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# Factorization and Regularization by Dimensional Reduction

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**Abstract**

Since an old observation by Beenakker et al, the evaluation of QCD processes in dimensional reduction has repeatedly led to terms that seem to violate the QCD factorization theorem. We reconsider the example of the process $gg \to t\bar{t}$ and show that the factorization problem can be completely resolved. A natural interpretation of the seemingly non-factorizing terms is found, and they are rewritten in a systematic and factorized form. The key to the solution is that the $D$- and $(4-D)$-dimensional parts of the 4-dimensional gluon have to be regarded as independent partons.

# 1 Introduction

Nearly 20 years ago, Ref. [1] observed a problem concerning factorization in conjunction with regularization by dimensional reduction (DRED) [2]. The partonic process $gg \to t\bar{t}$ with non-vanishing quark mass $m_t \equiv m$ was evaluated using both DRED and ordinary dimensional regularization (DREG) [3][2]. Contrary to expectations [4], the difference between the two regularization schemes could not be absorbed by a finite additional factorization, corresponding to a change in the parton distribution functions.

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[2]Sometimes, these regularization schemes are also abbreviated as "DR" and "CDR (conventional dimensional regularization)".

As a consequence it seems impossible to write the hadronic cross section $\sigma_{\rm had}$ as a convolution of parton distribution functions $f$ and the partonic cross section $\sigma^{\rm DRED}_{\rm parton}$. Schematically, the factorization problem can be expressed as

$$\sigma_{\rm had} = f \otimes \sigma^{\rm DRED}_{\rm parton} + \text{extra, non-factorizing terms.} \tag{1}$$

The extra terms vanish in the limit of vanishing quark mass $m = 0$. Moreover, in the case of only massless partons the transition between the two regularization schemes has been worked out for many examples [5, 6] and could always be performed as expected. Even for one-loop amplitudes involving massive partons the transition could be studied [7]. Nevertheless, the problem found in Ref. [1] for the massive case has remained unsolved. It has repeatedly shown up and has been stressed again e.g. in Refs. [8, 9].

There are two areas where DRED is traditionally applied with great benefit. One is the evaluation of purely massless amplitudes, especially within QCD. Here DRED and related methods allow the use of powerful helicity methods [10]. The most important application of DRED and its original purpose is the regularization of supersymmetric theories. It has been shown to preserve supersymmetry relations in many different cases at the one-loop [8, 11, 12] and the two-loop level [13, 14], and in [13] also further properties such as mathematical consistency have been established.

The factorization problem is particularly troublesome for the calculation of QCD corrections to supersymmetric processes involving hadrons. In spite of the advantages of DRED, it renders the use of DRED questionable (see e.g. the discussion in [8]). Resorting to DREG in such calculations introduces several disadvantages. Mainly, supersymmetry is broken and has to be restored by adding supersymmetry-restoring counterterms [8, 12] that do not correspond to multiplicative renormalization. In addition, the $\overline{DR}$ renormalization scheme, a very common definition of supersymmetry parameters, is naturally based on DRED but only awkward to realize using DREG. Clearly, a resolution of the factorization problem would be welcome for both fundamental and practical reasons [15, 16].

In this article we reconsider the problem found in Refs. [1, 9]. We show that, despite first appearances, the result of Ref. [1] in fact is perfectly consistent with factorization.

We begin in Sec. 2 with a detailed explanation of the calculation of the LO process $gg \to t\bar{t}$ and the real NLO correction $gg \to t\bar{t}g$. We consider the collinear limit of two of the gluons and recover the seemingly paradoxical result of Ref. [1]. An important ingredient of this collinear limit is the necessity to average over the unobservable azimuthal angle of the final state gluon. It distinguishes the massive from the massless case, and in the massive case it leads to the difference between the DREG- and the DRED-result.

In Sec. 3 we first describe the general idea that will lead to a re-interpretation of the result, showing that it is consistent with factorization. The crucial point to notice is that in DRED the 4-dimensional gluon is a composition of a $D$-dimensional part and a remaining $(4-D)$-dimensional part, and that these two parts behave as *two different partons* $g$ and $\phi$.

Finally it is demonstrated in detail how this idea leads to a resolution of the factorization problem. On the one hand, in the collinear limit the NLO cross section becomes equal to a linear combination of *two different* LO cross sections, with either $g$ or $\phi$ in the initial state. On the other hand, the appearing prefactors in this linear combination have a natural interpretation as splitting functions for the splitting processes $g \to gg$, $\phi \to g\phi$, etc. We will also explain why factorization works in the $m = 0$ case already without distinguishing between $g$ and $\phi$.

In Sec. 4 we give our conclusions.

# 2 Recovering the seemingly non-factorizing result

## 2.1 LO and NLO calculation

We consider hadroproduction of a quark pair $t\bar{t}$ via gluon fusion, the process for which the factorization problem has been reported in Refs. [1, 9]. In this section we will briefly describe the required tree-level calculations and recover the result of these references. At leading order (LO) we only need the $2 \to 2$ process $gg \to t\bar{t}$, whereas at next-to-leading order (NLO) we also have to consider the $2 \to 3$ process with an additional gluon in the final state.

We carry out the calculation using either DREG or DRED. In both cases, space-time, momenta and momentum integrals are treated in $D$ dimensions. In DREG, the gluon vector field is treated in $D$ dimensions as well, while in DRED the gluon field and $\gamma$-matrices remain 4-dimensional quantities.

At leading order the amplitude $\mathcal{A}_{\mathrm{RS}}(2{\to}2)$ is given by the diagrams sketched in Fig. 1a. The subscript RS denotes the regularization scheme, DREG or DRED. The incoming gluon momenta and colour indices are denoted by $k_{1,2}$ and $a_{1,2}$, respectively; the outgoing momenta are called $p_{1,2}$. We will use the kinematical variables

$$S = 2k_1k_2, \qquad T_1 = (k_1 - p_1)^2 - m^2, \qquad U_1 = (k_2 - p_1)^2 - m^2. \tag{2}$$

$\mathcal{A}_{\mathrm{RS}}(2{\to}2)$ can be decomposed into two colour structures as

$$\mathcal{A}_{\mathrm{RS}}(2{\to}2) = \mathcal{A}^{(12)}_{\mathrm{RS}}(2{\to}2) T^{a_1} T^{a_2} + \mathcal{A}^{(21)}_{\mathrm{RS}}(2{\to}2) T^{a_2} T^{a_1}. \tag{3}$$

The squared LO amplitude, summed over initial and final state polarizations and colours, can be decomposed as

$$\mathcal{M}_{\mathrm{RS}}(2\to 2) = \sum_{\mathrm{pols,col}} |\mathcal{A}_{\mathrm{RS}}(2\to 2)|^2$$
$$= \frac{(N^2-1)^2}{4N}\mathcal{M}^{(1)}_{\mathrm{RS}}(2\to 2) - \frac{N^2-1}{4N}\mathcal{M}^{(2)}_{\mathrm{RS}}(2\to 2), \tag{4}$$

where $N = 3$ is the number of colours. For the polarization sum corresponding to a gluon with polarization vector $\epsilon^\mu$ and momentum $k$, we use

$$\sum_{\mathrm{pols}} \epsilon^\mu \epsilon^{\nu *} \to -g^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{(nk)} - \frac{n^2 k^\mu k^\nu}{(nk)^2} \tag{5}$$

with an arbitrary gauge vector $n^\mu$ such that $nk \neq 0$.

We obtain the following results:

$$\mathcal{M}^{(1,2)}_{\mathrm{RS}}(2\to 2) = 8g^4 \left\{1 - \frac{2T_1U_1}{S^2},\ +\frac{2T_1U_1}{S^2}\right\} B_{\mathrm{QED}}, \tag{6a}$$

$$B_{\mathrm{QED}} = n_G^{\mathrm{RS}}\left(-1 + \frac{n_G^{\mathrm{RS}} S^2}{4T_1U_1}\right) + \frac{4m^2 S}{T_1^2 U_1^2}\left(T_1U_1 - m^2 S\right), \tag{6b}$$

in agreement with Ref. [1]. The difference between the calculation in DRED and DREG enters only through the number $n_G^{\mathrm{RS}}$ of gluon degrees of freedom,

$$n_G^{\mathrm{DREG}} = D - 2, \quad n_G^{\mathrm{DRED}} = 2. \tag{7}$$

Technically, $n_G^{\mathrm{RS}}$ appears in the form $n_G^{\mathrm{RS}} = g^\mu_\mu - 2$, where the metric tensor originates either from the numerator of a gluon propagator or the polarization sum (5).

At NLO, we restrict ourselves to the real corrections, corresponding to the process $gg \to t\bar{t}g$. This is sufficient for the discussion of the collinear divergences and the factorization problem [1,9]. The diagrams contributing to the amplitude $\mathcal{A}(2\to 3)$ are generically depicted in Fig. 1b. The outgoing momentum and colour indices of the additional final state gluon are denoted by $k_3$, $a_3$; in accordance with Ref. [1] we use the kinematical variables

$$\begin{aligned} s &= (k_1+k_2)^2, \quad s_4 = (k_3+p_1)^2 - m^2, \quad t' = (k_2-k_3)^2, \\ u' &= (k_1-k_3)^2, \quad u_6 = (k_2-p_1)^2 - m^2, \quad u_7 = (k_1-p_1)^2 - m^2, \end{aligned} \tag{8}$$

which satisfy $s + s_4 + t' + u' + u_6 + u_7 = 0$. It is useful to keep the distinction between these variables for the $2 \to 3$ process and the variables $S$, $T_1$, $U_1$ for the $2 \to 2$ process although $S$ and $s$ are the same functions of $k_{1,2}$. The explicit form of the full result for $\mathcal{M}_{\mathrm{RS}}(2\to 3)$ is lengthy and suppressed here.

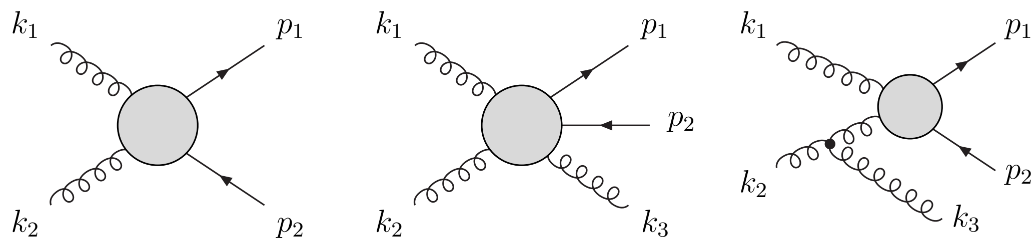


Figure 1: Generic structure of (a) LO diagrams, (b) NLO real correction diagrams, (c) NLO diagrams giving rise to a collinear divergence for $k_3 \to (1-x)k_2$.

In order to obtain the partonic cross sections, the squared amplitudes have to be averaged over the initial state polarizations and colours and divided by a flux factor. We denote these averaged quantities by

$$\langle \mathcal{M}_{\rm RS}(2\to2)\rangle = \frac{1}{2S}\frac{1}{[n_G^{\rm RS}(N^2-1)]^2}\mathcal{M}_{\rm RS}(2\to2), \tag{9a}$$

$$\langle \mathcal{M}_{\rm RS}(2\to3)\rangle = \frac{1}{2s}\frac{1}{[n_G^{\rm RS}(N^2-1)]^2}\mathcal{M}_{\rm RS}(2\to3). \tag{9b}$$

The differential cross sections are then given by ($P \equiv k_1 + k_2 - p_1 - p_2$)

$$d\sigma^{\rm RS}_{2\to2} = \langle \mathcal{M}_{\rm RS}(2\to2)\rangle \left(\prod_{p_f=p_{1,2}} \frac{d^{D-1}p_f}{2p_f^0(2\pi)^3}\right)(2\pi)^D\delta^{(D)}(P), \tag{10}$$

$$d\sigma^{\rm RS}_{2\to3} = \langle \mathcal{M}_{\rm RS}(2\to3)\rangle \left(\prod_{p_f=p_{1,2},k_3} \frac{d^{D-1}p_f}{2p_f^0(2\pi)^3}\right)(2\pi)^D\delta^{(D)}(P-k_3). \tag{11}$$

They depend on the regularization scheme at $\mathcal{O}(4-D)$ and at $\mathcal{O}((4-D)^0)$ due to soft and collinear divergences.

## 2.2 Collinear limit and azimuthal average

Now we consider the limit of $\langle \mathcal{M}_{\rm RS}(2\to3)\rangle$, where the unobserved final state gluon becomes collinear to one of the initial state gluons. To be specific we will concentrate on the collinear limit $2\|3$ of gluon 2 and gluon 3 and define the collinear limit $k_\perp \to 0$ by parametrizing the momenta $k_2^\mu$ and $k_3^\mu$ as follows:

$$k_3^\mu = (1-x)k_2^\mu + k_\perp^\mu - \frac{k_\perp^2}{1-x}\frac{n^\mu}{2k_2 n}, \tag{12}$$

where the auxiliary vector $n^\mu$ satisfies $n^2 = nk_\perp = 0$.

The collinear divergence in the NLO cross section originates from diagrams of the form shown in Fig. 1c where the virtual gluon becomes on-shell. In the squared amplitude this gives rise to terms of the order $1/t' \sim 1/k_\perp^2$, and such terms lead to singularities in the phase-space integral. As can be read off from Fig. 1c, one would expect the divergent NLO terms to become proportional to the LO terms with the identification

$$S \to xs, \qquad U_1 \to xu_6, \qquad T_1 \to -x(s+u_6). \tag{13}$$

However, this naive expectation does not take into account the following subtlety: not all poles $1/k_\perp^2$ of the squared amplitude are directly of the form $1/t'$. Some poles have a more involved structure. In particular, in our example, there are poles of the form $(ss_4 - u'u_6)^2/t'^2$. Upon taking the collinear limit, these terms depend on $k_\perp^\mu$. However, the transverse direction $k_\perp^\mu$ is unobservable in the collinear limit and will be azimuthally averaged over in the phase-space integral [6]. This averaging procedure affects only terms containing $1/t'^2$, and it yields

$$\frac{(ss_4 - u'u_6)^2}{t'^2} \overset{\langle 2\|3\rangle}{\longrightarrow} \frac{1}{D-2}\frac{-(1-x)}{x^2}\frac{4S\,(T_1U_1 - m^2S)}{t'} + \dots\ , \tag{14}$$

where the dots denote terms without a $1/t'$ singularity. The notation $\langle 2\|3\rangle$ implies that the average over the $(D-2)$-dimensional transverse space is taken in the collinear limit. The factor $(D-2)$ enters the denominator as a result of this averaging [6].

Taking the averaged collinear limit of $\langle \mathcal{M}_{\rm RS}(2{\to}3)\rangle$ we obtain

$$\langle \mathcal{M}_{\rm RS}(2{\to}3)\rangle \overset{\langle 2\|3\rangle}{\longrightarrow} \frac{-4g^2N}{t'}\left(\frac{(1-x+x^2)^2}{x(1-x)}\langle \mathcal{M}_{\rm RS}(2{\to}2)\rangle + \left(\frac{n_G^{\rm RS}}{D-2} - 1\right)\frac{(1-x)}{x}\langle \mathcal{M}_{\rm RS}(2{\to}2)\rangle|_m\right), \tag{15}$$

where $\mathcal{M}_{\rm RS}(2{\to}2)|_m \equiv \mathcal{M}_{\rm RS}(2{\to}2) - \mathcal{M}_{\rm RS}(2{\to}2)|_{m=0}$ denotes the mass terms of $\mathcal{M}_{\rm RS}(2{\to}2)$. This equation is equivalent to the result found in Refs. [1, 9].

The factorization theorem seems to suggest that the terms that are divergent in this collinear limit are proportional to the LO result. Whereas the first term on the right-hand side of (15) is in accordance with this expectation, the second term contains only the mass-dependent terms of the LO result and, therefore, seems to violate the factorization theorem. Due to the prefactor, this second term is absent in DREG, and the problem is only present in DRED. What we would expect in going from DREG, where factorization holds, to DRED is a change in the function multiplying the LO term, but not a change in the structure of the result.

As mentioned in Ref. [1, 9] the problematic term vanishes in the massless limit. However, this is not generally true but is peculiar to the process under consideration. The decisive feature is not the mass of the quarks but the presence of terms $\sim 1/t'^2$. In our case, the absence of $1/t'^2$ terms in the massless case can be explained by helicity conservation.

In the next section we will discuss the origin of the seemingly non-factorizing term and show that it can be rewritten in a way that is consistent with factorization.

# 3 Reconciling the NLO result with factorization

## 3.1 General idea

In the collinear limit, the NLO result in DRED (15) does not seem to factorize into a product of a splitting function and the LO result. In contrast, the NLO result in DREG does factorize. There is a simple argument that allows to understand why the two regularization schemes behave in such a different way. In the regularized expressions, the number of dimensions $D$ and of gluon degrees of freedom $n_G^{\mathrm{RS}}$ can be set to integers. For example, DREG with integer $D$ and $n_G^{\mathrm{DREG}} = D - 2$ simply corresponds to unregularized QCD in $D$ dimensions. Of course, factorization can be expected to hold in QCD with an arbitrary number of dimensions. This is the reason why Eq. (15) factorizes in the case of DREG.

In contrast, DRED with e.g. $D = 3$ does not lead to 3-dimensional QCD but rather to 4-dimensional QCD, dimensionally reduced to 3 dimensions. It is well known that in the process of dimensional reduction from 4 to 3 dimensions, the 4-dimensional gluon is decomposed into the 3-dimensional gluon $A^\mu$ ($\mu = 0, 1, 2$) and an extra scalar field $\phi \equiv A^3$. The resulting theory is 3-dimensional QCD, supplemented with a minimally coupled scalar $\phi$ in the adjoint representation.

The crucial point to be learnt from this discussion is that the dimensionally reduced theory contains *two distinct partons*, the 3-dimensional gluon $g$ and the scalar $\phi$. At LO there are therefore four distinct partonic processes for $t\bar{t}$ production:

$$gg \to t\bar{t}, \qquad g\phi \to t\bar{t}, \qquad \phi g \to t\bar{t}, \qquad \phi\phi \to t\bar{t}. \tag{16}$$

It is obvious that factorization can be expected to hold in this dimensionally reduced theory, but not in the same way as in DREG. On the right-hand side of Eq. (15) we do not expect one single term but instead a linear combination of all four partonic LO processes.

In DRED with arbitrary, non-integer $D$, the situation is similar. The regularized theory contains a $D$-dimensional gluon $g$ and $4-D$ additional scalar fields $\phi$, so-called $\epsilon$-scalars [11]. Again, $g$ and $\phi$ have to be viewed as independent partons, and the collinear limit is expected to contain all four LO processes.

In order to express this new expectation more formally, we denote the 4-dimensional gluon that has always been assumed in DRED in the previous section (and also in Refs. [1,9]) by $G$. The $2 \to 2$ and $2 \to 3$ processes considered in the previous section can then be written more explicitly as

$$\mathcal{M}_{\text{DRED}}(2{\to}2(3)) \equiv \mathcal{M}_{\text{DRED}}(GG{\to}t\bar{t}(G)). \tag{17}$$

Since the 4-dimensional gluon $G$ constitutes the combination $g + \phi$, the squared matrix elements satisfy the relation

$$\begin{aligned}\mathcal{M}_{\text{DRED}}(GG{\to}t\bar{t}) =& \mathcal{M}_{\text{DRED}}(gg{\to}t\bar{t}) + \mathcal{M}_{\text{DRED}}(g\phi{\to}t\bar{t}) \\ &+ \mathcal{M}_{\text{DRED}}(\phi g{\to}t\bar{t}) + \mathcal{M}_{\text{DRED}}(\phi\phi{\to}t\bar{t})\end{aligned} \tag{18}$$

and similarly for $\mathcal{M}_{\text{DRED}}(GG{\to}t\bar{t}G)$. This leads us to expect that the collinear limit in DRED can be written as

$$\langle\mathcal{M}_{\text{DRED}}(ij{\to}t\bar{t}k)\rangle \xrightarrow{\langle 2\|3\rangle} \frac{-2g^2}{t'}\left[\sum_{l=g,\phi} P_{j\to lk}\langle\mathcal{M}_{\text{DRED}}(il{\to}t\bar{t})\rangle\right]. \tag{19}$$

Contrary to the corresponding formula for DREG, the right-hand side of Eq. (19) is a linear combination involving more than one LO process.

In the following we will show that the seemingly non-factorizing term in Eq. (15) can be rewritten as a linear combination of the four partonic LO processes. Thus, factorization is valid in DRED in the form expected in Eq. (19) and we will see that the functions $P_{j\to lk}$ can be interpreted as splitting functions.

## 3.2 Collinear limit and LO result with $g$ or $\phi$ in the initial state

According to the idea discussed in the preceding subsection we evaluate all four partonic LO processes (16) individually. The algebraic expressions for the partonic processes involving $g$, $\phi$, or $G$ are distinguished by the values of the polarization vector $\epsilon^\mu$ and the corresponding polarization sum. The polarization sum corresponding to an external $G$ is the one given in Eq. (5); the ones corresponding to $g$ and $\phi$ read

$$g: \quad \sum_{\text{pols}} \epsilon^\mu \epsilon^{\nu *} \to -\hat{g}^{\mu\nu} + \frac{n^\mu k^\nu + k^\mu n^\nu}{(nk)} - \frac{n^2 k^\mu k^\nu}{(nk)^2}, \tag{20a}$$

$$\phi: \quad \sum_{\text{pols}} \epsilon^\mu \epsilon^{\nu *} \to -\tilde{g}^{\mu\nu}. \tag{20b}$$

The objects $\hat{g}^{\mu\nu}$ and $\tilde{g}^{\mu\nu}$ are the projectors on the $D$- and $(4-D)$-dimensional subspaces [2] (see also Ref. [13] for further details) and satisfy $\hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = D$, $\tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu} = 4 - D$ and the projector relations $g^{\mu\nu}\hat{g}_\nu{}^\rho = \hat{g}^{\mu\rho}$, $g^{\mu\nu}\tilde{g}_\nu{}^\rho = \tilde{g}^{\mu\rho}$. They are related to the 4-dimensional metric tensor by $g^{\mu\nu} = \hat{g}^{\mu\nu} + \tilde{g}^{\mu\nu}$.

We obtain the following results:

$$\mathcal{M}^{(1,2)}_{\text{DRED}}(ij\to t\bar{t}) = 8g^4 \left\{1 - \frac{2T_1U_1}{S^2},\ + \frac{2T_1U_1}{S^2}\right\} B_{ij}, \tag{21a}$$

$$B_{gg} = n_g^{\text{DRED}} \left(-1 + \frac{n_g^{\text{DRED}} S^2}{4T_1U_1}\right) + \frac{4m^2S}{T_1^2U_1^2}\left(T_1U_1 - m^2S\right), \tag{21b}$$

$$B_{\phi\phi} = n_\phi^{\text{DRED}} \left(-1 + \frac{n_\phi^{\text{DRED}} S^2}{4T_1U_1}\right), \tag{21c}$$

$$B_{g\phi} = n_g^{\text{DRED}} n_\phi^{\text{DRED}} \frac{S^2}{4T_1U_1}, \tag{21d}$$

$$B_{\phi g} = n_g^{\text{DRED}} n_\phi^{\text{DRED}} \frac{S^2}{4T_1U_1}. \tag{21e}$$

Here the symbols $n_g^{\text{DRED}}$ and $n_\phi^{\text{DRED}}$ denote the numbers of degrees of freedom corresponding to the partons $g$ and $\phi$:

$$n_g^{\text{DRED}} = D - 2, \quad n_\phi^{\text{DRED}} = 4 - D. \tag{22}$$

The fact that the 4-dimensional gluon $G$ is the combination of $g$ and $\phi$ is reflected in the equality $n_G^{\text{DRED}} = n_g^{\text{DRED}} + n_\phi^{\text{DRED}}$ and by the observation that, as already stated in Eq. (18), the sum of the four partial results (21) is equal to the result for the $GG$ initial state.

Note that the result for the $gg$ case is equal to the LO result in DREG because eqs. (6b) and (21b) have the same form and $n_G^{\text{DREG}} = n_g^{\text{DRED}}$. This equality can be understood as a consequence of the fact that in the simple process $gg \to t\bar{t}$ at tree level no $\epsilon$-scalars $\phi$ appear as virtual states in the Feynman diagrams.

In a next step we perform the calculation of all eight squared amplitudes $\mathcal{M}_{\text{DRED}}(ij\to t\bar{t}k)$ with $i, j, k = g, \phi$. We do not present the full analytic results but concentrate on the collinear limit $k_3 \to (1-x)k_2$, since we are interested in how the processes involving the individual partons $g$, $\phi$ behave as compared to the seemingly non-factorizing result (15) for the process involving only $G$. The averaged amplitudes are defined as in Eq. (9), replacing $n_G^{\text{DRED}}$ by $n_g^{\text{DRED}}$, $n_\phi^{\text{DRED}}$ where appropriate. We find the following results:

$$\langle\mathcal{M}_{\text{DRED}}(ig\to t\bar{t}g)\rangle \xrightarrow{\langle 2\|3\rangle} \frac{-4g^2N}{t'}\langle\mathcal{M}_{\text{DRED}}(ig\to t\bar{t})\rangle \frac{(1-x+x^2)^2}{x(1-x)}, \tag{23a}$$

$$\langle\mathcal{M}_{\text{DRED}}(i\phi\to t\bar{t}g)\rangle \xrightarrow{\langle 2\|3\rangle} \frac{-4g^2N}{t'}\langle\mathcal{M}_{\text{DRED}}(i\phi\to t\bar{t})\rangle \frac{x}{1-x}, \tag{23b}$$

$$\langle\mathcal{M}_{\text{DRED}}(ig\to t\bar{t}\phi)\rangle \xrightarrow{\langle 2\|3\rangle} \frac{-4g^2N}{t'}\langle\mathcal{M}_{\text{DRED}}(i\phi\to t\bar{t})\rangle \frac{n_\phi^{\text{DRED}}}{n_g^{\text{DRED}}}\, x(1-x), \tag{23c}$$

$$\langle\mathcal{M}_{\text{DRED}}(i\phi\to t\bar{t}\phi)\rangle \xrightarrow{\langle 2\|3\rangle} \frac{-4g^2N}{t'}\langle\mathcal{M}_{\text{DRED}}(ig\to t\bar{t})\rangle \frac{1-x}{x}. \tag{23d}$$

These results have precisely the form of Eq. (19) with

$$P_{g\to gg} = 2N\,\frac{(1-x-x^2)^2}{x(1-x)}, \tag{24a}$$

$$P_{\phi\to\phi g} = 2N\,\frac{x}{1-x}, \tag{24b}$$

$$P_{g\to\phi\phi} = 2N\,\frac{n_\phi^{\text{DRED}}}{n_g^{\text{DRED}}}\,x(1-x), \tag{24c}$$

$$P_{\phi\to g\phi} = 2N\,\frac{1-x}{x}, \tag{24d}$$

$$P_{j\to lk} = 0 \text{ otherwise}. \tag{24e}$$

They demonstrate clearly that all eight individual partonic processes factorize in the usual way into a product of a splitting function and a LO process, without any unusual terms. There are not even non-trivial linear combinations of LO processes on the right-hand sides. This fact and the origin of the splitting functions is discussed in the following subsections.

The eight results can now be combined to reconcile the collinear limit in Eq. (15) for DRED with factorization. Instead of Eq. (15) we now obtain

$$\begin{aligned}
\langle\mathcal{M}_{\text{DRED}}(GG\to t\bar{t}G)\rangle &= \sum_{i,j,k=g,\phi}\frac{n_i^{\text{DRED}}n_j^{\text{DRED}}}{(n_G^{\text{DRED}})^2}\langle\mathcal{M}_{\text{DRED}}(ij\to t\bar{t}k)\rangle \\
&\overset{\langle 2\|3\rangle}{\longrightarrow} \frac{-4g^2N}{t'}\Bigg[\langle\mathcal{M}_{\text{DRED}}(Gg\to t\bar{t})\rangle\left(\frac{n_g^{\text{DRED}}}{n_G^{\text{DRED}}}\frac{(1-x+x^2)^2}{x(1-x)}+\frac{n_\phi^{\text{DRED}}}{n_G^{\text{DRED}}}\frac{1-x}{x}\right) \\
&\qquad +\langle\mathcal{M}_{\text{DRED}}(G\phi\to t\bar{t})\rangle\frac{n_\phi^{\text{DRED}}}{n_G^{\text{DRED}}}\left(\frac{x}{1-x}+x(1-x)\right)\Bigg],
\end{aligned} \tag{25}$$

where the relations $\mathcal{M}(Gj\to t\bar{t}) = \mathcal{M}(gj\to t\bar{t}) + \mathcal{M}(\phi j\to t\bar{t})$ have been used. In this equation the collinear limit finally acquires a factorized structure although DRED is used. As expected in Sec. 3.1, a linear combination of LO processes appears on the right-hand side.

It is instructive to directly verify the equality of eqs. (25) and (15), the factorized and non-factorized version of the collinear limit respectively. Since the mass dependence in Eq. (21) enters only through the $gg$ result we can write

$$\langle\mathcal{M}_{\text{DRED}}(GG\to t\bar{t})\rangle|_m = \frac{n_g^{\text{DRED}}}{n_G^{\text{DRED}}}\left(\langle\mathcal{M}_{\text{DRED}}(Gg\to t\bar{t})\rangle - \langle\mathcal{M}_{\text{DRED}}(G\phi\to t\bar{t})\rangle\right), \tag{26a}$$

$$\langle\mathcal{M}_{\text{DRED}}(GG\to t\bar{t})\rangle|_{m=0} = \langle\mathcal{M}_{\text{DRED}}(Gg\to t\bar{t})\rangle|_{m=0} = \langle\mathcal{M}_{\text{DRED}}(G\phi\to t\bar{t})\rangle. \tag{26b}$$

Thus we see that the disturbing mass term in Eq. (15) indeed can be resolved as a linear combination of complete LO processes. Using eqs. (26) in Eq. (15) directly leads to Eq. (25).

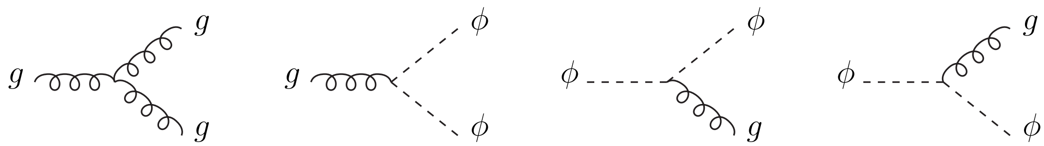


Figure 2: Tree level diagrams for the four splitting processes involving $g$ and $\phi$.

Finally we note that in the massless case (26b), several of the LO processes become equal, which is why the collinear limit then takes a simpler form and the problematic term in Eq. (15) disappears. This is however a peculiarity of the considered process and related to the absence of terms $\sim 1/t'^2$ discussed in Sec. 2, but it is not a generic feature of processes with massless partons.

### 3.3 Splitting functions involving $g$ and $\phi$

In this subsection we focus on the splitting functions appearing in Eq. (24), involving $g$ and $\phi$ as partons. In order to consolidate our understanding of factorization in DRED we will present an independent derivation of these splitting functions. Instead of reading them off from the collinear limits of particular NLO processes we directly evaluate the amplitudes for the splitting processes

$$\begin{aligned} g &\to g(x)\, g(1-x), \quad g \to \phi(x)\, \phi(1-x), \\ \phi &\to \phi(x)\, g(1-x), \quad \phi \to g(x)\, \phi(1-x). \end{aligned} \tag{27}$$

The corresponding diagrams are shown in Fig. 2. Note that the amplitudes for splitting processes involving an odd number of $\phi$ partons vanish at tree level. In each splitting process $i \to jk$ the momenta are assigned as $p_i \equiv k_2$, $p_k \equiv k_3$ as given in Eq. (12), and $p_j = k_2 - k_3$. In order to obtain the splitting probabilities, the amplitudes are squared and summed over colours and polarizations according to Eq. (20). Only particle $j$ is kept slightly off-shell, $p_j^2 \sim k_\perp^2$, and its Lorentz and colour indices are kept uncontracted. The result for each splitting process thus has the form $\mathcal{P}_{i\to jk}^{\rho\rho',aa'}$, where $\rho$, $\rho'$ and $a$, $a'$ are the open Lorentz and colour indices. Terms subleading in $k_\perp$ are neglected and the average over the $D-2$ transverse directions is performed. Finally, terms proportional to $(k_2-k_3)^\rho$ or $(k_2-k_3)^{\rho'}$ can be neglected, too. Due to the Ward identity they do not contribute if the splitting processes are part of a larger physical process where all particles are on-shell.

After these manipulations, the results $\mathcal{P}_{i\to jk}^{\rho\rho',aa'}$ take the form

$$\mathcal{P}_{i\to jk}^{\rho\rho',aa'} = P_{i\to jk}(x)\, \frac{2(k_2-k_3)^2}{x}\, \frac{n_i^{\mathrm{DRED}}}{n_j^{\mathrm{DRED}}}\, \delta_{aa'}\, g_{(j)}^{\rho\rho'}. \tag{28}$$

They are proportional to $\hat{g}^{\rho\rho'}$ if $j=g$ and to $\tilde{g}^{\rho\rho'}$ if $j=\phi$ (commonly abbreviated as $g_{(j)}^{\rho\rho'}$ here), and they are proportional to $\delta_{aa'}$ in colour space. As expected, the prefactors are given by the splitting functions $P_{i\to jk}(x)$ of Eq. (24), multiplied by additional factors that compensate for the different prefactors in cross sections with either $i$ or $j$ in the initial state.

Hence the functions given in Eq. (24) have a natural interpretation as universal splitting functions. The fact that only one term appears on each right-hand side of Eq. (23) is due to the vanishing of the splitting functions involving an odd number of $\phi$'s.

For future reference we introduce splitting functions corresponding to 4-dimensional gluons

$$n_G^{\rm DRED} P_{G\to jG} = \sum_{k=g,\phi} \left( n_g^{\rm DRED} P_{g\to jk} + n_\phi^{\rm DRED} P_{\phi\to jk} \right) \tag{29}$$

and note that the splitting functions satisfy the sum rule

$$n_G^{\rm DRED} P_{g\to gg} = \sum_{j=g,\phi} n_G^{\rm DRED} P_{G\to jG} = \sum_{i,j,k=g,\phi} n_i^{\rm DRED} P_{i\to jk}. \tag{30}$$

As in Eq. (28), the factors $n_G^{\rm DRED}$, $n_g^{\rm DRED}$ and $n_\phi^{\rm DRED}$ appear because we are considering the splitting of an initial state parton and, therefore, have to correct for the factors due to the average over polarizations.

We close the subsection with several remarks. First, note that $P_{g\to gg}$ is identical to the well-known gluon splitting function in DREG. Second, the splitting functions involving $\phi$ coincide with the splitting functions involving massless squarks and gluons, given in Ref. [7], if the colour factors for squarks $T_R$, $C_F$ are replaced by $C_A = N$. The particular splitting function $P_{g\to\phi\phi}$ has already been made use of in Ref. [6] in order to study the difference between DRED and DREG. And finally, $P_{\phi\to g\phi}$ is the prefactor of the puzzling term in Eq. (15), and it corresponds to the factor $K_g$ in Ref. [9]. The nature of $P_{\phi\to g\phi}$ as a splitting function explains the universal behaviour of $K_g$ described in this reference.

## 3.4 Final result

In the previous subsections we have seen that the real NLO processes with partons $g$, $\phi$ indeed factorize in the collinear limit. The $x$-dependent prefactors can be interpreted as the splitting functions $P_{i\to jk}$ corresponding to the parton splittings $g\to gg$, $g\to\phi\phi$, $\phi\to g\phi$, $\phi\to\phi g$. Thus the results for the collinear limits take a very systematic form:

$$\langle \mathcal{M}_{\rm DRED}(ij\to t\bar{t}k)\rangle \xrightarrow{\langle 2\|3\rangle} \frac{-2g^2}{t'} \left[ \sum_{l=g,\phi} P_{j\to lk} \langle \mathcal{M}_{\rm DRED}(il\to t\bar{t})\rangle \right], \tag{31}$$

where $i,j,k = g,\phi$. The sums on the right-hand side all collapse to one single term since only the four aforementioned splitting functions can contribute, while splitting functions with an odd number of $\phi$'s vanish at tree level. Similarly, using the combinations (29) of splitting functions involving $G$, the result for the process involving only 4-dimensional gluons can be expressed as

$$\langle\mathcal{M}_{\text{DRED}}(GG\to t\bar{t}G)\rangle \overset{\langle 2\|3\rangle}{\longrightarrow} \frac{-2g^2}{t'}\left[\sum_{j=g,\phi} P_{G\to jG}\langle\mathcal{M}_{\text{DRED}}(Gj\to t\bar{t})\rangle\right]. \tag{32}$$

Although there is a non-trivial structure on the right-hand side, this result has the form that is expected from the factorization theorem.

For comparison, we repeat the corresponding result for the case of DREG with adapted notation:

$$\langle\mathcal{M}_{\text{DREG}}(gg\to t\bar{t}g)\rangle \overset{\langle 2\|3\rangle}{\longrightarrow} \frac{-2g^2}{t'}\left[P_{g\to gg}\langle\mathcal{M}_{\text{DREG}}(gg\to t\bar{t})\rangle\right]. \tag{33}$$

In the remainder of this section we briefly discuss the relation between DRED and DREG for the cross sections. The results for the collinear limits can be elevated to the level of cross sections by performing the suitable phase-space integration and taking into account the second collinear limit $1\|3$. The singular terms in the collinear limits yield the subtraction terms that render the cross section finite. In DREG, the subtracted hard scattering cross section $d\hat{\sigma}^{\text{DREG}}$ at NLO is given by

$$\begin{aligned}\int d\hat{\sigma}^{\text{DREG}}_{gg\to t\bar{t}g} = \int d\sigma^{\text{DREG}}_{gg\to t\bar{t}g} & \\ +\Bigg[\int_0^{1-\delta} dx_1 & \left(\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{g\to gg}(x_1)\right) d\sigma^{\text{DREG}}_{gg\to t\bar{t}}(x_1k_1,k_2) \\ +\int_0^{1-\delta} dx_2 & \left(\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{g\to gg}(x_2)\right) d\sigma^{\text{DREG}}_{gg\to t\bar{t}}(k_1,x_2k_2)\Bigg]\end{aligned} \tag{34}$$

with $\alpha_s = g^2/(4\pi)$ and $D = 4-2\epsilon$. In DRED it can be defined analogously:

$$\begin{aligned}\int d\hat{\sigma}^{\text{DRED}}_{GG\to t\bar{t}G} = \int d\sigma^{\text{DRED}}_{GG\to t\bar{t}G} & \\ +\sum_{j=g,\phi}\Bigg[\int_0^{1-\delta} dx_1 & \left(\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{G\to jG}(x_1)\right) d\sigma^{\text{DRED}}_{jG\to t\bar{t}}(x_1k_1,k_2) \\ +\int_0^{1-\delta} dx_2 & \left(\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{G\to jG}(x_2)\right) d\sigma^{\text{DRED}}_{Gj\to t\bar{t}}(k_1,x_2k_2)\Bigg].\end{aligned} \tag{35}$$

In these equations, all integration regions are assumed to contain the same collinear regions. The small parameter $\delta > 0$ excludes the region around $x_i = 1$, which would lead to further infrared singularities that cancel only by adding the virtual NLO corrections.

These subtracted cross sections are free of collinear singularities and, by construction, the non-singular remainders in both regularization schemes are equal up to terms of $\mathcal{O}(4-D)$:[3]

$$\int d\hat{\sigma}^{\text{DREG}}_{gg\to t\bar{t}g} = \int d\hat{\sigma}^{\text{DRED}}_{GG\to t\bar{t}G} + \mathcal{O}(4-D). \tag{36}$$

Eqs. (34)–(36) can also be derived directly from the puzzling result Eq. (6.28) in Ref. [1] by inserting our expression (26a) for the disturbing mass term.

This shows that the final hadronic cross section, which is obtained from $d\hat{\sigma}$ through convolution with parton distribution functions, can be evaluated both using DREG or using DRED. In particular, eq. (36) shows that the same factorization scheme can be realized using either DREG or DRED, and therefore the same parton distribution functions (e.g. defined in the $\overline{MS}$ factorization scheme) have to be used in both cases. The structure of the calculation is the same. The only difference is the appearance of the two independent partons $g$, $\phi$ in the subtraction terms for DRED that lead from $d\sigma$ to $d\hat{\sigma}$.

# 4 Conclusions

We have considered the factorization problem of DRED that has repeatedly shown up in the literature [1,8,9]. Eq. (15) exhibits the seemingly non-factorizing terms in the collinear limit of the process $gg \to t\bar{t}g$. We have shown that the problem can be completely solved.

The key to the solution is to consider the 4-dimensional gluon $G$ in DRED as a combination of the $D$-dimensional gluon $g$ and $4-D$ $\epsilon$-scalars $\phi$. If $g$ and $\phi$ are treated as independent partons as in Eq. (25), the collinear limit acquires a factorized form. The problematic terms on the right-hand side are replaced by a linear combination of several LO processes involving $g$ and $\phi$. Furthermore we have shown that the coefficients in this linear combination have a natural interpretation as splitting functions.

The final form of the collinear limit is displayed in eqs. (31) and (32). We have shown that the result for the collinear limit can be transferred to the level of cross sections and that the hadron cross section can be evaluated using both DREG or DRED. All results have a very systematic and natural structure.

[3] Note that the factorization scheme has been implicitly fixed in eqs. (34), (35). Different factorization schemes can be realized by adding identical terms in the brackets multiplying the $d\sigma_{ij\to t\bar{t}}$ in eqs. (34), (35). The resulting subtracted cross sections in DREG and DRED are then still equal.

In summary, the factorization problem of DRED, i.e. the presence of seemingly non-factorizing terms, is not a problem but a signal that the distinction between $g$ and $\phi$ as independent partons cannot be ignored. The solution does not affect the computation of the NLO diagrams itself. Only the expectation from the collinear limit and the structure of the subtraction terms needed to obtain the hard scattering cross section have to reflect this distinction. Although we have only considered the process $gg \to t\bar{t}g$ as an example and ignored virtual NLO corrections, one can expect that factorization in DRED holds in general and even in higher orders. The details of the general construction of finite, regularization-independent hard scattering cross sections will be left for future work.

An interesting remaining question is for which processes the factorization problem and the decomposition of the 4-dimensional gluon as $G = g + \phi$ is relevant in general. While a general answer to this question is beyond the scope of the present article, we can give two criteria, based on the analysis of the considered process, where the problem disappears for $m = 0$.

From the point of view of Sec. 2, for $m = 0$ the terms of the order $1/t'^2$ vanish. In this case, no average over the transverse direction of the collinear gluon has to be performed. Therefore, the result in DRED is trivially the $D = 4$ limit of the DREG-result, and in both regularizations factorization holds in the naive way.

From the point of view of Sec. 3, in the massless case the LO processes with $GG$, $Gg$ or $G\phi$ in the initial state all become equal, see Eq. (26b). As a result, in the collinear limit (32) no distinction between the different LO processes has to be made, and the prefactors combine to the sum $P_{G\to gG} + P_{G\to\phi G}$, which is simply equal to $P_{g\to gg}$ according to the sum rule (30). Hence the collinear limit in DRED again reduces to the naive form involving only 4-dimensional gluons and one splitting function $P_{g\to gg}$.

The situation is different for the process with one more leg, $gg \to t\bar{t}g$ with a hard gluon in the final state. We have checked that for this example, e.g. $\langle\mathcal{M}_{\rm DRED}(Gg{\to}t\bar{t}G)\rangle \neq \langle\mathcal{M}_{\rm DRED}(G\phi{\to}t\bar{t}G)\rangle$ already for $m = 0$ in contrast to Eq. (26b). Therefore, the factorization problem is not generally linked to the presence of massive partons but rather to sufficiently complicated kinematics.

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