A monoidal interval of isotone clones on a finite chain

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Abstract

Let $\underline{k}$ denote a k-element chain, $k \geq 3$. Let M denote the clone generated by all unary isotone operations on $\underline{k}$ and let $Pol \leq$ denote the clone of all isotone operations on $\underline{k}$. We investigate the interval of clones $[M, Pol \leq]$. Among other results, we describe completely those clones which contain only join (or meet) homomorphisms, and describe the interval completely for $k \leq 4$.

1 Introduction

Let A be a finite set. A *clone* on A is a set of finitary operations on A closed under superposition (composition) and containing all projections. If X is an $m \times n$-matrix with entries from A, and f is an n-ary operation, then the column $f(X)$ is calculated row-wise. For an m-ary relation θ on A, the clone $Pol\,\theta$ consists of all operations f such that $f(X)$ belongs to θ whenever all columns of X do. If f belongs to $Pol\,\theta$ we say that f *preserves* the relation θ.

Let ρ be a preorder on A, i.e. a binary relation on A which is reflexive and transitive. Let M denote the clone generated by all unary operations preserving ρ and as above, denote the clone of all operations preserving ρ by $Pol\,\rho$. It is proved in [4] that if the interval of clones $[M, Pol\,\rho]$ is finite then the preorder must be a chain, and that for $|A| = 3$ it is indeed the case that the interval is finite. Our purpose in this note is to further study the monoidal interval $[M, Pol \leq]$ where $\leq$ is the natural ordering on the set $\underline{k} = \{1, 2, \ldots, k\}$ for $k \geq 3$. We refer the reader to [3, 4] and Chapter 3 of [11] for a discussion of the general problem of determining monoidal intervals, and to [8, 9, 11] for standard results and notation.

Before we state our results, we need some notation. Let $3 \leq h \leq k$ and let μ_h denote the h-ary relation consisting of all tuples $(a_1, \ldots, a_h)$ such that $a_1 \leq a_2 \leq \ldots \leq a_h$ and such that $|\{a_1, \ldots, a_h\}| < h$. For $1 \leq h \leq k$ let P_h denote the clone of all isotone operations f which are either essentially unary or such that the image of f contains at most h elements. Notice that $P_1 = M$ and that $P_k = Pol \leq$.

Let $\vee^\circ$ denote the 3-ary relation consisting of all tuples $(a, b, a \vee b)$ where $\vee$ denotes the join operation of the chain, and similarly for the relation $\wedge^\circ$ where $\wedge$ is the meet operation of the chain $\underline{k}$. Notice that since the order we consider is a chain, we have that $M \subseteq Pol \vee^\circ$ and $M \subseteq Pol \wedge^\circ$.

It is difficult to state our main result in one short theorem. Therefore we shall refer to Figure 1 and describe its main properties and where in the text their proofs can be found. The figure depicts the (partial) Hasse diagram of the interval $[M, Pol \leq]$ for $k \geq 3$.

1. The interval has three maximal elements, $Pol \vee^\circ$, $Pol \wedge^\circ$ and $Pol\,\mu_k$; this is proved in Lemma 2.4.

2. Each solid line segment indicates, as usual, a covering relation. This follows from Lemmas 2.5 and 2.6 and Theorem 3.15.

3. Let C be a clone in the interval $[M, Pol \leq]$. Suppose that C in not one of M, $Pol \leq$, P_h, $Pol\ \vee^\circ \cap P_h$, $Pol\ \wedge^\circ \cap P_h$, $Pol\,\mu_h$, for any h. Then C is contained in an interval $[P_h, Pol\,\mu_{h+1}]$ for some $3 \leq h \leq k-1$. This is Theorem 3.15. These intervals are depicted by curved lines in Figure 1.

Notice that the above is sufficient to describe the interval if $k = 3$ (this was first done in [4]), see Figure 2. In section 4 we describe completely the interval for the case $k = 4$:

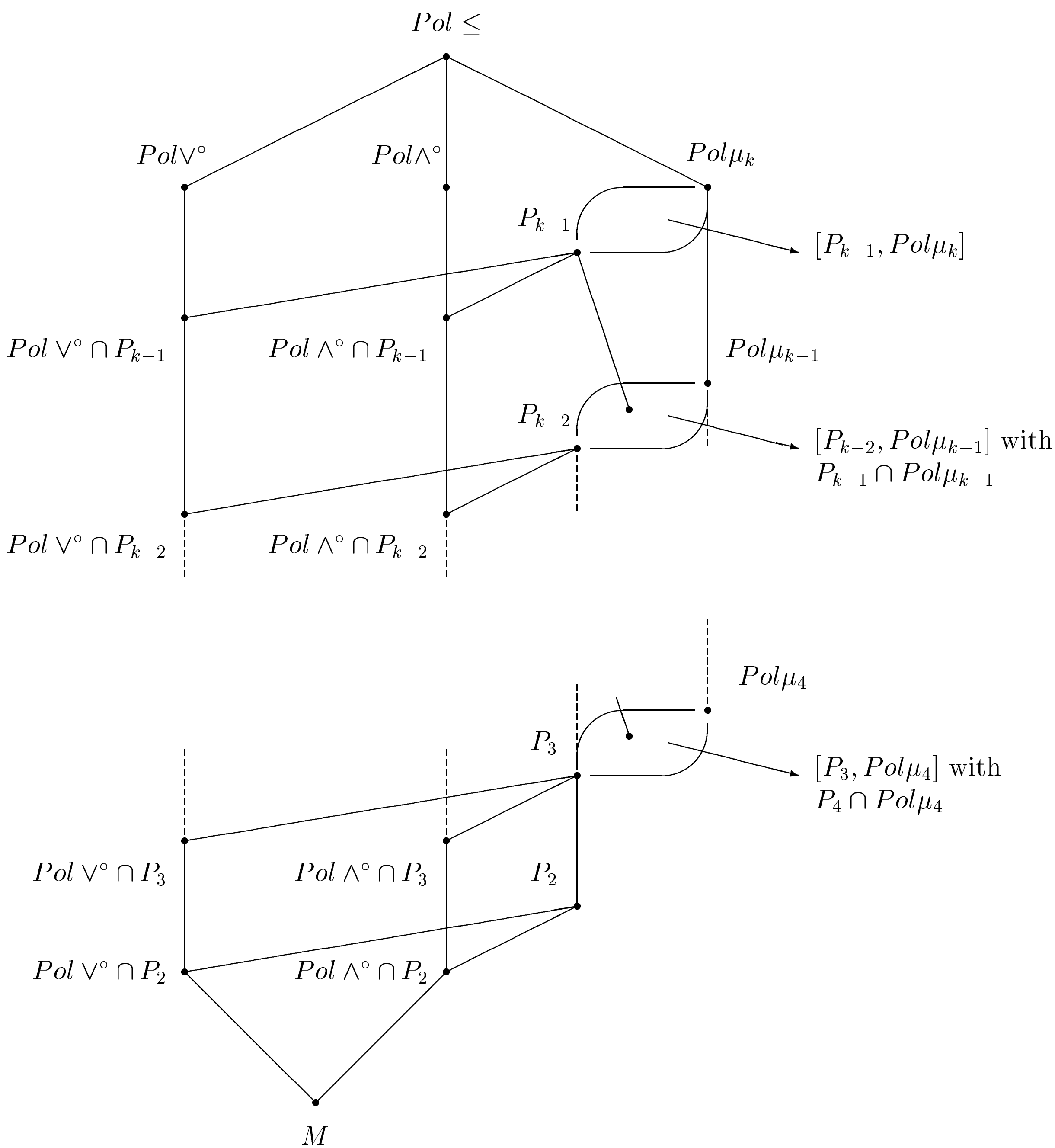

Figure 1: The interval $[M, Pol\le]$.

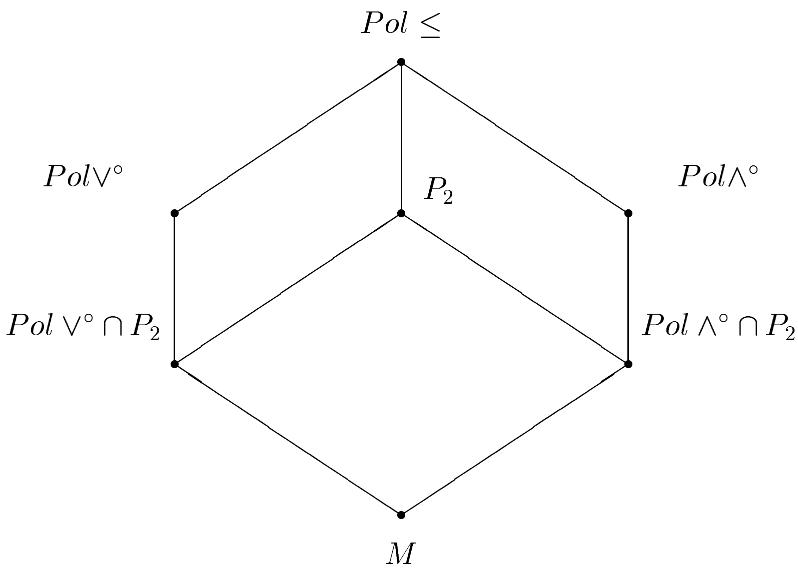

Figure 2: The interval $[M, Pol \leq]$ for $k = 3$.

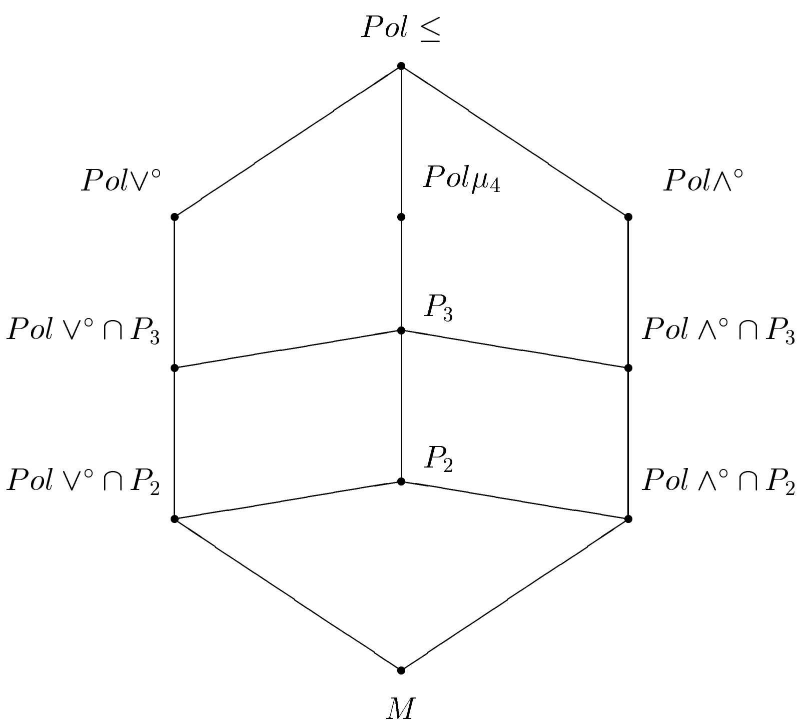

Figure 3: The interval $[M, Pol \leq]$ for $k = 4$.

Theorem 1.1 *For $k = 4$, the interval $[M, Pol \leq]$ consists of exactly 11 clones, as shown in Figure 3.*

The next section presents some basic results and definitions we shall need. In section 3 we prove all results that lead up to our description of the interval $[M, Pol \leq]$ in the general case. Then in section 4 we prove Theorem 1.1. We conclude with a few comments on the structure of the interval for $k \geq 5$.

2 Preliminaries

We begin with a few auxiliary results and definitions. In the following, the symbol $\subset$ shall denote strict inclusion. If F is a set of operations on $\underline{k}$ then $\langle F \rangle$ shall denote the clone generated by F. To simplify notation we shall write $\langle M, f_1, \ldots, f_n \rangle$ instead of $\langle M \cup \{f_1, \ldots, f_n\} \rangle$.

Definition. Let θ be an r-ary relation on $\underline{k}$, $r \geq 1$. Let i and j be distinct, $1 \leq i, j \leq r$. Then let θ_{ij} denote the set of all pairs (a_i, a_j) such that there exists $(b_1, \ldots, b_r) \in \theta$ with $b_i = a_i$ and $b_j = a_j$. The relation θ is *irredundant* if θ_{ij} is not the equality relation for any $i \neq j$.

Lemma 2.1 *Let θ be an irredundant r-ary relation on $\underline{k}$, $r \geq 2$. If $M \subseteq Pol\,\theta$ then θ_{ij} is one of $\leq$, $\geq$ or $\underline{k}^2$.*

Proof. This is straightforward.

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Lemma 2.2 *(Extension Lemma) Let P be any finite poset and D a non-empty subset of P. Let $f : D \to \underline{k}$ be an isotone map. Then there exists a map $g : P \to \underline{k}$ such that (i) g is isotone, (ii) the restriction of g to D is f and (iii) g and f have the same image.*

Proof. For each $x \in P$ let $D_x = \{y \in D : y \leq x\}$. Let T denote the image of f and let a_0 denote the least element in T. Now define

$$g(x) = \begin{cases} \max\{f(y) : y \in D_x\} & \text{if } D_x \neq \emptyset, \\ a_0 & \text{otherwise.} \end{cases}$$

It is easy to see that g satisfies all the requirements.

■

Lemma 2.3 *[7] An n-ary operation f is in $Pol \vee^\circ$ if and only if*

$$f(x_1, \ldots, x_n) = f_1(x_1) \vee \cdots \vee f_n(x_n)$$

for some $f_i \in M$. (Mutatis mutandis for the clone $Pol \wedge^\circ$.)

■

Lemma 2.4 *The maximal subclones of $Pol \le$ containing M are $Pol \vee^\circ$, $Pol \wedge^\circ$ and $Pol\, \mu_k$.*

Proof. We refer the reader to [6] for terminology, notation, and auxiliary results used in this proof. The three clones in question are maximal subclones by Theorem 3.4 of [6]. Now we prove that there are no others. If θ is a binary relation and $M \subseteq Pol\, \theta$ then by Lemma 2.1 $Pol\, \theta$ is equal to $Pol \le$ or the clone of all operations on $\underline{k}$. Then by Lemma 3.1 of [6], if C is a maximal subclone of $Pol \le$ containing M then it is of type (C,h), (R,h) or (MI,h) for $h \ge 3$. Suppose that C is equal neither to $Pol \vee^\circ$ nor to $Pol \wedge^\circ$. By Lemmas 3.2 and 3.3 of [6] we may assume that $C = Pol\, \theta$ where θ is a chain-like, essential relation of arity $h \ge 3$. By Lemma 2.5 of [6], θ must contain μ_h. On the other hand, if θ contains some h-tuple not in μ_h, say $(a_1, \ldots, a_h)$ such that $a_1 < a_2 < \ldots < a_h$, let $(b_1, \ldots, b_h)$ be any tuple such that $b_1 < \ldots < b_h$. Then it is easy to find, using the extension lemma above, an $f \in M$ that will map $(a_1, \ldots, a_h)$ to $(b_1, \ldots, b_h)$. Hence θ is full (i.e. $Pol\, \theta = Pol \le$), a contradiction. Thus $\theta = \mu_h$.

■

Lemma 2.5 *1. $P_{h-1} \subset P_h$ for all $2 \le h \le k-1$.*

2. $P_{h-1} \subseteq Pol\, \mu_h$ for every $3 \le h \le k$.

3. $P_h \not\subseteq Pol\, \mu_h$ for all $3 \le h \le k$.

4. $Pol\, \mu_h \cap P_h \not\subseteq Pol\, \mu_{h-1}$ for every $4 \le h \le k$.

5. $Pol\, \mu_h \subset Pol\, \mu_{h+1}$ for every $3 \le h \le k-1$.

6. *Pol* $\mu_4 \not\subseteq P_h$, *for every* $3 \leq h \leq k-1$.

7. *Pol* $\mu_3 = P_2$.

8. *Pol* $\mu_h \cap P_h \not\subseteq P_{h-1}$ *for every* $4 \leq h \leq k$.

Proof.
1) This is trivial.
2) This inclusion is easy.
3) This is simple, define a binary operation f as follows:

$$f(x,y) = \begin{cases} x+1 & \text{if } y = k \text{ and } 1 \leq x \leq h-1, \\ h & \text{if } y = k \text{ and } x \geq h, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $f \in P_h$ and easy to see that $f \notin Pol\,\mu_h$.

4) This follows from 1), 2) and 3).

5) Note that $Pol\,\mu_h \not\subseteq Pol\,\mu_{h-1}$ follows from 4). We prove the inclusion as follows: consider the $(h+1)$-ary relation θ consisting of all tuples $(a_1,\ldots,a_{h+1})$ such that $a_1 \leq a_2 \leq \ldots \leq a_{h+1}$ and such that there exists $x \in \underline{k}$ with $(x,a_3,\ldots,a_{h+1}) \in \mu_h$ and $(a_1,a_2,x,a_4,\ldots,a_{h-1}) \in \mu_h$. Since this relation is constructed using only μ_h and $\leq$ we have that $Pol\,\mu_h \subseteq Pol\,\theta$. It remains to show that $\theta = \mu_{h+1}$. Let $(a_1,\ldots,a_{h+1}) \in \theta$ and suppose that the a_i are pairwise distinct. Then for some $x \in \underline{k}$ we have $(x,a_3,\ldots,a_{h+1}) \in \mu_h$ and $(a_1,a_2,x,a_4,\ldots,a_{h-1}) \in \mu_h$. From the first we have that $x = a_3$ and from the second we have that $x = a_2$ or $x = a_4$, a contradiction. Hence θ is contained in μ_{h+1}. The other inclusion is easy.

6) It suffices by 1) to show that $Pol\,\mu_4 \not\subseteq P_{k-1}$. Define a binary operation f on $\underline{k}$ as follows: let S be the set of pairs (x,y) such that $x+y = k+1$ and $2 \leq x \leq k-1$. Let

$$f(x,y) = \begin{cases} x & \text{if } (x,y) \in S, \\ 1 & \text{if } (x,y) < (a,b) \text{ for some } (a,b) \in S, \\ k & \text{otherwise.} \end{cases}$$

It is easy to see that f is isotone and that $f \notin P_{k-1}$. However f is in $Pol\,\mu_4$: indeed, suppose it is not; then there exist tuples $(a_1,\ldots,a_4)$ and $(b_1,\ldots,b_4)$ in μ_4 such that f maps $((a_1,b_1),\ldots,(a_4,b_4))$ to some tuple $(c_1,\ldots,c_4)$ not in μ_h. Since f is isotone, this means that $c_1 < c_2 < c_3 < c_4$.

This means that $1 < c_2 < k$ and $1 < c_3 < k$ and so (a_2, b_2) and (a_3, b_3) are in S; but S is an antichain in $\underline{k}^2$ so this is impossible by definition of μ_4.

7) By 2) it suffices to prove that $Pol\,\mu_3 \subseteq P_2$. By a well-known result of Burle [1] it will suffice to show that $Pol\,\mu_3 \subseteq Pol\,\theta$ where θ is the 3-ary relation consisting of all (a, b, c) with $|\{a, b, c\}| \leq 2$. Construct the following 3-ary relation: let α be the set of all (x_{13}, x_{22}, x_{31}) such that there exist x_{ij}, $1 \leq i, j \leq 3$ satisfying the following:

$$x_{ij} \leq x_{kl} \text{ if } i \leq j \text{ and } k \leq l \tag{1}$$

$$(x_{11}, x_{13}, x_{33}) \in \mu_3 \tag{2}$$

$$(x_{11}, x_{31}, x_{33}) \in \mu_3 \tag{3}$$

$$(x_{12}, x_{22}, x_{32}) \in \mu_3 \tag{4}$$

$$(x_{21}, x_{22}, x_{23}) \in \mu_3 \tag{5}$$

Clearly $Pol\,\mu_3 \subseteq Pol\,\alpha$. We show that $\alpha \subseteq \theta$, the other inclusion is easy. Suppose that there exists $(a, b, c) \in \alpha$ with a, b and c distinct. Suppose first that a or c is neither the largest nor the smallest of a, b and c. Without loss of generality, we may assume that $a > \min\{a, b, c\}$ and $a < \max\{a, b, c\}$. Then by condition (1) we have that $x_{11} < a < x_{33}$ and thus condition (2) is not satisfied. Hence we may assume without loss of generality that $a < b < c$. But then

$$x_{12} \leq a < b < c \leq x_{32}$$

by condition (1) so condition (4) fails.

8) If $h = 4$ consider the binary operation

$$f(x, y) = \begin{cases} 2 & \text{if } (x, y) = (k, 1), \\ 3 & \text{if } (x, y) = (k-1, 2), \\ 1 & \text{if } (x, y) < (k, 1) \text{ or } (x, y) < (k-1, 2), \\ 4 & \text{otherwise.} \end{cases}$$

It is easy to see that f is in $Pol\,\mu_4 \cap P_4$ but not in P_3.

Now assume that $h \geq 5$. Define a binary operation as follows: let S be the set of all pairs (x, y) such that $x = k - 1$ and $2 \leq y \leq h - 3$, and let T

be the set of pairs (x, y) such that $x = k$ and $2 \leq y \leq h-3$. Let

$$g(x,y) = \begin{cases} 2 & \text{if } (x,y) = (k,1), \\ 3 & \text{if } (x,y) \in S, \\ y+2 & \text{if } (x,y) \in T, \\ h & \text{if } y \geq h-2, \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that $g \in P_h$ and $g \notin P_{h-1}$. tuples $\overline{x} = (1, 2, 4, 5, ..., h)$ and The argument that shows that $g \in Pol\, \mu_h$ is very similar to the one used in 6).

■

Lemma 2.6 *The clone M is the intersection of the clones $Pol \vee^\circ$ and $Pol \wedge^\circ$. In fact, $M = Pol\, \rho$ where ρ consists of all 4-tuples of the form (a, a, b, b) with $a \leq b$ or of the form (a, b, a, b) with $a \leq b$.*

Proof. Notice that an n-ary operation f is in $Pol \vee^\circ \cap Pol \wedge^\circ$ if and only if it is a lattice homomorphism $f : \underline{k}^n \to \underline{k}$. In particular, the kernel θ of f is a congruence of $\underline{k}^n$. But then θ must be of the form $\theta = \theta_1 \times \theta_2 \times \cdots \times \theta_n$ where each θ_i is a congruence of the lattice $\underline{k}$ (see for example [8], Theorem 2.70).

Suppose that f is not constant, i.e. that some θ_i is not equal to $\underline{k}^2$. Without loss of generality, we may assume that there are $a_1 < b_1$ such that a_1 and b_1 are not congruent modulo θ_1. Now suppose that there are $a_2 < b_2$ with a_2 and b_2 not congruent modulo θ_2. Then

$$(a_1, b_2, 0, \ldots, 0) \vee (b_1, a_2, 0, \ldots, 0) = (b_1, b_2, 0, \ldots, 0).$$

But $\underline{k}^n/\theta$ is isomorphic to a chain, hence the join operation is the 'maximum', so $(b_1, b_2, 0, \ldots, 0)/\theta = (a_1, b_2, 0, \ldots, 0)/\theta$ or $(b_1, b_2, 0, \ldots, 0)/\theta = (b_1, a_2, 0, \ldots, 0)/\theta$. But by choice of the a_i, b_i this is not the case. Hence $\theta_2 = \underline{k}^2$ and by the same argument the same holds for all θ_i with $i \geq 2$. This means that f depends only on its first variable, so $f \in M$ and we are done.

For the second statement: We have that $Pol\, \rho \subseteq Pol\, \rho_{234}$ and $Pol\, \rho \subseteq Pol\, \rho_{231}$ where

$$\rho_{234} = \{(u, v, w) : (x, u, v, w) \in \rho \text{ for some } x\}$$

and

$$\rho_{231} = \{(u,v,w) : (w,u,v,x) \in \rho \text{ for some } x\}$$

But $(x,u,v,w) \in \rho$ iff either $u = w \geq v$ or $v = w \geq u$ iff $w = u \vee v$. In other words, $\rho_{234} = \vee^\circ$. In the same manner one sees that $\rho_{231} = \wedge^\circ$.

Hence by the result above we have that $Pol\,\rho \subseteq M$. On the other hand it is clear that $M \subseteq Pol\,\rho$ so we are done.

■

3 The interval $[M, Pol \leq]$, $k \geq 3$

The next few lemmas will be used to prove the following result:

Theorem 3.1 *Let $f \in Pol \vee^\circ$ be essentially at least binary, and suppose the image of f has h elements, $2 \leq h \leq k$. Then $\langle M, f\rangle = Pol \vee^\circ \cap P_h$. (Mutatis mutandis for $Pol \wedge^\circ$).*

Lemma 3.2 *Let f be an n-ary operation in $Pol \vee^\circ$, say $f(x_1, \ldots, x_n) = f_1(x_1) \vee \cdots \vee f_n(x_n)$ with $f_i \in M$ for all $1 \leq i \leq n$. Then f depends on x_i if and only if there exist $u < v$ in the image of f_i and t_j in the image of f_j for all $j \neq i$ such that $t_j < v$ for all j.*

Proof. Suppose that f depends on x_i, i.e. there exist $x_i < x_i'$ and x_j ($j \neq i$) such that $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) < f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$. Let $u = f_i(x_i)$ and $v = f_i(x_i')$ and $t_j = f_j(x_j)$ for all $j \neq i$. Then

$$t_1 \vee t_2 \vee \cdots t_{i-1} \vee u \vee t_{i+1} \vee \cdots \vee t_n < t_1 \vee t_2 \vee \cdots t_{i-1} \vee v \vee t_{i+1} \vee \cdots \vee t_n$$

implies that $u < v$ and that no t_j is greater or equal to v.

Conversely, suppose that there exist u, v, t_j as in the statement of the lemma. Let $f_i(x_i) = u$ and $f_i(x_i') = v$ and $f(x_j) = t_j$ for all $j \neq i$. Then

$$f(x_1, \ldots, x_n) = t_1 \vee t_2 \vee \cdots t_{i-1} \vee u \vee t_{i+1} \vee \cdots \vee t_n <$$

$$< v = t_1 \vee t_2 \vee \cdots t_{i-1} \vee v \vee t_{i+1} \vee \cdots \vee t_n =$$

$$= f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n).$$

■

Lemma 3.3 *Let ϕ be an n-ary operation in $Pol\vee^\circ$. Then there exist $g_i \in M$ such that ϕ and $r(t) = \phi(g_1(t), \dots, g_n(t))$ have the same image.*

Proof. Let $T = \{a_1 < a_2 < \dots < a_h\}$ be the image of ϕ. We find elements $b_1, \dots, b_h$ of $\underline{k}^n$ such that (1) $\phi(b_i) = a_i$ for all $1 \le i \le h$ and (2) $b_i \le b_{i+1}$ for all $1 \le i \le h-1$. Indeed, choose $c_1, \dots, c_h$ such that $\phi(c_i) = a_i$ for all $1 \le i \le h$. Let $b_i = c_1 \vee c_2 \vee \dots \vee c_i$ for all $1 \le i \le h$. Certainly the b_i's satisfy the second condition, and to see that they satisfy the first, just notice that

$$\phi(b_i) = \phi(c_1 \vee \dots \vee c_i) = \phi(c_1) \vee \dots \vee \phi(c_i) = a_1 \vee \dots \vee a_i = a_i.$$

Now it suffices to define the maps g_i ($i \in \underline{n}$) as follows: consider the set of first coordinates of the tuples $b_1, \dots, b_h$, say $B_1 = \{b_1(1) \le b_2(1) \le \dots \le b_h(1)\}$. We may find an isotone map g_1 from $\underline{k}$ onto B_1 such that $g_1(i) = b_i(1)$ for all $1 \le i \le h$ (easy). Do the same for each coordinate. Then of course $b_i = (g_1(i), g_2(i), \dots, g_n(i))$ for all i, so $\phi(g_1(i), \dots, g_n(i)) = \phi(b_i) = a_i$ for all i and we are done.

■

Lemma 3.4 *Let $\phi \in Pol\vee^\circ$ be an essentially at least binary operation. Then there exists $\psi \in \langle M, \phi\rangle$ such that (1) ψ is essentially binary and (2) ψ and ϕ have the same image.*

Proof. Let ϕ be an n-ary operation in $Pol\vee^\circ$, and suppose without loss of generality that $n \ge 3$ and that ϕ depends on its first two variables. Let T be the image of ϕ. By Lemma 2.3 we have that $\phi(x_1, \dots, x_n) = f_1(x_1) \vee \cdots f_n(x_n)$ for some $f_i \in M$. Consider the operation $F(x_2, \dots, x_n) = f_2(x_2) \vee \cdots f_n(x_n)$, and let B denote its image. By Lemma 3.3, we may find $g_2, \dots, g_n \in M$ such that the map $h(t) = F(g_2(t), \dots, g_n(t))$ has image equal to B.

We claim that the map $\psi(x, y) = f_1(x) \vee h(y)$ is the one we're looking for. Indeed, ψ is in the clone $\langle M, \phi\rangle$ since $\psi(x, y) = \phi(x, g_2(y), \dots, g_n(y))$.

The image of ϕ is clearly the set of all $z_1 \vee \dots \vee z_n$ such that z_i is in the image of f_i. Similarly, the image of F is the set of all $z_2 \vee \dots \vee z_n$ such that z_i is in the image of f_i for $2 \le i \le n$. Hence the image of ϕ is equal to the set of all $z_1 \vee b$ such that z_1 is in the image of f_1 and $b \in B$, which is also the image of ψ. Hence ϕ and ψ have the same image.

We now show that ψ depends on both variables.

We prove that ψ depends on x: ϕ depends on x_1, so by Lemma 3.2, there exist $u < v$ in the image of f_1 and t_j in the image of f_j $(2 \leq j \leq n)$ such that $t_j < v$ for all $j \geq 2$. Let $b = t_2 \vee \cdots \vee t_n$. Then $b \in B$, and $b < v$ so by Lemma 3.2 ψ depends on its first variable.

We prove that ψ depends on y: ϕ depends on x_2, so by Lemma 3.2, there exist $z < z'$ in the image of f_2 and t_j in the image of f_j $(j \neq 2)$ such that $t_j < z'$ for all $j \neq 2$. Let $u = z \vee t_3 \vee \cdots \vee t_n$ and $v = z' \vee t_3 \vee \cdots \vee t_n$. Notice that both u and v are in B. Now $v = z'$ and $u < z'$ so $u < v$, and t_1 is an element of the image of f_1 which is less than v, By Lemma 3.2, ψ depends on its second variable.

■

Definition. Let $2 \leq h \leq k$ and let $T = \{a_1 < a_2 < \ldots < a_h\}$ be a subset of $\underline{k}$. Define an element α_T of M by

$$\alpha_T(x) = \begin{cases} a_1 & \text{if } x \leq a_1, \\ a_i & \text{if } a_{i-1} < x \leq a_i \text{ for some } 1 < i \leq h-1, \\ a_h & \text{otherwise.} \end{cases}$$

Notice that α_T is a retraction onto T, i.e. $\alpha_T^2 = \alpha_T$.

For each $n \geq 2$ we define n-ary operations $J_T^{(n)} = J_T$ and $M_T^{(n)} = M_T$ as follows: $J_T(x_1, \ldots, x_n) = \alpha_T(x_1 \vee x_2 \vee \cdots \vee x_n)$ and $M_T(x_1, \ldots, x_n) = \alpha_T(x_1 \wedge x_2 \wedge \cdots \wedge x_n)$ for all $x_i \in \underline{k}$. Notice that we have $J_T(x_1, \ldots, x_n) = \alpha_T(x_1) \vee \cdots \vee \alpha_T(x_n)$ and similarly for M_T. (J and M stand for 'join' and 'meet'). Notice also that we have nice 'identities' such as $J_T(x, J_T(y, z)) = J_T(x, y, z)$, etc. (hence the convenient abuse of notation).

Lemma 3.5 *Let $\phi \in Pol \vee^\circ$ be an essentially binary operation, say $\phi(x, y) = f(x) \vee g(y)$ where $f, g \in M$. Let T denote the image of g. Then the operation $\Gamma(x, y) = f(x) \vee \alpha_T(y)$ is in $\langle M, \phi \rangle$, it has the same image as ϕ and depends on both variables.*

Proof. Let $T = \{c_1 < \ldots < c_r\}$. Choose $b_i \in \underline{k}$ such that $g(b_i) = c_i$ for all $1 \leq i \leq r$. Of course we have that $b_1 < \ldots < b_r$. Define

$$h(t) = \begin{cases} b_1 & \text{if } t \leq c_1, \\ b_i & \text{if } c_{i-1} < t \leq c_i \text{ for some } 1 < i \leq r-1, \\ b_r & \text{otherwise.} \end{cases}$$

Then $gh(y) = \alpha_T(y)$, hence $\phi(x, h(y)) = f(x) \vee \alpha_T(y)$. In particular, this operation is in $\langle M, \phi \rangle$. Since the image of ϕ consists of all $u \vee v$ with u in the image of f and v in the image of g, it is clear that Γ has the same image. By Lemma 3.2 it is clear that Γ depends on both variables since ϕ does.

■

Lemma 3.6 *Let $\phi \in Pol \vee^\circ$ be an essentially at least binary operation and let T denote its image. Let a_1 denote the least element of T. Then there is an operation $F(x, y) = f(x) \vee g(y)$ in the clone $\langle M, \phi \rangle$ such that (1) the image of F is T, (2) the images of f and g are contained in T, (3) a_1 is in the image of f and g and (4) F is essentially binary.*

Proof. By Lemma 3.4 we may assume without loss of generality that ϕ is essentially binary, say $\phi(x, y) = p(x) \vee q(y)$ for some $p, q \in M$. Since the map α_T is a retraction onto T we have that

$$\begin{aligned} \phi(x, y) &= \alpha_T(\phi(x, y)) \\ &= \alpha_T p(x) \vee \alpha_T q(y) \end{aligned}$$

Let $f(x) = \alpha_T p(x)$ and $g(y) = \alpha_T q(y)$. Then $F = \phi$ satisfies the conclusion of the lemma. Indeed, it is clear that the image of f and of g is contained in T. This implies that $f(x) \vee g(y) \geq a_1$ for all x and y, and since a_1 is in the image of ϕ, a_1 must be in the image of f and of g. Since ϕ satisfies (1) and (4) we are done.

■

Lemma 3.7 *Let $\phi \in Pol \vee^\circ$ be an essentially at least binary operation and let T denote its image. Let a_1 denote the least element of T. Then there exists a subset D of T with $|D| \geq 2$ and containing a_1 such that the operation $G(x, y) = \alpha_T(x) \vee \alpha_D(y)$ is in $\langle M, \phi \rangle$. Furthermore, G depends on both variables and has image equal to T.*

Proof. By Lemma 3.6, there exists an operation $F \in \langle M, \phi \rangle$ such that $F(x,y) = f(x) \vee g(y)$ and such that T contains the image of f and g, a_1 is contained in the image of f and g, F is essentially binary and has image equal to T. Let U and V denote the image of f and g respectively. By Lemma 3.5, we have that the operation $F'(x,y) = f(x) \vee \alpha_V(y)$ is in $\langle M, \phi \rangle$, is essentially binary and has image equal to T. Applying Lemma 3.5 again, we get that the operation $F''(x,y) = \alpha_U(x) \vee \alpha_V(y)$ is in $\langle M, \phi \rangle$, is essentially binary and has image equal to T.

For convenience, let us put $f = \alpha_U$ and $g = \alpha_V$ and $\phi = F''$.

We may assume without loss of generality that a_h is in the image of g. Consider the operation

$$G(x,y,z) = \phi(\phi(x,y),z).$$

Clearly G is in the clone $\langle M, \phi \rangle$. Now we have

$$\begin{aligned} G(x,y,z) &= f(f(x) \vee g(y)) \vee g(z) \\ &= f(x) \vee fg(y) \vee g(z) \\ &= f(x) \vee g(z) \vee fg(y) \\ &= \phi(x,z) \vee fg(y) \end{aligned}$$

By Lemma 3.3 we may find operations h_1 and h_2 in M such that $f'(t) = \phi(h_1(t), h_2(t))$ has the same image as ϕ, namely T. So we can construct the operation

$$H(x,y) = G(h_1(x), y, h_2, (x)) = \phi(h_1(x), h_2(x)) \vee fg(y) = f'(x) \vee fg(y)$$

where the image of f' is T. Notice that H depends on both variables: indeed, we have that $fg(1) = f(a_1) = f(f(1)) = f(1) = a_1$. Thus by Lemma 3.2 H depends on x. To show that H depends on y it suffices to find some element in the image of fg which is greater than a_1. If this is not the case, then we have that fg is constant so $a_1 = fg(1) = fg(k) = f(a_h)$. Hence $f(a) = a_1$ for all $a \in T$. However, the image of f is contained in T and since the map ϕ depends on x the image of f must contain at least two elements; since f is a retraction onto its image, this is a contradiction. Furthermore, the new operation H also has image T. Indeed, we saw above that $fg(1) = a_1$; if $a \in T$ and x is such that $f'(x) = a$ then $H(x,1) = f'(x) \vee fg(1) = a \vee a_1 = a$. Now we may apply Lemma 3.5 to construct the operation $\psi(x, y) = \alpha_T(x) \vee fg(y)$. By Lemma 3.5, ψ has image equal to T and depends on both variables.

Let D denote the image of fg. We've seen above that the image of fg contains at least two elements, that it is contained in T and contains a_1. Now apply Lemma 3.5 to the operation ψ to obtain that the operation $G(x, y) = \alpha_T(x) \vee \alpha_D(y)$ is in $\langle M, \phi\rangle$, that it depends on both variables and has image T.

■

Lemma 3.8 *Let $\phi \in Pol \vee^\circ$ be an essentially at least binary operation and let T denote its image. Then the operation J_T is in the clone $\langle M, \phi\rangle$.*

Proof.

Let $T = \{a_1 < a_2 < \ldots a_h\}$. By Lemma 3.7 there exists a subset D of T such that $G(x, y) = \alpha_T(x) \vee \alpha_D(y)$ is in $\langle M, \phi\rangle$, is essentially binary and has image equal to T. Furthermore, D contains at least 2 elements, contains a_1 and is contained in T. If D is equal to T, then $G = J_T$ and we are done. Thus we will assume that D is properly contained in T. We shall build an operation $\alpha_T(x) \vee \alpha_{D'}(y)$ where D' is a subset of T that contains D properly.

Let $b_2 < b_3 < \ldots < b_s$ be the elements of T not in D; then of course $2 \leq s < h$. Also note that $a_1 < b_2$ since a_1 is in D.

Define $\sigma \in M$ as follows:

$$\sigma(t) = \begin{cases} a_1 & \text{if } t \leq a_1, \\ b_i & \text{if } a_{i-1} < t \leq a_i, \text{ for } 2 \leq i < s \\ b_s & \text{otherwise.} \end{cases}$$

Let $\psi(x, y) = \sigma G(x, y)$ and define

$$\Delta(x, y, z, w) = G(\psi(x, y), G(z, w)).$$

Clearly Δ is in the clone $\langle M, \phi\rangle$. We have

$$\begin{aligned} \Delta(x, y, z, w) &= \alpha_T(\sigma\alpha_T(x) \vee \sigma\alpha_D(y)) \vee \alpha_D(\alpha_T(z) \vee \alpha_D(w)) \\ &= [\alpha_T\sigma\alpha_T(x) \vee \alpha_D\alpha_T(z)] \bigvee [\alpha_T\sigma\alpha_D(y) \vee \alpha_D(w)] \end{aligned}$$

Let

$$\delta(x, z) = \alpha_T\sigma\alpha_T(x) \vee \alpha_D\alpha_T(z)$$

and

$$\epsilon(y,w) = \alpha_T\sigma\alpha_D(y) \vee \alpha_D(w).$$

We claim that (1) δ has image equal to T and that (2) the image of ϵ contains D properly. It is immediate that the images of δ and ϵ are contained in T.

(1) Let $a \in T$. If $a \in D$ then

$$\begin{aligned}\delta(1,a) &= \alpha_T\sigma\alpha_T(1) \vee \alpha_D\alpha_T(a) \\ &= \alpha_T\sigma(a_1) \vee \alpha_D(a) \\ &= \alpha_T(a_1) \vee a \\ &= a_1 \vee a = a.\end{aligned}$$

If $a \notin D$ then $a = b_i$ for some $2 \leq i \leq s$. Then

$$\begin{aligned}\delta(a_i,1) &= \alpha_T\sigma\alpha_T(a_i) \vee \alpha_D\alpha_T(1) \\ &= \alpha_T\sigma(a_i) \vee \alpha_D(a_1) \\ &= \alpha_T(b_i) \vee a_1 \\ &= b_i \vee a_1 = b_i.\end{aligned}$$

(2) First we show that D is contained in the image of ϵ. Let $d \in D$. Then

$$\begin{aligned}\epsilon(1,d) &= \alpha_T\sigma\alpha_D(1) \vee \alpha_D(d) \\ &= \alpha_T\sigma(a_1) \vee d \\ &= \alpha_T(a_1) \vee d \\ &= a_1 \vee d = d.\end{aligned}$$

Now we show that the image of ϵ must contain b_i for some $2 \leq i \leq s$. Suppose first that $a_i \in D$ for some $2 \leq i \leq s$. Then

$$\begin{aligned}\epsilon(a_i,1) &= \alpha_T\sigma\alpha_D(a_i) \vee \alpha_D(1) \\ &= \alpha_T\sigma(a_i) \vee a_1 \\ &= \alpha_T(b_i) \vee a_1 \\ &= b_i \vee a_1 = b_i.\end{aligned}$$

Otherwise D must contain a_i for some $i > s$. Then

$$\epsilon(a_i,1) = \alpha_T\sigma\alpha_D(a_i) \vee \alpha_D(1)$$

$$\begin{aligned} &= \alpha_T\sigma(a_i) \vee a_1 \\ &= \alpha_T(b_s) \vee a_1 \\ &= b_s \vee a_1 = b_s. \end{aligned}$$

Let D' denote the image of ϵ. By Lemma 3.3 there exist operations $f_i, g_i \in M$, $1 \leq i \leq 2$, such that $P(x) = \delta(f_1(x), f_2(x))$ has image T and $Q(y) = \epsilon(g_1(y), g_2(y))$ has image D'. Then the operation $R(x,y) = P(x) \vee Q(y)$ is in the clone $\langle M, \phi\rangle$ since

$$P(x) \vee Q(y) = \delta(f_1(x), f_2(x)) \vee \epsilon(g_1(y), g_2(y)) = \Delta(f_1(x), g_1(y), f_2(x), g_2(y)).$$

Since D' contains a_1 and at least two elements, it is clear that R has image equal to T and depends on both variables. We may apply Lemma 3.5 twice to R to obtain that the clone $\langle M, \phi\rangle$ contains the operation $G'(x,y) = \alpha_T(x) \vee \alpha_{D'}(y)$. Repeating the above argument to this operation will eventually yield the operation J_T.

■

Lemma 3.9 *Let T be any h-element subset of $\underline{k}$ with $2 \leq h \leq k$. Then $\langle M, J_T\rangle = Pol \vee^\circ \cap P_h$.*

Proof. It obviously suffices to prove that $Pol \vee^\circ \cap P_h \subseteq \langle M, J_T\rangle$. Let f be an n-ary operation in $Pol \vee^\circ \cap P_h$. We may assume that f is essentially at least binary. Let $B = \{b_1 < b_2 < \ldots < b_t\}$ be the image of f, where $2 \leq t \leq h$. Let $T = \{a_1 < a_2 < \ldots a_h\}$. Define a map σ as follows:

$$\sigma(t) = \begin{cases} b_1 & \text{if } t \leq a_1, \\ b_i & \text{if } a_{i-1} < t \leq a_i, \text{ for } 2 \leq i < t \\ b_t & \text{ otherwise.} \end{cases}$$

Consider the operation $F(x,y) = \sigma(J_T(x,y))$. Clearly it is in $\langle M, J_T\rangle$. It has image equal to B since the image of J_T is T and σ maps T onto B. Furthermore, F is essentially binary. Indeed, we have that

$$\begin{aligned} F(1,1) &= \sigma(J_T(1,1)) \\ &= \sigma(\alpha_T(1 \vee 1)) \\ &= \sigma(a_1) = b_1, \end{aligned}$$

and

$$\begin{aligned} F(1,k) &= \sigma(J_T(1,k)) \\ &= \sigma(\alpha_T(1 \vee k)) \\ &= \sigma(a_h) = b_t. \end{aligned}$$

and

$$\begin{aligned} F(k,1) &= \sigma(J_T(k,1)) \\ &= \sigma(\alpha_T(k \vee 1)) \\ &= \sigma(a_h) = b_t. \end{aligned}$$

By Lemma 3.8 we obtain that $J_B \in \langle M, J_T \rangle$.

By Lemma 2.3 we may write $f(x_1, \ldots, x_n) = f_1(x_1) \vee \ldots \vee f_n(x_n)$ for some $f_i \in M$. Since α_B is a retraction onto the image B of f we obtain that

$$\begin{aligned} f(x_1, \ldots, x_n) &= \alpha_B(f(x_1, \ldots, x_n)) \\ &= \alpha_B(f_1(x_1) \vee \ldots \vee f_n(x_n)) \\ &= J_B(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)). \end{aligned}$$

Hence $f \in \langle M, J_B \rangle \subseteq \langle M, J_T \rangle$ and this completes the proof.

■

We may now prove the result mentioned at the beginning of this section:

Proof of Theorem 3.1: Let f be an essentially binary operation in $Pol \vee^\circ$ whose image has h elements, $2 \leq h \leq k$. By Lemma 3.8 the clone $\langle M, f \rangle$ contains the operation J_T where T is the image of f. Hence by Lemma 3.9 we have that $Pol \vee^\circ \cap P_h \subseteq \langle M, f \rangle$. The other inclusion is trivial.

■

Corollary 3.10 *The only clones C such that $M \subset C \subseteq Pol \vee^\circ$ are those of the form $Pol \vee^\circ \cap P_h$ with $2 \leq h \leq k$. (Mutatis mutandis for $Pol \wedge^\circ$).*

Proof: Let C be a clone that contains M properly and contained in $Pol\vee^\circ$. Then C contains an operation f which is essentially at least binary and has largest image T, say $|T| = h$ where $2 \leq h \leq k$. Clearly $C \subseteq Pol\ \vee^\circ \cap P_h$. By Theorem 3.1 C must contain $Pol\ \vee^\circ \cap P_h$ and this completes the proof.

■

Corollary 3.11 *Let $f \in Pol\vee^\circ$ be essentially at least binary, and suppose the image of f has h elements, $2 \leq h \leq k$. Let $g \in Pol\wedge^\circ$ be essentially at least binary, and suppose the image of g has h elements. Then $\langle M, f, g\rangle = P_h$.*

Proof: It follows from Lemmas 2.3 and 2.4 that $Pol \leq = \langle M, \vee, \wedge\rangle$. In fact, we claim that the n-ary operations in $Pol \leq$ are those operations of the form

$$f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) \wedge f_2(x_1, \ldots, x_n) \wedge \cdots \wedge f_s(x_1, \ldots, x_n)$$

where the f_i are n-ary operations in $Pol\vee^\circ$. Indeed, it is clear that operations of this form are in $Pol \leq$. It thus suffices to prove that this set of operations is closed under the operations in M (easy) and under the operations $\wedge$ (obvious) and $\vee$: indeed, just use the distributive law for this last case.

Let F be an n-ary operation in P_h and denote its image by T. Let $C = \langle M, f, g\rangle$ where the operations f and g are as in the statement of the corollary. By Theorem 3.1 (and its dual) C contains $Pol\ \vee^\circ \cap P_h$ and $Pol\ \wedge^\circ \cap P_h$. In particular, C contains M_T.

Write

$$F(\overline{x}) = f_1(\overline{x}) \wedge f_2(\overline{x}) \wedge \cdots \wedge f_s(\overline{x})$$

where $\overline{x} = (x_1, \ldots, x_n)$, and $f_i \in Pol\vee^\circ$. Since the image of F is T, we have that

$$\begin{aligned} F(\overline{x}) &= \alpha_T(F(\overline{x})) \\ &= \alpha_T^2(F(\overline{x})) \\ &= \alpha_T^2(f_1(\overline{x}) \wedge f_2(\overline{x}) \wedge \cdots \wedge f_s(\overline{x})) \\ &= \alpha_T(\alpha_T(f_1(\overline{x})) \wedge \alpha_T(f_2(\overline{x})) \wedge \cdots \wedge \alpha_T(f_s(\overline{x}))) \\ &= M_T(\alpha_T(f_1(\overline{x})), \alpha_T(f_2(\overline{x})), \ldots, \alpha_T(f_s(\overline{x}))) \end{aligned}$$

where each $\alpha_T(f_i(\overline{x}))$ is in $Pol\ \vee^\circ \cap P_h$. Hence F is in the clone C and we are done.

■

The following result improves on Corollary 3.11. It states that, if a clone C above M contains non-trivial (i.e. non-unary) operations in both $Pol \vee^\circ$ and $Pol \wedge^\circ$, then it contains P_h where h is the maximum value for which *either* $Pol \vee^\circ \cap P_h \subseteq C$ or $Pol \wedge^\circ \cap P_h \subseteq C$.

Theorem 3.12 *Let $f \in Pol \vee^\circ$ be essentially at least binary and assume its image contains h elements, $2 \leq h \leq k$. Let $g \in Pol \wedge^\circ$ be essentially at least binary and assume its image contains r elements, $2 \leq r \leq k$. Then the clone $\langle M, f, g\rangle$ contains P_t where $t = \max\{h, r\}$.*

Proof. We shall prove the result for $r \leq h$ (the other case follows easily by dualising the argument). By Corollary 3.11 we may assume without loss of generality that $r < h$. Let $C = \langle M, f, g\rangle$. Let $U = \{1, 2, \ldots, h\}$ and let $V = \{1, 2, \ldots, r\}$. By Theorem 3.1 the clone C contains the operations J_U and M_V. By Corollary 3.11 the clone C contains P_2, and hence contains the operation

$$f(x, y) = \begin{cases} 1 & \text{if } x \leq r \text{ or } y \leq r, \\ r+1 & \text{otherwise.} \end{cases}$$

Then C contains the operation

$$\phi(x, y) = J_U(M_V(x, y), f(x, y)).$$

We claim that $\phi = M_D$ where $D = \{1, 2, \ldots, r+1\}$. Indeed, we have by definition that

$$M_V(x, y) = \begin{cases} x \wedge y & \text{if } x \leq r \text{ or } y \leq r, \\ r & \text{otherwise.} \end{cases}$$

On the other hand, it easy to see that

$$M_D(x, y) = \begin{cases} x \wedge y & \text{if } x \leq r \text{ or } y \leq r, \\ r+1 & \text{otherwise.} \end{cases}$$

Suppose that $x \leq r$ or $y \leq r$. Then $M_V(x, y) = x \wedge y$ and $f(x, y) = 1$. Thus $\phi(x, y) = J_U(x \wedge y, 1) = x \wedge y$. Otherwise we have that $M_V(x, y) = r$ and $f(x, y) = r + 1$ so $\phi(x, y) = J_U(r, r + 1) = r + 1$.

Thus the clone C contains M_D where D contains $r + 1$ elements. If $r + 1 < h$ then repeat the above construction until the operation M_U is shown to be in C. By Corollary 3.11 we conclude that C contains P_h.

■

Lemma 3.13 *Let f be an isotone operation not in $Pol \vee^\circ$. Then $Pol \wedge^\circ \cap P_2 \subseteq \langle M, f \rangle$. (Mutatis mutandis for the dual).*

Proof. Let f satisfy the hypothesis of the lemma. Then permuting variables if necessary, we may assume that there exist $a_i \leq b_i$ in $\underline{k}$, $1 \leq i \leq n$ such that

$$\begin{aligned} f(a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n) &= u \\ f(b_1, b_2, \ldots, b_k, a_{k+1}, \ldots, a_n) &= v \\ f(b_1, b_2, \ldots, b_k, b_{k+1}, \ldots, b_n) &= w \end{aligned}$$

where $u \vee v \neq w$. Since f is isotone we actually have that $u \vee v < w$. For $1 \leq i \leq n$ define

$$f_i(t) = \begin{cases} a_i & \text{if } t < k, \\ b_i & \text{otherwise.} \end{cases}$$

and define

$$h(t) = \begin{cases} 1 & \text{if } t \leq u \vee v, \\ k & \text{otherwise.} \end{cases}$$

Consider the operation defined by

$$\phi(x, y) = hf(f_1(x), \ldots, f_k(x), f_{k+1}(y), \ldots, f_n(y)).$$

Clearly ϕ is in $\langle M, f \rangle$. Let $x = y = k$. Then $\phi(x, y) = h(f(b_1, \ldots, b_n)) = h(w) = k$. If $x = k$ and $y < k$ then $\phi(x, y) = h(f(b_1, \ldots, b_k, a_{k+1}, \ldots, a_n)) = h(v) = 1$. If $x < k$ and $y = k$ then $\phi(x, y) = h(f(a_1, \ldots, a_k, b_{k+1}, \ldots, b_n)) = h(u) = 1$. Finally if $x < k$ and $y < k$ then $\phi(x, y) = h(f(a_1, \ldots, a_n)) \leq h(v) = 1$. Hence $\phi(x, y) = k$ if $x = y = k$ and $\phi(x, y) = 1$ otherwise. This is obviously an essentially binary operation in $Pol \wedge^\circ$, so by Theorem 3.1 we are done.

■

Theorem 3.14 *Let C be a clone containing M and contained in $Pol \leq$. Suppose that C is contained neither in $Pol \vee^\circ$ nor in $Pol \wedge^\circ$. Let $3 \leq h \leq k$. If C is not contained in $Pol\, \mu_h$ then C contains P_h.*

Proof. Let C be a clone containing M and contained in $Pol \leq$, and suppose that C is contained neither in $Pol \vee^\circ$ nor in $Pol \wedge^\circ$. To prove the theorem, it will suffice to prove the following equivalent statement:

for all $3 \leq h \leq k$, if C contains P_{h-1} and is not contained in $Pol\, \mu_h$ then P_h is contained in C. (*)

We first prove by induction on h that statement (*) implies our result. Assume that (*) holds for all $3 \leq h \leq k$. Let $h = 3$. By Lemma 3.13 (and its dual) C must contain $Pol\ \vee^\circ \cap P_2$ and $Pol\ \wedge^\circ \cap P_2$. Hence by Theorem 3.12 C contains P_2 and we conclude from (*) that C contains P_3. Now assume the result holds for $h-1$. If C is not contained in $Pol\, \mu_h$ then by Lemma 2.5 (1) C is not contained in $Pol\, \mu_{h-1}$. By induction hypothesis we then have that $P_{h-1} \subseteq C$. We then conclude from (*) that C contains P_h and we are done.

We now proceed to prove statement (*). Since C contains P_2, it will suffice by Theorem 3.12 to find an essentially at least binary operation $\phi \in C$ such that ϕ is in $Pol \wedge^\circ$ and whose image contains (at least) h elements. There exists an n-ary operation $f \in C$ that does not preserve μ_h, i.e. there are elements $a_{ij} \in \underline{k}$, $1 \leq i \leq n$, $1 \leq j \leq h$ such that $(a_{i1}, \ldots, a_{ih}) \in \mu_h$ for all i and such that $(u_1, \ldots, u_h) = (f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1h}, \ldots, a_{nh}))$ is not in μ_h. Notice that by definition of μ_h we have that $a_{ij} \leq a_{i(j+1)}$ for all i and j. But $f \in Pol \leq$ so it follows that $u_1 < u_2 < \ldots < u_h$. Since C contains M, we may assume that $u_i = i$ for all $1 \leq i \leq h$ (simply compose f with an operation $g \in M$ that maps u_i to i). For each $1 \leq i \leq n$ define an operation $g_i \in M$ as follows:

$$g_i(j) = \begin{cases} a_{ij} & \text{if } i < h, \\ a_{ih} & \text{if } i \geq h. \end{cases}$$

Let $T = \{1, 2, \ldots, h\}$ and for convenience let $\overline{x}$ stand for $(x_1, \ldots, x_n)$. We claim that the following operation is the one we seek:

$$\phi(\overline{x}) = f(g_1 M_T(\overline{x}), \ldots, g_n M_T(\overline{x}))$$

where M_T is the 'partial meet' operation defined earlier. We will prove that (1) ϕ is in C, (2) ϕ depends on all its variables, (3) the image of ϕ contains T and (4) ϕ is in $Pol \wedge^\circ$.

(1) By definition of μ_h the set $\{a_{i1}, \ldots, a_{ih}\}$ contains at most $h-1$ elements, hence the operation $g_i M_T$ is in P_{h-1} for all i. It follows that $\phi \in C$.

(2) For any $1 \leq i \leq n$ we have that

$$\begin{aligned} \phi(2, 2, \ldots, 2, 1, 2, \ldots, 2) &= f(g_1(1), \ldots, g_n(1)) \\ &= f(a_{11}, \ldots, a_{n1}) \\ &= 1 \end{aligned}$$

(where the lone 1 appears in the i-th place) and

$$\begin{aligned} \phi(2, \ldots, 2) &= f(g_1(2), \ldots, , g_n(2)) \\ &= f(a_{12}, \ldots, a_{n2}) \\ &= 2. \end{aligned}$$

(3) Let $1 \leq j \leq h$. Then

$$\begin{aligned} \phi(j, \ldots, j) &= f(g_1(j), \ldots, g_n(j)) \\ &= f(a_{1j}, \ldots, a_{nj}) \\ &= j. \end{aligned}$$

Hence the image of ϕ contains T.

(4) We start with a simple observation: for any $\overline{x} \in \underline{k}^n$, there exists $1 \leq j \leq n$ such that

$$(g_1 M_T(\overline{x}), \ldots, g_n M_T(\overline{x})) = (a_{1j}, \ldots, a_{nj}).$$

Notice also that the tuples $(a_{1j}, \ldots, a_{nj})$, $1 \leq j \leq n$ form a chain in $\underline{k}^n$ (this follows from the definition of μ_h).

Suppose for a contradiction that there exist $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$ such that $\phi(\overline{x}) \wedge \phi(\overline{y}) \neq \phi(\overline{x} \wedge \overline{y})$. Since ϕ is isotone it implies that $\phi(\overline{x} \wedge \overline{y})$ is distinct from $\phi(\overline{x})$ and $\phi(\overline{y})$. However, there exist j and r such that

$$(g_1 M_T(\overline{x}), \ldots, g_n M_T(\overline{x})) = (a_{1j}, \ldots, a_{nj})$$

and

$$(g_1 M_T(\overline{y}), \ldots, g_n M_T(\overline{y})) = (a_{1r}, \ldots, a_{nr}).$$

Since these n-tuples are comparable, assume without loss of generality that

$$(g_1 M_T(\overline{x}), \ldots, g_n M_T(\overline{x})) \leq (g_1 M_T(\overline{y}), \ldots, g_n M_T(\overline{y})).$$

Hence

$$\begin{aligned}\phi(\overline{x} \wedge \overline{y}) &= f(g_1 M_T(\overline{x} \wedge \overline{y}), \ldots, g_n M_T(\overline{x} \wedge \overline{y})) \\ &= f(g_1 M_T(\overline{x}) \wedge g_1 M_T(\overline{y}), \ldots, g_n M_T(\overline{x}) \wedge g_n M_T(\overline{y})) \\ &= f(g_1 M_T(\overline{x}), \ldots, g_n M_T(\overline{x})) \\ &= \phi(\overline{x}),\end{aligned}$$

and this is a contradiction. Hence ϕ preserves the meet and we are done.

■

Theorem 3.15 *Let C be a clone in the interval $[M, Pol \leq]$. Suppose that C in not one of M, $Pol \leq$, P_h, $Pol \vee^\circ \cap P_h$, $Pol \wedge^\circ \cap P_h$, $Pol\, \mu_h$, for any h. Then C is contained in an interval $[P_h, Pol\, \mu_{h+1}]$ for some $3 \leq h \leq k-1$.*

Proof. By Corollary 3.10 C can be contained neither in $Pol \vee^\circ$ nor in $Pol \wedge^\circ$. Hence C contains $Pol \vee^\circ \cap P_2$ and $Pol \wedge^\circ \cap P_2$, by Lemma 3.13. Then by Theorem 3.12 C contains P_2, which is equal to $Pol\, \mu_3$ by Lemma 2.5 (4). Since C is not equal to $Pol\, \mu_3$, it follows by Theorem 3.14 that C contains P_3. Now let h be the largest integer such that $P_h \subseteq C$. Clearly $h \geq 3$. Since C does not contain P_{h+1}, we conclude from Theorem 3.14 again that C is contained in $Pol\, \mu_{h+1}$, which concludes the proof.

■

4 The case $k = 4$

(In the following we shall assume throughout that $k = 4$.) We shall now prove Theorem 1.1. By Theorem 3.15 it will suffice to prove that $Pol\, \mu_4$ covers P_3 (Lemma 4.5). We start with a few basic remarks concerning relations θ such that $P_3 \subseteq Pol\, \theta$.

Let θ be an irredundant relation of arity $r \geq 2$ such that $M \subseteq Pol\, \theta$. By Lemma 2.1, there exists a partial ordering $\langle \underline{r}, \sqsubseteq \rangle$ of the indices $\{1, 2, \ldots, r\}$ such that $i \sqsubseteq j$ iff $\theta_{ij} = \leq$. By permuting the indices of θ (this does not affect the clone $Pol\, \theta$) we may assume that the natural ordering $\underline{r}$ is a linear extension of $\langle \underline{r}, \sqsubseteq \rangle$. We shall say that an r-tuple $\overline{a} = (a_1, \ldots, a_r)$ *respects the ordering of* θ if $a_i \leq a_j$ whenever $i \sqsubseteq j$.

Lemma 4.1 *Let $M \subseteq Pol\,\theta$, where θ is an irredundant r-ary relation. Then $P_h \subseteq Pol\,\theta$ if and only if θ contains every $\overline{a}$ which respects the ordering of θ and $|\{a_1, \ldots, a_r\}| \leq h$.*

Proof. ($\Rightarrow$) For $i \sqsubseteq j$ ($i \neq j$) we may find an element $w \in \theta$ such that $w_i < w_j$. For every pair of incomparable elements i and j in $\langle \underline{r}, \sqsubseteq \rangle$, we may find elements u and v in θ such that $u_i < u_j$ and $v_i > v_j$. Consider the matrix X whose columns are all these tuples, say of size $r \times m$. Certainly the rows of X form a subposet of $\underline{k}^m$ isomorphic to $\langle \underline{r}, \sqsubseteq \rangle$.

Let $\overline{a} = (a_1, \ldots, a_r)$ be an r-tuple that respects the ordering of θ and such that $|\{a_1, \ldots, a_r\}| \leq h$. Then the map f which sends row i of matrix X to a_i is isotone. By the extension lemma,there is an isotone map g that extends f and whose image contains at most h elements. Hence this map is in P_h. Since the columns of X are in θ, it follows that $\overline{a}$ must also be in θ.

($\Leftarrow$) Let $f \in P_h$. If f is unary then we are done. Otherwise let Y be a matrix whose columns are in θ. We must show that $f(Y) \in \theta$. Since f is isotone, $f(Y)$ respects the ordering of θ, and since f is not essentially unary $f(Y)$ contains at most h distinct entries. Hence $f(Y) \in \theta$ and we are done.

■

The following lemma follows from a more general result [5] we shall discuss in the next section. At any rate, the proof of this very special case is not difficult. Let $Q = \langle \underline{4}, \sqsubseteq \rangle$ be an ordering of $\{1, 2, 3, 4\}$. Consider the 4-ary relation μ_Q consisting of all (a_1, a_2, a_3, a_4) that satisfy (i) $a_i \leq a_j$ if $i \sqsubseteq j$ and (ii) $|\{a_1, a_2, a_3, a_4\}| \leq 3$. Notice that $\mu_4 = \mu_Q$ when Q is the usual ordering of $\underline{4}$. The next lemma states that an operation preserves μ_Q if and only if f is unary or the image of any copy of Q under f contains at most 3 elements.

Lemma 4.2 *An n-ary operation f is in $Pol\,\mu_Q$ if and only if either (i) f is unary or (ii) $|f(e(Q))| \leq 3$ for any isotone map $e : Q \to \underline{4}^n$.*

■

Let θ be an r-ary relation such that $P_3 \subseteq Pol\,\theta \subseteq Pol\,\mu_4$. Let $\langle \underline{r}, \sqsubseteq \rangle$ denote the ordering of the indices described above. Let $\overline{a}$ be an r-tuple. We shall say that $\overline{a}$ is *fine for θ* if it satisfies the following condition: if $a_i = 2$ and $a_j = 3$ then i and j are incomparable in $\langle \underline{r}, \sqsubseteq \rangle$.

Lemma 4.3 *Let θ be an irredundant r-ary relation such that $P_3 \subseteq Pol\,\theta$. Then $Pol\,\mu_4 \subseteq Pol\,\theta$ if and only if θ contains every $\overline{a}$ which respects the ordering of θ and is fine for θ.*

Proof. ($\Rightarrow$) Suppose that $Pol\,\mu_4 \subseteq Pol\,\theta$. Notice that $M \subseteq Pol\,\theta$. Proceeding just as in the proof of Lemma 4.1 we may find a matrix X whose columns are in θ and whose rows $\{\overline{x}_1, \ldots \overline{x}_r\}$ form a subposet of $\underline{k}^m$ isomorphic to $\langle \underline{r}, \sqsubseteq \rangle$, the ordering of θ. Let $\overline{a}$ be an r-tuple which respects the ordering of θ and which is fine for θ. Define an operation as follows:

$$f(\overline{y}) = \begin{cases} a_i & \text{if } \overline{y} = \overline{x}_i, \\ 4 & \text{if } \overline{y} > \overline{x}_i \text{ with } a_i > 1, \\ 1 & \text{ otherwise.} \end{cases}$$

Clearly this map is isotone and $f(X) = \overline{a}$. Since $\overline{a}$ is fine for θ, f maps chains to at most 3 elements and hence by Lemma 4.2 it belongs to $Pol\,\mu_4$. Since the columns of X are in θ it follows that $\overline{a} \in \theta$.

($\Leftarrow$) Suppose that θ contains all tuples which satisfy the desired conditions. By Lemma 4.1 θ also contains every $\overline{a}$ which respects its ordering and such that $|\{a_1, \ldots, a_r\}| \leq 3$. Let $f \in Pol\,\mu_4$. If f is unary then we are done. Otherwise we may suppose by Lemma 4.2 that f maps every chain to at most 3 elements. Let X be a matrix with columns in θ. We must show that $\overline{a} = f(X) \in \theta$. Clearly $\overline{a}$ respects the ordering of θ since f is isotone. If $|\{a_1, \ldots, a_r\}| \leq 3$ then we are done, so we may suppose that f is onto. In particular, it is clear that $f(1, \ldots, 1) = 1$ and $f(4, \ldots, 4) = 4$. Then $\overline{a}$ must be fine for θ; indeed, suppose the contrary so that $a_i = 2$ and $a_j = 3$ where i and j are comparable in $\langle \underline{r}, \sqsubseteq \rangle$. Since f is isotone this implies that $i \sqsubseteq j$, which means that $\overline{x}_i \leq \overline{x}_j$ where $\overline{x}_l$ denotes the l-th row of X. But then f maps the chain $\{(1, \ldots, 1), \overline{x}_i, \overline{x}_j, (4, \ldots, 4)\}$ onto 4 elements, a contradiction.

■

We define two relations of arity 4 on $\underline{k}$: let ξ consist of all 4-tuples (a_1, a_2, a_3, a_4) such that (i) $a_1 \leq a_i \leq a_4$ for every i and (ii) $|\{a_1, a_2, a_3, a_4\}| \leq 3$. Let $\beta = \xi \cup \{(1, 3, 2, 4)\}$. (Note that $\xi = \mu_Q$ where Q is described by $1 \sqsubseteq i \sqsubseteq 4$ for all i).

Lemma 4.4 $Pol\,\xi = Pol\,\beta = P_3$.

Proof. By Lemma 4.1 we have that $P_3 \subseteq Pol\,\xi$ and $P_3 \subseteq Pol\,\beta$. Next we show that $Pol\,\xi \subseteq P_3$ using Lemma 4.2. Let $f \in Pol\,\xi$; if f is unary we are done. Otherwise, suppose for a contradiction that f is onto. Then certainly $f(1,\ldots,1) = 1$ and $f(4,\ldots,4) = 4$ and it follows that f will either map a chain or a copy of Q onto 4 elements, which is impossible.

Now suppose that there is some f in $Pol\,\beta$ which is not in $Pol\,\xi$. This means there exists a matrix X with columns in ξ such that $f(X)$ is not in ξ. Since $f \in Pol\,\beta$ and β contains ξ it follows that $f(X) = (1,3,2,4)^T$. Now consider the matrix Y obtained from X by exchanging the two middle rows. Clearly the columns of Y are in ξ and hence in β; however, $f(Y) = (1,2,3,4)^T$ which is not in β, a contradiction.

■

Lemma 4.5 *Let C be a clone such that $P_3 \subseteq C \subseteq Pol\,\mu_4$. Then $C = P_3$ or $C = Pol\,\mu_4$.*

Proof. We may write $C = \cap_{i \in I} Pol\,\theta_i$ where each θ_i is irredundant. If $C \neq Pol\,\mu_4$ then there is some i such that $Pol\,\mu_4 \not\subseteq Pol\,\theta_i$. For convenience let $\theta = \theta_i$. We shall show that $Pol\,(\theta, \leq) = P_3$, from which $C = P_3$ follows. Let r denote the arity of θ and let $\langle \underline{r}, \sqsubseteq'' \rangle$ denote the partial ordering of the indices of θ. By Lemma 4.1 θ must contain every $\overline{b}$ which respects this ordering and such that $|\{b_1,\ldots,b_r\}| \leq 3$. In particular $r \geq 4$. By Lemma 4.3 there exists a tuple $\overline{a} = (a_1, a_2, \ldots, a_r)$ which respects the ordering of θ and which is fine for θ such that $\overline{a} \notin \theta$. We construct a 4-ary relation as follows: let ρ consist of all tuples $\overline{x} = (x_1, x_2, x_3, x_4)$ such that $(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) \in \theta$. It is clear that $Pol\,\theta \subseteq Pol\,\rho$.

Claim 1. $(1,2,3,4) \notin \rho$.

Indeed, if $x_i = i$ for all i then $(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) = (a_1, a_2, \ldots, a_r)$ which is not in θ.

Let $\langle \underline{4}, \sqsubseteq' \rangle$ denote the partial ordering of the indices of ρ. Also, let $Q = \langle \underline{4}, \sqsubseteq \rangle$ denote the partial ordering defined by $1 \sqsubseteq i \sqsubseteq 4$ for all i (i.e. this is the ordering of the relation ξ defined earlier).

Claim 2. $\langle \underline{4}, \sqsubseteq' \rangle$ admits $\langle \underline{4}, \sqsubseteq \rangle$ as an extension, i.e. if $i \sqsubseteq' j$ then $i \sqsubseteq j$.

It is easy to see it suffices to show that $(1,2,3,3)$ and $(1,3,2,3)$ belong to ρ. By the definition of ρ, if $(x_{a_1}, x_{a_2}, \ldots, x_{a_r}) \in \theta$ then $\overline{x} = (1,2,3,3) \in \rho$. Since there are only three distinct entries, it suffices to prove that $\overline{x}$ respects the ordering of θ. Now clearly $x_j = a_j$ if $j = 1, 2, 3$ and $x_4 = 3$ implies that $\overline{x}$ is obtained from $\overline{a}$ by replacing occurrences of 4 by 3. If $i \sqsubseteq'' j$ then $a_i \leq a_j$ and hence $x_i \leq x_j$. Now consider the case of $(1, 3, 2, 3)$. As above it suffices to show that $\overline{x}$ respects the ordering of θ. Now $\overline{x}$ is obtained from $\overline{a}$ as follows: replace all occurrences of 2 by 3 and occurrences of 3 by 2, then replace all occurrences of 4 by 3. Let $i \sqsubseteq'' j$. Then $a_i \leq a_j$ and since $\overline{a}$ is fine for θ, either $a_i \neq 2$ or $a_j \neq 3$. It is easy to see that $x_i \leq x_j$ (the correspondence $a_i \mapsto x_i$ is order-preserving except for the pair (2,3)).

We construct a 4-ary relation as follows: let γ consist of all (x_1, x_2, x_3, x_4) in ρ such that $x_1 \leq x_i \leq x_4$ for all i. Clearly $Pol\,(\theta, \leq) \subseteq Pol\,\gamma$. Hence to finish our proof it will suffice to prove $Pol\,\gamma = P_3$. To do this, we prove that γ is one of ξ or β and invoke Lemma 4.4.

Claim 3. $\gamma = \xi$ or $\gamma = \beta$.

Indeed: by Claim 2 and its proof, it is easy to see that the ordering of γ is Q. By Lemma 4.1 γ contains every tuple that respects Q and has at most 3 entries. The only other tuples that can be in γ are $(1, 2, 3, 4)$ and $(1, 3, 2, 4)$. By Claim 1, $(1, 2, 3, 4) \notin \gamma$. Hence $\gamma = \xi$ if it does not contain $(1, 3, 2, 4)$ and $\gamma = \beta$ otherwise.

■

5 Comments on the structure of the interval for $k \geq 5$

It appears that the structure of the interval $[M, Pol \leq]$ is much more complicated for $k \geq 5$ than the cases $k = 3$ and $k = 4$ would let us believe. Indeed, consider the following generalisation of the relation μ_h: let $3 \leq r$ and $h \geq 2$. Let $Q = \langle \underline{r}, \sqsubseteq \rangle$ be a partial ordering and define $\mu_{Q,h}$ as the set of all r-tuples $\overline{a}$ that respect the ordering Q and such that $|\{a_1, \ldots, a_r\}| \leq h$. It is clear that we may suppose that $h < \max\{r, k\}$, otherwise $Pol \leq$ is contained in $Pol\,\mu_{Q,h}$. If Q is an $h+1$-element chain then of course we find $\mu_{Q,h} = \mu_{h+1}$ and if Q is an antichain then $Pol\,\mu_{Q,h}$ is a Burle clone. From now on we shall assume without loss of generality that there is always at least some comparability in Q.

Lemma 5.1 *$Pol\,\mu_{Q,h} = \cap_{\alpha \in \mathcal{A}} Pol\,\alpha$ where $\mathcal{A}$ is the set of all restrictions of $\mu_{Q,h}$ to $h+1$ indices. Moreover, each $\alpha \in \mathcal{A}$ is of the form $\alpha = \mu_{Q',h}$ for some partial ordering Q'.*

Proof. The inclusion $\subseteq$ is trivial. Now let f be an n-ary operation that preserves every $\alpha \in \mathcal{A}$ and let X be an $r \times n$ matrix whose columns are in $\mu_{Q,h}$. Since Q is non-trivial f is isotone. Hence $f(X)$ respects Q. If $|f(X)| > h$ then there must be a subset I of $\underline{r}$ with $h+1$ elements such that $|f(X')| > h$ where X' is the matrix obtained from X by deleting rows whose index is not in I. Hence f does not preserve α, the restriction of $\mu_{Q,h}$ to I, and this is a contradiction.

For the second statement: let I be a subset of $\underline{r}$ with $h+1$ elements. We prove that $(\mu_{Q,h})_I = \mu_{Q',h}$ where Q' is the restriction of Q to I. The inclusion $\subseteq$ is easy. Now let $\overline{b}$ respect the ordering Q' and $|\{b_1, \ldots, b_{h+1}\}| \leq h$. Consider the partial map $i \mapsto b_i$ from $\underline{r}$ to $\underline{k}$. By the extension lemma, there exists an isotone map $i \mapsto a_i$ from $\underline{r}$ to $\underline{k}$ that extends $\overline{b}$ and such that $\overline{a} \in \mu_{Q,h}$.

■

If Q is an ordering of $\underline{h+1}$ then we denote $\mu_{Q,h}$ simply by μ_Q.

Lemma 5.2 *Let Q be an ordering of $\underline{h+1}$. Then $\mu_Q = \cap_{Q' \in \mathcal{B}} Pol\,\mu_{Q'}$ where $\mathcal{B}$ is the set of all bounded extensions Q' of Q.*

Proof. As in the previous result we need only prove that if f is an n-ary operation that preserves $\mu_{Q'}$ for every $Q' \in \mathcal{B}$ then f preserves μ_Q. Certainly f is isotone. Let X be an $(h+1) \times n$ matrix whose columns are in μ_Q. Then $f(X)$ respects Q. Now suppose that $|f(X)| = h+1$. Let $\overline{x}_i$ and $\overline{x}_j$ be the rows of X such that $f(\overline{x}_i) = \min\{f(X)\}$ and $f(\overline{x}_j) = \max\{f(X)\}$. Consider the new matrix X' obtained from X by replacing $\overline{x}_i$ by the tuple $(u_1, \ldots, u_n)$ where u_l is the least element appearing in column l, and replacing $\overline{x}_j$ by the tuple $(v_1, \ldots, v_n)$ where v_l is the greatest element appearing in column l. We claim that the columns of X' are in μ_Q. Since f is isotone and one-to-one on X it is clear by definition of i and j that $l \sqsubseteq i$ for no l and $j \sqsubseteq l$ for no l. It follows that each column respects Q. If column l of X' is equal to column l of X then of course it is in μ_Q; otherwise it means that column l of X' must contain a repetition and hence is in μ_Q. Now consider the ordering Q'

obtained from Q by adding the comparibilities $i \sqsubseteq' m \sqsubseteq' j$ for all m. This is obviously a bounded extension of Q, and it is clear that the columns of X' are all in $\mu_{Q'}$. But since f is isotone it is clear that $|f(X')| = h + 1$ so $f(X') \notin \mu_{Q'}$, a contradiction.

■

There is a nice characterisation of the operations in $Pol\,\mu_{Q,h}$ which helps in comparing these clones. It is a generalisation of a result of Jablonskii [2] (see also [10], p. 152) which we mentioned before Lemma 4.2. Notice that the two previous lemmas allow us to reduce the proof of this result to the case $Pol\,\mu_{Q'}$ where Q' is bounded.

Lemma 5.3 *[5] An n-ary operation f is in* $Pol\,\mu_{Q,h}$ *if and only if either (i) f is unary or (ii)* $|f(e(Q))| \leq h$ *for any isotone map* $e : Q \to (\underline{h+1})^n$.

■

These results show that it suffices to consider clones of the form $Pol\,\mu_{Q'}$ where Q' is a bounded ordering of $\underline{h+1}$ if we want to classify the clones $Pol\,\mu_{Q,h}$. Moreover, notice that as a result, *there are only finitely many clones* $Pol\,\mu_{Q,h}$. On the other hand, it would appear that these are not the only clones in the interval $[M, Pol\leq]$. Furthermore, for large k, even the poset of clones $Pol\,\mu_{Q'}$ seems difficult to characterize. As a simple example, consider, for any $k \geq 6$, the partial ordering Q of $\{1, 2, 3, 4, 5\}$ given by $1 \sqsubset 2 \sqsubset 3 \sqsubset 4$. It is a simple exercise to verify that $P_4 \subset Pol\,\mu_Q \subset Pol\,\mu_5$, and that in fact the clones $Pol\,\mu_Q$ and $P_5 \cap Pol\,\mu_5$ are incomparable elements of $[M, Pol\leq]$.

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