensional Yang-Mills theory we need an operator that acts on an element, $\tilde{\sigma}$, of a space $\tilde{\Sigma}$ of two-dimensional integrals of local functions of $\bar{A}$ to produce other elements of that space. Proceeding as before

we define an operator $\tilde{\Delta}_{\tilde{\phi}}$ so that its action on $\tilde{\sigma}$ coincides with that of the unregulated Laplacian when the latter is finite. Thus

$$\tilde{\Delta}_{\tilde{\phi}}\tilde{\sigma} = \quad Limit \quad s \downarrow 0 \quad s\int_0^\infty dp\, p^{s-1}\tilde{\phi}(p)\,\tilde{T}_p\tilde{\sigma},$$

$$\tilde{T}_t = \int d^2x\, d^2y\, \frac{\delta}{\delta\bar{A}^R(\bar{x})}\cdot\bar{K}^{RS}(\bar{x},\bar{y},t)\cdot\frac{\delta}{\delta\bar{A}^S(\bar{y})}. \tag{2.46}$$

To ensure gauge and reparametrisation invariance the kernel is chosen as $\bar{K} = exp\, -t\bar{\Lambda}\circ$ with $\bar{\Lambda}$ a tensor in the space of gauge-potentials,

$$\bar{\Lambda}^{RS}(\bar{x},\bar{y}) \equiv \frac{\delta}{\delta\bar{A}^R(\bar{x})}\frac{\delta}{\delta\bar{A}^S(\bar{y})}\tilde{\mathcal{B}}. \tag{2.47}$$

As before, $\bar{K}$ can be expressed in terms of simpler heat-kernels

$$\bar{K}(\bar{x},\bar{y}) = \tilde{G}(\bar{x},\bar{y}) - \int_0^t d\tau\tilde{\mathcal{D}}\tilde{H}(\bar{x},\bar{y})\tilde{\mathcal{D}}\cdot, \tag{2.48}$$

with $\tilde{\mathcal{D}}f = \bar{\nabla}f + [\bar{A}, f]$ and

$$\frac{\partial}{\partial t}\tilde{H} = \tilde{\mathcal{D}}\cdot\tilde{\mathcal{D}}\tilde{H}, \quad \frac{\partial}{\partial t}\bar{G} = (\tilde{\mathcal{D}}\cdot\tilde{\mathcal{D}}\bar{1} + 2\bar{F})\cdot\bar{G}, \tag{2.49}$$

Here $F_{ij} \equiv \epsilon_{ij}B_{adj}$ is the field strength in the adjoint representation. $\bar{G}$ and H are assumed to have small t expansions of the form

$$\bar{G}^{BC}(\bar{x},\bar{y},t) = \frac{e^{-(\bar{x}-\bar{y})^2/4t}}{4\pi t}\sum_{n=0}^{\infty}t^n\bar{b}_n^{BC}(\mathbf{x},\mathbf{y}),$$

$$H^{BC}(\bar{x},\bar{y},t) = \frac{e^{-(\bar{x}-\bar{y})^2/4t}}{4\pi t}\sum_{n=0}^{\infty}t^n\tilde{a}_n^{BC}(\mathbf{x},\mathbf{y}). \tag{2.50}$$

The calculation of the expansion coefficents is the same as in the previous case, leading to

$$\tilde{a}_1(\bar{x},\bar{x}) = 0, \quad \tilde{a}_2(\bar{x},\bar{x}) = \frac{1}{6}B_{adj}^2,$$

$$\bar{b}_1(\bar{x},\bar{x}) = 2\bar{F}, \quad \bar{b}_2(\bar{x},\bar{x}) = -\frac{11}{6}B_{adj}^2\bar{1} + \frac{1}{3}\tilde{\mathcal{D}}\cdot\tilde{\mathcal{D}}\bar{F} \tag{2.51}$$

Using these expansions we find

$$\tilde{T}_\epsilon\tilde{\mathcal{B}} = \int d^2x\left(\tilde{\alpha}_1\,\epsilon^{-2} + \tilde{\alpha}_2\,tr\,B^2 + ...\right) \tag{2.52}$$

with $\tilde{\alpha}_1 = \frac{n_g}{4\pi}$, $\quad\tilde{\alpha}_2 = \frac{23C}{24\pi}$, from which follows (1.28).

3. Conclusions

We have constructed a regulated form of Schrödinger's equation for four-dimensional continuum Yang-Mills theory in which the Hamiltonian operator has a finite action on integrals of local functions of **A**. We solved this for the logarithm of the ground-state wave-functional as a power series in $1/g^4$ on the assumption that the terms in this series are themselves integrals of local functions of **A**. Ground-state expectation values calculated using the leading order result reduce to calculations in two-dimensional Yang-Mills theory. This is known to confine and be directly related to certain string theories. Consequently the Wilson loop in the four-dimensional theory satisfies an area law, and the coupling is seen to grow as the mass-scale, μ, is reduced, which is infra-red slavery. It is the intrinsic non-linearity of Yang-Mills theory that is responsible for this. We would not have been able to solve Schrödinger's equation for an Abelian gauge theory using the *ansatz* of locality. Our results are preliminary, and many questions remain to be answered. We need to know how to ensure that all physical quantities are independent of the arbitrary function ϕ that was introduced to regulate, and partially renormalise the theory. We need to compute corrections, and to construct the excited states. We also need to know how to incorporate fermions. We expect that other quantum theories such as CP^n-models and String Theories may be treated in a similar fashion.

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4. References

[1] K.G.Wilson Phys.Rev.D 10(1974)2445
[2] D.J.Gross, F.Wilczek Phys. Rev. Lett. 30(1973)1343
H.D.Politzer Phys. Rev. Lett. 30 (1973) 1346
[3] D.J.Gross, W.Taylor PUPT-1376 LBL-33458 UCB-PTH-93/02
[4] R.Jackiw Rev. Mod. Phys. 52(1980)661
[5] R.P. Feynman Nucl. Phys. B188(1981)479
[6] E.Corrigan, P.Goddard, H.Osborn, S.Templeton Nucl. Phys. B159(1979)469
J.S.Dowker, R.Critchley Phys. Rev. D13(1976)3224, D16(1977)3390
S.Hawking Comm. Math. Phys. 55(1977)133
D.B.Ray, I.M.Singer Adv. Math. 7(1971)145
[7] B.S. deWitt Phys. Reports 19(1975)295
H.P.McKean, I.M.Singer, J. Diff. Geom. 5(1971)233
P.B.Gilkey, J. Diff. Geom. 10(1975)601, Proc. Symp. Pure Math. 27 (1975)265