THE GEOMETRY OF THE EISENSTEIN-PICARD MODULAR GROUP

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Abstract

The Eisenstein-Picard modular group $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ *is defined to be the subgroup of* $\mathrm{PU}(2,1)$ *whose entries lie in the ring* $\mathbb{Z}[\omega]$, *where* ω *is a cube root of unity. This group acts isometrically and properly discontinuously on* $\mathbf{H}^2_{\mathbb{C}}$, *that is, on the unit ball in* $\mathbb{C}^2$ *with the Bergman metric. We construct a fundamental domain for the action of* $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ *on* $\mathbf{H}^2_{\mathbb{C}}$, *which is a 4-simplex with one ideal vertex. As a consequence, we elicit a presentation of the group (see Theorem 5.9). This seems to be the simplest fundamental domain for a finite covolume subgroup of* $\mathrm{PU}(2,1)$.

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1. Introduction

Lattices in rank one symmetric spaces have been studied for a long time with important results concerning rigidity and arithmeticity. Among symmetric spaces, the complex ball is a particularly challenging case. In particular, very few examples of lattices have been constructed. Perhaps the first example for the complex two-dimensional ball, the group $\mathrm{PU}(2,1;\mathbb{Z}[\omega]) \subset \mathrm{PU}(2,1)$, is due to Picard [Pi1], [Pi2]; here $\omega = (-1+i\sqrt{3})/2$ is a primitive cube root of unity (see Sections 2, 3 for notation). This group generalises the modular group $\mathrm{PSL}(2,\mathbb{Z})$ in complex dimension one. We call $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ the Eisenstein-Picard modular group due to the important role of Eisenstein integers $\mathbb{Z}[\omega]$.

Our goal in this article is to obtain a fundamental domain for the Eisenstein-Picard group along with a presentation. Of course, fundamental domains exist and were studied in some generality (see [GR]), but the actual construction of a concrete example is not easy. Curiously, this has not yet been done for the Eisenstein-Picard group, maybe because the simplest way to obtain fundamental domains—namely, by the Dirichlet method—gives rise to combinatorially complicated objects.

Studies of lattices using Dirichlet fundamental domains were made by Giraud [G] and Mostow [M]. The calculations are difficult because bisectors are not totally geodesic submanifolds, and, in fact, Mostow used computers. Moreover, it is not clear whether his proof is independent of some numerical analysis (see the discussion in [D]). Other fundamental domains for Mostow's groups are given in [DFP].

In this article, we abandon Dirichlet domains and instead construct a remarkably simple fundamental domain. In fact, it is the simplest possible combinatorial structure, being a 4-simplex with one ideal vertex (the group has only one cusp) inside the two-dimensional complex ball $\mathbf{H}^2_{\mathbb{C}}$ (see Theorem 5.9). In fact, we construct the Ford domain with a centre parabolic fixed point, that is, the intersection of the exteriors of isometric spheres of all elements not fixing infinity. As is well known, the Ford domain is a fundamental domain for the coset space of Γ_∞ (the parabolic group stabilising the ideal vertex; see, e.g., [L, page 58]). In order to construct a fundamental domain, we must intersect the Ford domain with a fundamental domain for Γ_∞. The fact that our fundamental domain is a simplex follows from the fact that there is a single Γ_∞-orbit of isometric spheres with maximal radius, and the boundary of the Ford domain consists of Γ_∞-equivalent tetrahedral faces. This leads us to a choice of fundamental domain for Γ_∞, namely, the geodesic cone from the boundary of one of these tetrahedra to the centre of the Ford domain.

This construction is completely analogous to the famous 2-simplex with one ideal vertex which is the fundamental domain for the classical modular group $\mathrm{PSL}(2,\mathbb{Z})$ in the hyperbolic plane $\mathbf{H}^1_{\mathbb{C}}$. The proofs we give, wherever possible, follow those for $\mathrm{PSL}(2,\mathbb{Z})$ (see [L, pages 59 – 60]; readers may find it helpful to keep this example in mind). For $\mathrm{PSL}(2,\mathbb{Z})$, the boundary of the Ford domain consists of arcs of Euclidean circles with radius 1 centred at the integers. These arcs are equivalent under the action of integer translations, and so a fundamental domain for $\mathrm{PSL}(2, \mathbb{Z})$ is obtained by intersecting the Ford domain with a strip of (Euclidean) width 1. If this strip is centred on an integer, then the resulting domain is a hyperbolic triangle. Moreover, it is the geodesic cone from infinity to one of the edges of the Ford domain.

The relation between the groups $\mathrm{PSL}(2, \mathbb{Z})$ and $\mathrm{PU}(2, 1; \mathbb{Z}[\omega])$ is given in Proposition 5.10; $\mathrm{PU}(2, 1; \mathbb{Z}[\omega])$ is obtained from a representation of $\mathrm{PSL}(2, \mathbb{Z})$ by adjoining one element (see also [FP]). Finally, we show that as well as its geometric presentation, the Eisenstein-Picard modular group admits a presentation with two generators (see Proposition 5.11). Furthermore, this presentation falls into the same pattern as the family of the groups constructed by Mostow in [M] (see Corollary 5.13).

The orbifold $\mathbf{H}^2_{\mathbb{C}}/\mathrm{PU}(2, 1; \mathbb{Z}[\omega])$ has volume $\pi^2/27$ (this follows from the work of Holzapfel; see [H1, page 151]). This is conjectured to be the smallest volume of a cusped, complex hyperbolic orbifold. The fact that the Eisenstein-Picard group is a basic lattice in complex dimension two is also shown by the fact that a smallest-volume complex hyperbolic two-manifold can be obtained from an index 72 subgroup of the Eisenstein-Picard group (see [P1]). These facts are again direct analogies of the corresponding results for $\mathrm{PSL}(2, \mathbb{Z})$.

Our construction uses bisectors (see [M] and [Go]) and a version of Poincaré's polyhedron theorem, following [M]. It involves a careful study of a fundamental domain for the parabolic subgroup fixing the cusp inside the Heisenberg group that is the ideal boundary of complex hyperbolic space. The finite face of our polyhedron is contained in an isometric sphere (that is, a vertical bisector; see [Go, Section 5.1.9]), but all four faces containing the ideal vertex are not contained in a bisector; rather, they are contained in the geodesic cone over a lower-dimensional face. A different construction of a fundamental polyhedron for the Eisenstein-Picard modular group is given in [P2]. This polyhedron consists of two simplices with a common face, and so it has eight faces. The advantage of this construction is that all eight faces are contained in bisectors.

The other Picard modular groups are $\mathrm{PU}(2, 1; \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$ for any positive square-free integer d. It would be interesting to find a strategy to obtain fundamental domains for $\mathrm{PU}(2, 1; \mathcal{O}_d)$, as was done by Swan [Sw] for the Bianchi groups $\mathrm{PSL}(2, \mathcal{O}_d)$.

2. Complex hyperbolic space and its isometries

2.1. The Siegel domain

We consider the Hermitian form $\langle z, w\rangle = w^* J_0 z$ on $\mathbb{C}^3$ with signature (2, 1) defined by the matrix

$$J_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consider the following subspaces of $\mathbb{C}^3$:

$$V_0 = \left\{z \in \mathbb{C}^3 - \{0\} \,:\, \langle z, z\rangle = 0\right\},$$
$$V_- = \left\{z \in \mathbb{C}^3 \,:\, \langle z, z\rangle < 0\right\}.$$

Let $\mathbb{P} : \mathbb{C}^3 - \{0\} \to \mathbb{C}P^2$ be the canonical projection onto complex projective space. Then $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_-)$ is a complex hyperbolic space. Using nonhomogeneous coordinates, we obtain $\mathbf{H}^2_{\mathbb{C}}$ as the Siegel domain

$$\left\{ \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \in \mathbb{C}P^2 :\ 2\mathrm{Re}\,(z_1) + |z_2|^2 < 0 \right\}.$$

Recall that the Heisenberg group is $\mathfrak{N} = \mathbb{C} \times \mathbb{R}$ with the group law

$$(z_1, t_1)(z_2, t_2) = \big(z_1 + z_2, t_1 + t_2 + 2\ \mathrm{Im}(z_1\overline{z}_2)\big).$$

Complex hyperbolic space is parametrised in *horospherical coordinates* by $\mathfrak{N} \times \mathbb{R}^+$:

$$(z, t, u) \to \begin{bmatrix} \frac{-|z|^2 - u + it}{2} \\ z \\ 1 \end{bmatrix}. \tag{1}$$

The point at infinity is

$$q_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\mathbb{P}(V_0) = \partial\mathbf{H}^2_{\mathbb{C}} = (\mathfrak{N} \times \{0\}) \cup \{q_\infty\}$.

The horosphere based at q_∞ of height u is the hypersurface $H_u = \mathfrak{N} \times \{u\}$, which bounds the horoball $B_u = \mathfrak{N} \times (u, \infty)$. In horospherical coordinates, the geodesics with endpoint q_∞ are the vertical lines

$$\left\{(z_0, t_0, u) : u \in (0, \infty)\right\}.$$

2.2. Complex hyperbolic isometries

The group of biholomorphic transformations of $\mathbf{H}^2_{\mathbb{C}}$ is then PU(2, 1), the projectivisation of the unitary group U(2, 1) preserving the Hermitian form given by J_0. The general form of an element of $A \in \mathrm{PU}(2,1)$ and its inverse are

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} \overline{j} & \overline{f} & \overline{c} \\ \overline{h} & \overline{e} & \overline{b} \\ \overline{g} & \overline{d} & \overline{a} \end{bmatrix}. \tag{2}$$

If A fixes q_∞, then it is upper triangular. We now examine the subgroup of $\mathrm{PU}(2,1)$ fixing q_∞. First, for $(z_0, t_0) \in \mathfrak{N}$, *Heisenberg translation* by (z_0, t_0) is given by

$$\begin{bmatrix} 1 & -\overline{z}_0 & \frac{-|z_0|^2 + it_0}{2} \\ 0 & 1 & z_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Any Heisenberg translation by $(0, t_0) \in \mathfrak{N}$ is called a *vertical translation*.

For $e^{i\theta} \in S^1$, *Heisenberg rotation* by θ fixing the complex line $(0, t, u) \subset \mathbf{H}^2_{\mathbb{C}}$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

All other Heisenberg rotations fixing q_∞ may be obtained from such a map by conjugating by a Heisenberg translation.

For $\lambda \in \mathbb{R}_+$, *Heisenberg dilation* by λ fixing q_∞ and $q_o = (0,0,0) \in \partial\mathbf{H}^2_{\mathbb{C}}$ is given by

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}.$$

All other Heisenberg dilations fixing q_∞ may be obtained by conjugating by a Heisenberg translation.

All Heisenberg rotations and translations preserve each horosphere based at q_∞, but all nontrivial Heisenberg dilations map each horosphere in $\mathbf{H}^2_{\mathbb{C}}$ to another one. The group generated by all Heisenberg translations, rotations, and dilations is the stabiliser of q_∞ in $\mathrm{PU}(2,1)$. The *Heisenberg isometry group* $\mathrm{Isom}(\mathfrak{N})$ is the subgroup generated by all Heisenberg translations and rotations. We can write $\mathrm{Isom}(\mathfrak{N})$ as $\mathfrak{N} \rtimes \mathrm{U}(1)$. In particular, each element of $\mathrm{Isom}(\mathfrak{N})$ preserves every horosphere.

We define *vertical projection* $\Pi : \mathfrak{N} \longrightarrow \mathbb{C}$ by $\Pi : (z,t) \longmapsto z$. Using the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{N} \xrightarrow{\Pi} \mathbb{C} \longrightarrow 0,$$

we obtain the exact sequence (see Scott [S, page 467])

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(\mathfrak{N}) \xrightarrow{\Pi_*} \mathrm{Isom}(\mathbb{C}) \longrightarrow 1. \tag{3}$$

Here $\mathrm{Isom}(\mathbb{C})$ is the group of orientation-preserving Euclidean isometries of $\mathbb{C}$.

Observe that elements in $\mathrm{Isom}(\mathbb{C})$ can be represented by matrices in $\mathrm{GL}(2, \mathbb{C})$ of the form

$$\begin{bmatrix} e^{i\theta} & z_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} e^{i\theta} z + z_0 \\ 1 \end{bmatrix}.$$

Therefore, the map Π_* can be explicitly given by

$$\Pi_* : \begin{bmatrix} 1 & -\overline{z}_0 e^{i\theta} & \frac{-|z_0|^2 + it_0}{2} \\ 0 & e^{i\theta} & z_0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} e^{i\theta} & z_0 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

It is clear that

$$\ker(\Pi_*) = \left\{ \begin{bmatrix} 1 & 0 & \frac{it_0}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : t_0 \in \mathbb{R} \right\},$$

the group of vertical translations fixing q_∞.

2.3. Isometric spheres

Given an element $A \in \mathrm{PU}(2, 1)$ such that $A(q_\infty) \neq q_\infty$, we define the *isometric sphere* of A to be the hypersurface

$$\left\{ z \in \mathbf{H}^2_{\mathbb{C}} \;:\; |\langle z, q_\infty \rangle| = |\langle z, A^{-1}(q_\infty) \rangle| \right\}.$$

For example,

$$S_0 = \left\{ (z, t, u) : \left| |z|^2 + u + it \right| = 2 \right\} \tag{5}$$

is the isometric sphere of

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Both the isometric sphere S_0 and the map R play crucial roles in our constructions. All other isometric spheres are images of S_0 by Heisenberg dilations, rotations, and translations. Thus the isometric sphere with radius r and centre $(z_0, t_0, 0)$ is given by

$$\Big\{(z,t,u) : \big||z-z_0|^2 + u + it - it_0 + 2i\ \mathrm{Im}(z\bar{z}_0)\big| = r^2\Big\}. \tag{6}$$

(The factor r^2 in this expression is because we are using the Cygan metric to measure the radius; see, e.g., [P1].) Thus if A has the form (2), then $A(q_\infty) \neq q_\infty$ if and only if $g \neq 0$. The isometric sphere of A has radius $r = \sqrt{2/|g|}$ and centre $A^{-1}(q_\infty)$, which in horospherical coordinates is

$$(z_0, t_0, 0) = \left(\frac{\overline{h}}{\overline{g}}, 2\mathrm{Im}\Big(\frac{\overline{j}}{\overline{g}}\Big), 0\right).$$

Isometric spheres are examples of *bisectors* and, as such, have a very nice foliation by two different families of totally geodesic submanifold. There is a geodesic called the *spine* of the bisector. Mostow [M] showed that a bisector is the preimage of the spine under orthogonal projection onto the unique complex line containing the spine. The fibres of this projection are complex lines called the *slices* of the bisector. Goldman [Go] showed that a bisector is the union of all totally real Lagrangian planes containing the spine. Such Lagrangian planes are called the *meridians*. Together the slices and meridians give *geographical coordinates* on the bisector. Specifically, we begin by writing $|z|^2 + u - it = 2e^{i\theta}$ for $\theta \in [-\pi/2, \pi/2]$ (this ensures that $|z|^2 + u \geq 0$); in particular, $|z| \leq \sqrt{2\cos(\theta)}$. We also write z in polar coordinates, and we choose its argument in a way that is adapted to the decomposition of S_0 into meridians. We achieve this by writing $z = re^{i\alpha + i\theta/2}$ for $r \in [-\sqrt{2\cos(\theta)}, \sqrt{2\cos(\theta)}]$ and $\alpha \in [-\pi/2, \pi/2)$. We remark that it might seem more natural to keep r nonnegative and allow α to vary over $[-\pi, \pi)$. As we show in Proposition 2.1, we made this choice so that meridians of S_0 correspond to a fixed α. In geographical coordinates, S_0 is given by

$$S_0 = \left\{ \begin{bmatrix} -e^{i\theta} \\ re^{i\alpha + i\theta/2} \\ 1 \end{bmatrix} : \theta \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big], \alpha \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big), \right.$$
$$\left. r \in [-\sqrt{2\cos(\theta)}, \sqrt{2\cos(\theta)}] \right\}. \tag{7}$$

In horospherical coordinates, the point of S_0 with geographical coordinates (r, θ, α) is given by $(re^{i\alpha + i\theta/2}, -2\sin(\theta), 2\cos(\theta) - r^2)$.

We now find the spine, slices, and meridians of S_0 in terms of geographical coordinates.

PROPOSITION 2.1

The isometric sphere S_0 with coordinates given by (7) is a bisector. Moreover,

- *the spine of S_0 is given by $r = 0$;*
- *the slices of S_0 are given by $\theta = \theta_0$ for fixed $\theta_0 \in [-\pi/2, \pi/2]$;*
- *the meridians of S_0 are given by $\alpha = \alpha_0$ for fixed $\alpha_0 \in [-\pi/2, \pi/2)$.*

Proof

All isometric spheres are bisectors. The spine of S_0 is given by the intersection of the bisector with its complex spine, that is, the complex line spanned by q_∞ and $R(q_\infty)$. This complex line has equation $z = 0$, and the first part follows.

Given a point $(0, -2\sin(\theta_0), 2\cos(\theta_0))$ on the spine of S_0, the slice through this point is given by the inverse image of orthogonal projection onto the complex spine. Such points are given by

$$\left\{ \begin{bmatrix} -e^{i\theta_0} \\ z \\ 1 \end{bmatrix} \in \mathbb{P}(V_-) \right\}.$$

The second part follows immediately.

The meridians of S_0 are the fixed-point loci of antiholomorphic involutions fixing the spine. For $\alpha_0 \in [-\pi/2, \pi/2)$, these maps are given by

$$\iota_{\alpha_0} : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} \overline{z_3} \\ -e^{2i\alpha_0}\overline{z_2} \\ \overline{z_1} \end{bmatrix}.$$

Applying ι_{α_0} to a point of S_0 and taking horospherical coordinates, we obtain

$$\iota_{\alpha_0}\bigl(re^{i\alpha+i\theta/2}, -2\sin(\theta), 2\cos(\theta) - r^2\bigr) = \bigl(re^{2i\alpha_0-i\alpha+i\theta/2}, -2\sin(\theta), 2\cos(\theta) - r^2\bigr).$$

Therefore, the meridian fixed by ι_{α_0} is given by $\alpha = \alpha_0$. □

3. The Eisenstein-Picard modular group

Let $\mathcal{O}_d$ be the ring of integers in the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$, where d is a positive square-free integer. If $d \equiv 1, 2 \pmod 4$, then $\mathcal{O}_d = \mathbb{Z}[i\sqrt{d}]$, and if $d \equiv 3 \pmod 4$, then $\mathcal{O}_d = \mathbb{Z}[(1 + i\sqrt{d})/2]$. The subgroup of PU(2, 1) with entries in $\mathcal{O}_d$ is called the *Picard modular group* for $\mathcal{O}_d$ and is written $\mathrm{PU}(2, 1; \mathcal{O}_d)$ (see [H2]). (In fact, [H2] uses a different Hermitian form. However, the two Picard modular groups are conjugate; see [P1, page 452].)

We are only interested in the case where $d = 3$. Let ω denote the cube root of unity $(-1 + i\sqrt{3})/2$. Then $\mathcal{O}_3 = \mathbb{Z}[\omega]$ is the set of Eisenstein integers. Thus the Picard modular group in this case is $\Gamma = \mathrm{PU}(2, 1; \mathbb{Z}[\omega])$, which we call the *Eisenstein-Picard* *modular group*. The goal of this section is to prove Theorem 3.5, which gives generators for $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$. In later sections, we go on to give a presentation.

3.1. The stabiliser of q_∞

First, we want to analyse Γ_∞, the stabiliser of q_∞ in $\Gamma = \mathrm{PU}(2,1;\mathbb{Z}[\omega])$. Every element of Γ_∞ is upper triangular, and its diagonal entries are units in $\mathbb{Z}[\omega]$. Therefore, Γ_∞ contains no dilations and so is a subgroup of $\mathrm{Isom}(\mathfrak{N})$; thus it fits into the exact sequence (3) as

$$0 \longrightarrow \mathbb{R} \cap \Gamma_\infty \longrightarrow \Gamma_\infty \xrightarrow{\Pi_*} \Pi_*(\Gamma_\infty) \longrightarrow 1.$$

We now find the image and kernel in this exact sequence.

PROPOSITION 3.1
The stabiliser Γ_∞ of q_∞ in $\Gamma = \mathrm{PU}(2,1;\mathbb{Z}[\omega])$ satisfies

$$0 \longrightarrow 2\sqrt{3}\,\mathbb{Z} \longrightarrow \Gamma_\infty \xrightarrow{\Pi_*} \Delta(2,3,6) \to 1,$$

where $\Delta(2,3,6)$ denotes the triangle group comprising orientation-preserving symmetries of $\mathbb{Z}[\omega]$.

Proof
From our explicit construction (4) of Π_*, we see that for $A \in \Gamma_\infty$,

$$\Pi_*(A) = \begin{bmatrix} (-\omega)^m & z_0 \\ 0 & 1 \end{bmatrix},$$

where $z_0 \in \mathbb{Z}[\omega]$. Thus $\Pi_*(\Gamma_\infty)$ is the group of orientation-preserving symmetries of $\mathbb{Z}[\omega] \subset \mathbb{C}$. This is well known to be the triangle group $\Delta(2,3,6)$.

Likewise, the kernel of Π_* is easily seen to consist of those vertical translations in Γ, that is, Heisenberg translations by $(0,\ 2\sqrt{3}\,n) \in \mathfrak{N}$ for $n \in \mathbb{Z}$. □

This enables us to find generators for Γ_∞.

PROPOSITION 3.2
Γ_∞ is generated by

$$P = \begin{bmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

Proof
The triangle group $\Delta(2, 3, 6)$ is generated by

$$\Pi_*(P) : z \longmapsto \omega z - \omega, \qquad \Pi_*(Q) : z \longmapsto -z + 1.$$

Hence we only need to show that P and Q generate $\mathbb{R} \cap \Gamma_\infty = 2\sqrt{3}\,\mathbb{Z}$. Observe that

$$P^3 = Q^2 = \begin{bmatrix} 1 & 0 & i\sqrt{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

This is precisely the generator of $\mathbb{R} \cap \Gamma_\infty = 2\sqrt{3}\,\mathbb{Z}$. □

As a first step toward the construction of a fundamental domain for the Eisenstein-Picard modular group Γ, we construct a fundamental domain for the parabolic subgroup Γ_∞ acting on the Heisenberg group. As Γ_∞ preserves horospheres, a fundamental domain for Γ_∞ acting on $\mathbf{H}^2_{\mathbb{C}}$ is obtained by taking the bundle of vertical geodesics (in horospherical coordinates) over a fundamental domain in the Heisenberg group. In other words, the fundamental domain in $\mathbf{H}^2_{\mathbb{C}}$ is the geodesic cone over a fundamental domain in $\mathfrak{N}$.

We want to describe the action of P and Q on each horosphere. To do so, we use the identification (1) between $\mathfrak{N} \times \mathbb{R}^+ = (\mathbb{C} \times \mathbb{R}) \times \mathbb{R}^+$ and a subset of complex projective space. Then using the matrices (8), we obtain the following action of P,

$$\begin{bmatrix} 1 & 1 & \omega \\ 0 & \omega & -\omega \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-|z|^2 - u + it}{2} \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-|z|^2 - u + it}{2} + z + \omega \\ \omega z - \omega \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-|\omega z - \omega|^2 - u + it + 2i\ \mathrm{Im}(z) + i\sqrt{3}}{2} \\ \omega z - \omega \\ 1 \end{bmatrix},$$

and Q,

$$\begin{bmatrix} 1 & 1 & \omega \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-|z|^2 - u + it}{2} \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-|z|^2 - u + it}{2} + z + \omega \\ -z + 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-|-z + 1|^2 - u + it + 2i\ \mathrm{Im}(z) + i\sqrt{3}}{2} \\ -z + 1 \\ 1 \end{bmatrix}.$$

Therefore, in horospherical coordinates,

$$P : (z, t, u) \longmapsto \big(\omega z - \omega, t + 2\,\mathrm{Im}(z) + \sqrt{3}, u\big)$$

and

$$Q : (z, t, u) \longmapsto \big(-z + 1, t + 2\,\mathrm{Im}(z) + \sqrt{3}, u\big).$$

This action preserves each horosphere, that is, the set of points where u is constant. Thus we may drop the dependence on u, and we obtain the action on $\mathfrak{N} = \mathbb{C} \times \mathbb{R}$.

Consider $\mathbf{T}_*$, the equilateral triangle in $\mathbb{C}$ with vertices at the points 0, 1, and $-\omega$. The map $\Pi_*(P)$ is the rotation by $2\pi/3$ about the centre of this triangle, and $\Pi_*(Q)$ is the rotation by π around the midpoint of the side joining 0 to 1. Observe that a fundamental domain for $\Pi_*(\Gamma_\infty) = \Delta(2, 3, 6)$ acting on $\mathbb{C}$ is one-third of $\mathbf{T}_*$. Starting from 0, one can define the vertices of $\mathbf{T}_*$ as 0, $\Pi_*(P)(0) = -\omega$, and $\Pi_*(P^2)(0) = 1$. This action of $\Pi_*(P)$ and $\Pi_*(Q)$ may be lifted to give a geometrical interpretation of the action of P and Q. Specifically, writing $z = (3 - i\sqrt{3}\,)/6 + \zeta$, we see

$$P : \Big(\frac{1}{2} - \frac{i\sqrt{3}}{6} + \zeta, t, u\Big) \longmapsto \Big(\frac{1}{2} - \frac{i\sqrt{3}}{6} + \omega\zeta, t + 2\,\mathrm{Im}(\zeta) + \frac{2}{\sqrt{3}}, u\Big).$$

Hence the action of the parabolic element P is a (Heisenberg) rotation by $2\pi/3$ around the vertical line that projects to $(3 - i\sqrt{3}\,)/6$, the centre of $\mathbf{T}_*$, followed by an upward vertical translation by $2/\sqrt{3}$. From the Euclidean point of view, P also involves a shear. Likewise, writing $z = 1/2 + \zeta$, we see that

$$Q : \Big(\frac{1}{2} + \zeta, t, u\Big) \longmapsto \Big(\frac{1}{2} - \zeta, t + 2\,\mathrm{Im}(\zeta) + \sqrt{3}, u\Big).$$

Thus the action of the parabolic element Q is a (Heisenberg) rotation by π about the vertical line that projects to $1/2$ followed by an upward vertical translation by $\sqrt{3}$.

The map PQ^{-1} is

$$PQ^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

In horospherical coordinates, this action is just

$$PQ^{-1} : (z, t, u) \longmapsto (-\omega z, t, u).$$

This is just rotation about the vertical axis by $-\pi/3 = \arg(-\omega)$. In particular, $(PQ^{-1})^6 = 1$.

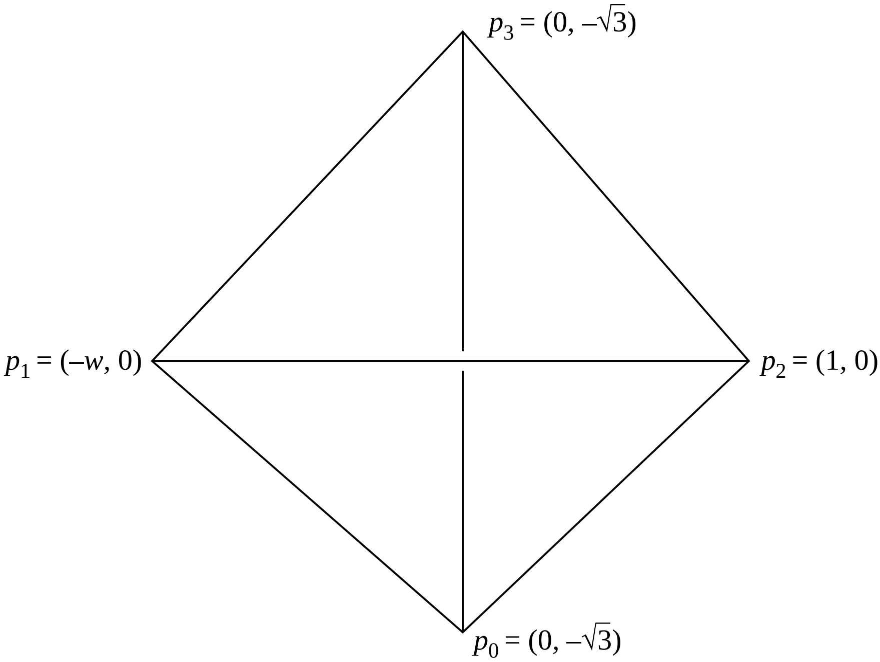

Figure 1. The tetrahedron **T** in the Heisenberg group

Let **T** be the affine tetrahedron, shown in Figure 1, in $\mathfrak{N}$ with vertices at $p_0 = (0, -\sqrt{3})$, $p_1 = P(p_0) = (-\omega, 0)$, $p_2 = P^2(p_0) = (1, 0)$, and $p_3 = P^3(p_0) = (0, \sqrt{3})$.

Observe that PQ^{-1} fixes p_0, p_3 and that $p_1 = PQ^{-1}(p_2)$. Denoting the faces of **T** by the ordered triples of their vertices, this gives the following side-pairing maps for **T**:

$$P : (p_0, p_1, p_2) \longmapsto (p_1, p_2, p_3),$$
$$PQ^{-1} : (p_0, p_2, p_3) \longmapsto (p_0, p_1, p_3).$$

Similarly, denoting the edges of **T** by the ordered pairs of their endpoints, the edge cycles given by these side-pairings are

$$(p_0, p_3) \xrightarrow{PQ^{-1}} (p_0, p_3),$$
$$(p_0, p_1) \xrightarrow{P} (p_1, p_2) \xrightarrow{P} (p_2, p_3) \xrightarrow{PQ^{-1}} (p_1, p_3) \xrightarrow{P^{-1}} (p_0, p_2) \xrightarrow{PQ^{-1}} (p_0, p_1).$$

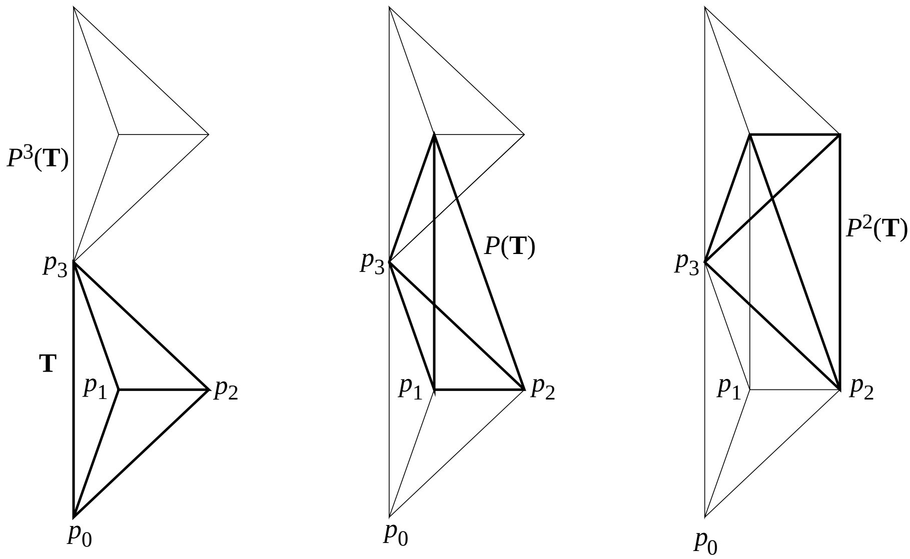

Figure 2. Tessellating the Heisenberg group with **T**

This has used all the edges of **T**. The first of these cycles gives the relation $(PQ^{-1})^6 = 1$, and the second gives the relation $Q^{-2}P^3 = 1$. These relations follow from equations (9) and (10).

We now show that the images of **T** under Γ_∞ tessellate $\mathfrak{N}$ (see Fig. 2).

LEMMA 3.3
The images of **T** *under* $\langle P \rangle$ *are disjoint except for common faces and fill the prism whose vertical projection under* Π *is* $\mathbf{T}_*$, *the equilateral triangle with vertices* $0, 1, -\omega$.

Proof
It is clear that the vertical sides of **T**, namely, (p_0, p_1, p_3) and (p_0, p_2, p_3), are contained in the vertical sides of the prism. Moreover, $P(\mathbf{T})$ is an affine tetrahedron with vertices p_1, p_2, p_3, and $p_4 = P(p_3) = (-\omega, 2\sqrt{3})$. The vertical sides of this tetrahedron are contained in the vertical sides of the prism. The two tetrahedra **T** and $P(\mathbf{T})$ share a common face (p_1, p_2, p_3). Otherwise, they are disjoint. A similar result holds for $P^2(\mathbf{T})$, which shares a face with $P(\mathbf{T})$. The three tetrahedra **T**, $P(\mathbf{T})$, and $P^2(\mathbf{T})$ together form a finite piece of the prism with parallel top and bottom faces (p_0, p_1, p_2) and $P^3(p_0, p_1, p_2)$. Since P^3 is a vertical translation, the result follows immediately. □

PROPOSITION 3.4
The images of $\mathbf{T}$ *under* Γ_∞ *tessellate* $\mathfrak{N}$. *Moreover,* Γ_∞ *has the presentation*

$$\Gamma_\infty = \langle P,\ Q \mid (PQ^{-1})^6 = 1,\ P^3 = Q^2 \rangle.$$

Proof
Let $\mathbf{T}_*$ be the equilateral triangle with vertices 0, 1, and $-\omega$ in $\mathbb{C}$. The complex plane is tessellated by images of this equilateral triangle, each of which consists of three copies of a fundamental domain for $\Delta(2,3,6) = \Pi_*(\Gamma_\infty)$. The preimage of $\mathbf{T}_*$ under Π is tessellated by images of $\mathbf{T}$. Applying an appropriate word in Γ_∞, we see that the preimages under Π of each of the other equilateral triangles are also tessellated by images of $\mathbf{T}$. Hence the images of $\mathbf{T}$ under Γ_∞ cover $\mathfrak{N}$.

It remains to check which words in Γ_∞ give rise to the same tetrahedron. Suppose that A and B are two such words. Then the words $\Pi_*(A)$ and $\Pi_*(B)$ give the same element of $\Delta(2,3,6)$. In other words, $\Pi_*(AB^{-1})$ is in the normal closure of the group generated by $\Pi_*(P^3)$, $\Pi_*(Q^2)$, $\Pi_*((PQ^{-1})^6)$. Because $\ker(\Pi_*) = \langle P^3 \rangle$ is central, we see that AB^{-1} is the corresponding word in the normal closure of P^3, Q^2, $(PQ^{-1})^6$ times a power of P^3. Since $P^3 = Q^2$ and $(PQ^{-1})^6 = 1$, we see that AB^{-1} is a power of P^3. (We have again used the fact that P^3 is central.) Since A and B gave rise to the same tetrahedron and since P^3 is a translation, we see that $AB^{-1} = 1$. Hence the images of $\mathbf{T}$ under Γ_∞ have disjoint interiors, and so they tessellate $\mathfrak{N}$. □

3.2. Generators for $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$
As in Section 2.3, let R be given by

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{11}$$

Recall that R has isometric sphere S_0 given by (5), which we equip with geographical coordinates. Observe that R maps S_0 to itself, sending the point with coordinates (r,θ,α) to the point with coordinates $(r,-\theta,\alpha)$, fixing the slice of S_0 corresponding to $\theta = 0$. Moreover, R swaps the inside and the outside of S_0. Similarly, PQ^{-1} maps S_0 to itself and sends the point (r,θ,α) to $(r,\theta,\alpha-\pi/3)$, fixing the spine of S_0.

We now show that adjoining R to Γ_∞ gives the full Eisenstein-Picard modular group.

THEOREM 3.5
The Eisenstein-Picard modular group $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ *is generated by* P, Q, *and* R.

Proof
We first show that $\langle P, Q, R\rangle$ has only one cusp. (The fact that $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ has only one cusp is already known; see [H2, page 30].) Our fundamental domain for $\Gamma_\infty = \langle P, Q\rangle$ is an affine simplex $\mathbf{T}$ whose vertices all lie inside the Heisenberg sphere $||z|^2 + it| = 2$. Since this Heisenberg sphere is convex, the whole of $\mathbf{T}$ lies inside the sphere. There is a fundamental domain for $\langle P, Q, R\rangle$ lying outside the isometric sphere of R and inside the fundamental domain (in $\mathbf{H}^2_{\mathbb{C}}$) for $\langle P, Q\rangle$. This intersection meets $\partial\mathbf{H}^2_{\mathbb{C}}$ only in q_∞. Hence $\langle P, Q, R\rangle$ has only one cusp.

Clearly, the group generated by P, Q, R is a subgroup of $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$. As both groups have cofinite volume, $\langle P, Q, R\rangle$ must have finite index, say, d, in $\Gamma = \mathrm{PU}(2,1;\mathbb{Z}[\omega])$. Hence the stabiliser of q_∞ in $\langle P, Q, R\rangle$ must have index d in Γ_∞ as well. Since the stabiliser of q_∞ in both groups is $\langle P, Q\rangle$, we must have $d = 1$, and so $\langle P, Q, R\rangle = \mathrm{PU}(2,1;\mathbb{Z}[\omega])$. □

We remark that in [Pi2, page 181], Picard gave generators for the congruence subgroup of Γ comprising those $T \in \Gamma$ such that the entries of $T - I$ lie in $i\sqrt{3}\,\mathbb{Z}[\omega]$ (see [H2, Proposition 6.3.13]). In terms of our generators, matrices corresponding to Picard's generators are

$$(P^{-1}QP^{-1})^2 = \begin{bmatrix} 1 & \omega-\overline{\omega} & \overline{\omega}-1 \\ 0 & \overline{\omega} & 1-\omega \\ 0 & 0 & 1 \end{bmatrix},$$

$$(Q^{-1}P)^2 = \begin{bmatrix} 1 & 1-\overline{\omega} & \overline{\omega}-1 \\ 0 & \overline{\omega} & 1-\overline{\omega} \\ 0 & 0 & 1 \end{bmatrix},$$

$$(QP^{-1})^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(RPQ)^2 = \begin{bmatrix} 1 & 0 & \omega-\overline{\omega} \\ 0 & \overline{\omega} & 0 \\ \omega-\overline{\omega} & 0 & -2 \end{bmatrix},$$

$$RPQ^{-1}(P^{-1}QP^{-1})^2QP^{-1}R = \begin{bmatrix} 1 & 0 & 0 \\ \omega-\overline{\omega} & \overline{\omega} & 0 \\ \overline{\omega}-1 & 1-\omega & 1 \end{bmatrix}.$$

4. A fundamental domain for $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$

We now construct a fundamental domain of $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$. A priori, there is no reason to expect that the fundamental domain is the intersection of the outside of the isometric sphere S_0 of R with the fundamental domain we have already constructed for Γ_∞. Indeed, this is not the case.

For example, consider the map $P^2Q^{-1}RP$ and, for small δ, the point

$$z_\delta = \begin{bmatrix} -1-\delta \\ 1-\omega-i\overline{\omega} \\ 1 \end{bmatrix}.$$

Then, after scaling so that its last coordinate is 1, the point $P^2Q^{-1}RP(z_\delta)$ is

$$\begin{bmatrix} \omega & -1 & 1 \\ -\omega & 1-\omega & \omega \\ 1 & 1 & \omega \end{bmatrix} \begin{bmatrix} -1-\delta \\ 1-\omega-i\overline{\omega} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -1+i\delta \\ 1-\omega-i\overline{\omega}+\delta+i\omega\delta \\ 1 \end{bmatrix} + O(\delta^2).$$

($O(\delta^2)$ denotes a vector in $\mathbb{C}^3$ whose entries have absolute values bounded by a constant multiple of δ^2 for small δ.) For sufficiently small δ, both z_δ and $P^2Q^{-1}RP(z_\delta)$ lie outside S_0 and inside the fundamental domain we constructed for Γ_∞.

In fact, we show that by making suitable modifications to the fundamental domain of Γ_∞, it is possible to produce a fundamental domain for Γ that is the intersection of a fundamental domain for Γ_∞ with the outside of S_0. If this is the case, then it is clearly necessary that the points of S_0 in the boundary of our fundamental domain lie outside every other isometric sphere.

The modifications consist of introducing totally geodesic skeletons whenever possible. The vertices of the fundamental domain are the same as those for the intersection of S_0 (the isometric sphere of R) with the fundamental domain we have already constructed for Γ_∞. The edges are geodesics joining the vertices (the point q_∞ is an ideal vertex). The 2-faces are totally geodesic whenever possible. In our case, as all 2-faces are triangles, they are totally geodesic if and only if their three vertices are contained in a totally geodesic subspace. The triangles containing the ideal vertex are foliated by geodesics starting at the ideal vertex and arriving at the opposite edge.

To determine the remaining 2-faces and 3-faces, we observe that the finite edges (those not containing the ideal vertex) are all contained in the isometric sphere S_0. Two of the 2-faces are meridians of S_0, and the two remaining 2-faces are defined as intersections of S_0 with appropriate images of themselves by elements of Γ_∞. In this way, we guarantee the pairing between the faces.

One of the 3-faces (the finite one) is contained in S_0. The other four 3-faces are cones based at the 2-faces of that 3-face with the cone point the ideal vertex.

To this end, we begin by investigating the intersection of S_0 with its neighbouring isometric spheres.

4.1. The intersection of S_0 and its neighbours

We have already considered the points $p_n \in \mathfrak{N}$ for $n = 0, \ldots, 3$. Consider the geodesic γ_n through p_n with one end q_∞, and let z_n be the intersection of γ_n with S_0. Then

$$z_0 = \begin{bmatrix} \overline{\omega} \\ 0 \\ 1 \end{bmatrix}, \qquad z_1 = \begin{bmatrix} -1 \\ -\omega \\ 1 \end{bmatrix}, \qquad z_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \qquad z_3 = \begin{bmatrix} \omega \\ 0 \\ 1 \end{bmatrix}.$$

In horospherical coordinates, these fixed points are given by

$$z_0 = (0, -\sqrt{3}, 1), \qquad z_1 = (-\omega, 0, 1), \qquad z_2 = (1, 0, 1), \qquad z_3 = (0, \sqrt{3}, 1).$$

We see that these points all lie in the horosphere H_1. Making the canonical identification between H_1 and $\mathfrak{N}$ identifies z_n with p_n for $n = 0,\ 1,\ 2,\ 3$. Instead of joining these vertices with affine subspaces to form the simplex $\mathbf{T}$ in $\mathfrak{N}$ as we did before, we now join them with subspaces reflecting the geometry of complex hyperbolic space to obtain a simplex $\mathbf{T}_0$ contained in S_0.

In terms of the geographical coordinates on S_0, these points are given by the following.

- The point z_0 has $r = 0$, so it lies on the spine of S_0 and on the slice of S_0 with $\theta = \pi/3$.
- The point z_1 has $r = 1$ and lies on the slice of S_0 with $\theta = 0$ and the meridian with $\alpha = -\pi/3$.
- The point z_2 has $r = 1$ and lies on the slice of S_0 with $\theta = 0$ and the meridian with $\alpha = 0$.
- The point z_3 has $r = 0$, so it lies on the spine of S_0 and on the slice of S_0 with $\theta = -\pi/3$.

We observe that since $p_n = P^n(p_0)$ for $n = 0, \ldots, 3$ and since the points z_n all lie on a horosphere, we immediately have $z_n = P^n(z_0)$. Alternatively, we could have verified this directly. This means that $P^{-m}(z_n) = z_{n-m}$ lies on $P^{-m}(S_0)$ for each $n - 3 \leq m \leq n$. This immediately gives the following lemma.

LEMMA 4.1

We have

$$z_0 \in S_0 \cap P^{-1}(S_0) \cap P^{-2}(S_0) \cap P^{-3}(S_0), \qquad z_1 \in P(S_0) \cap S_0 \cap P^{-1}(S_0) \cap P^{-2}(S_0),$$
$$z_2 \in P^2(S_0) \cap P(S_0) \cap S_0 \cap P^{-1}(S_0), \qquad z_3 \in P^3(S_0) \cap P^2(S_0) \cap P(S_0) \cap S_0.$$

For each pair of distinct $m,\ n \in \{0, 1, 2, 3\}$, let $\gamma_{mn} = \gamma_{nm}$ be the geodesic arc joining z_n and z_m.

LEMMA 4.2
We have

$$\begin{aligned} &\gamma_{01} \subset S_0 \cap P^{-1}(S_0) \cap P^{-2}(S_0), && \gamma_{12} \subset P(S_0) \cap S_0 \cap P^{-1}(S_0), \\ &\gamma_{23} \subset P^2(S_0) \cap P(S_0) \cap S_0, && \gamma_{02} \subset S_0 \cap P^{-1}(S_0), \\ &\gamma_{13} \subset P(S_0) \cap S_0, && \gamma_{03} \textit{ in the spine of } S_0. \end{aligned}$$

Proof
As z_0 and z_3 lie on the spine of S_0, then, by definition, so does the geodesic arc joining them. Hence z_0 and z_3 must lie on every meridian.

The points z_0, z_3, and z_1 all lie on the meridian of S_0 with $\alpha = -\pi/3$. Since meridians are totally geodesic, this implies that γ_{01} and γ_{13} both lie on this meridian. Applying P, we see that $z_1 = P(z_0)$ and $z_2 = P(z_1)$ lie on a meridian of $P(S_0)$. Hence γ_{12} lies on this meridian. Similarly, γ_{23} lies on a meridian of $P^2(S_0)$. Applying P^{-1}, we see that $z_0 = P^{-1}(z_1)$ and $z_2 = P^{-1}(z_3)$ lie on the same meridian of $P^{-1}(S_0)$. Hence γ_{02} lies on this meridian.

Likewise, γ_{02} and γ_{23} lie on the meridian of S_0 with $\alpha = 0$. Applying powers of P, we see that γ_{13} lies on a meridian of $P(S_0)$, γ_{12} lies on a meridian of $P^{-1}(S_0)$, and γ_{01} lies on a meridian of $P^{-2}(S_0)$.

Observe that z_1 and z_2 lie on the slice of S_0 with $\theta = 0$. Since slices are totally geodesic, we see that γ_{12} lies on this slice. Applying P, we see that γ_{23} lies on a slice of $P(S_0)$; likewise, γ_{01} lies on a slice of $P^{-1}(S_0)$.

Putting all this together gives the result. □

We now investigate the intersection of S_0 and $S_{-1} = P^{-1}(S_0)$ a little more closely. A brief computation shows that S_{-1} is given by

$$S_{-1} = \left\{(z, t, u) \in \mathbf{H}^2_{\mathbb{C}} : \left||z|^2 + u - it - 2z - 2\omega\right| = 2 \right\}. \tag{12}$$

LEMMA 4.3
A point (r, θ, α) of S_0 written in geographical coordinates with $-\pi/3 \le \alpha \le 0$ does not intersect the interior of S_{-1}, provided that

$$r \le 2\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)}$$

with equality if and only if the point lies in $S_0 \cap S_{-1}$.

Proof
Changing to geographical coordinates in (12), we see that a point of S_0 does not intersect the interior of S_{-1} if and only if

$$1 \leq |e^{i\theta} - re^{i(\alpha+\theta/2)} + e^{-i\pi/3}| = \left| re^{i(\alpha+\pi/6)} - 2\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right) \right| \tag{13}$$

with equality if and only if the point lies on $S_0 \cap S_{-1}$. Expanding out the right-hand side, we see that this is equivalent to

$$0 \leq r^2 - 4r\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) + 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right) - 1.$$

This is satisfied for all points of S_0 with

$$r \leq 2\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)}$$

or

$$r \geq 2\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) + \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)}.$$

We claim that when $-\pi/3 \leq \alpha \leq 0$, the second of these solutions is always greater than $\sqrt{2\cos(\theta)}$ and so does not correspond to a point of S_0. In order to see this, observe that $-\pi/3 \leq \alpha \leq 0$ implies $2\cos(\alpha + \pi/6) \geq \sqrt{3}$ and $4\sin^2(\alpha + \pi/6) \leq 1$. Thus

$$\begin{aligned}
&2\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) + \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)} \\
&\geq \sqrt{3}\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right) + \sin\left(\frac{\theta}{2} + \frac{\pi}{6}\right) \\
&= 2\cos\left(\frac{\theta}{2}\right) \\
&= \sqrt{2\cos(\theta) + 2} \\
&> \sqrt{2\cos(\theta)}.
\end{aligned}$$

This proves the result. □

We can now characterise the geodesic arcs γ_{mn} in terms of geographical coordinates.

LEMMA 4.4
In terms of geographical coordinates, we have the following.

- *The geodesic arc* γ_{01} *consists of those points of* S_0 *with* $\alpha = -\pi/3$, $r = 2\cos(\theta/2 + \pi/3)$, *and* $0 \leq \theta \leq \pi/3$.

- *The geodesic arc γ_{12} consists of those points of S_0 with $\theta = 0$, $-\pi/3 \le \alpha \le 0$, and*

$$r = \sqrt{3}\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 3\sin^2\left(\alpha + \frac{\pi}{6}\right)};$$

that is, $re^{i\alpha}$ lies on the circle centred at $1-\omega$ of radius 1.
- *The geodesic arc γ_{02} consists of those points of S_0 with $r = 2\cos(\theta/2 + \pi/3)$, $\alpha = 0$, and $0 \le \theta \le \pi/3$.*
- *The geodesic arc γ_{23} consists of those points of S_0 with $r = 2\cos(\theta/2 - \pi/3)$, $\alpha = 0$, and $-\pi/3 \le \theta \le 0$.*
- *The geodesic arc γ_{13} consists of those points of S_0 with $r = 2\cos(\theta/2 - \pi/3)$, $\alpha = -\pi/3$, and $-\pi/3 \le \theta \le 0$.*
- *The geodesic arc γ_{03} consists of those points of S_0 with $-\pi/3 \le \theta \le \pi/3$ and $r = 0$.*

Proof
Since γ_{03} lies in the spine of S_0, its expression in geographical coordinates follows immediately.

We have already seen that γ_{01}, γ_{12}, and γ_{02} all lie in $S_0 \cap S_{-1}$. We know that $\alpha = -\pi/3$ for each point of γ_{01}. Substituting into Lemma 4.3 and requiring equality gives

$$r = \sqrt{3}\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right) - \sqrt{1 - \cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)} = 2\cos\left(\frac{\theta}{2} + \frac{\pi}{3}\right).$$

We know that $\theta = \pi/3$ at z_0 and $\theta = 0$ at z_1. This gives the first part.

The coordinates for γ_{02} follow similarly, using $\alpha = 0$.

The geodesic arc γ_{12} lies in the slice of S_0 given by $\theta = 0$. We know that $\alpha = -\pi/3$ at z_1 and $\alpha = 0$ at z_0, and so $-\pi/3 \le \alpha \le 0$ on γ_{12}. Using Lemma 4.3 and setting $\theta = 0$ gives

$$r = \sqrt{3}\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 3\sin^2\left(\alpha + \frac{\pi}{6}\right)}.$$

Recall that, as in Section 3.2, R acts on S_0 by $R : (r, \theta, \alpha) \longmapsto (r, -\theta, \alpha)$, and so $R(z_0) = z_3$ and R fixes z_1 and z_2. Thus to find γ_{13} and γ_{23}, we should replace θ with $-\theta$ in the expressions for γ_{01} and γ_{02}, respectively. This gives the result. □

4.2. The basic tetrahedron
We are now ready to define the tetrahedron $\mathbf{T}_0$.

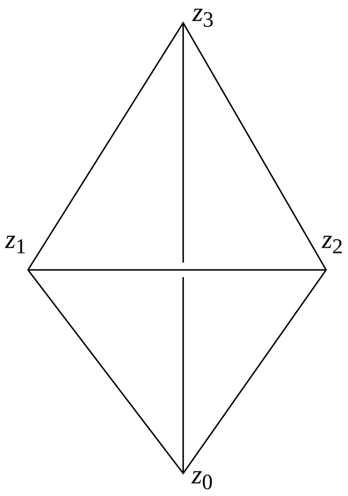

Figure 3. A schematic view of the 1-skeleton of the basic tetrahedron $\mathbf{T}_0$.
The fundamental domain $\mathbf{D}$ has a boundary that is a union of five tetrahedra: $\mathbf{T}_0$ and four tetrahedra constructed as cones at ∞ over the four faces of $\mathbf{T}_0$.

Definition 4.5
Using geographical coordinates from (7), the tetrahedron $\mathbf{T}_0$ comprises those points of S_0 for which $-\pi/3 \le \theta \le \pi/3$, $-\pi/3 \le \alpha \le 0$, and

$$0 \le r \le 2\cos\left(\frac{|\theta|}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 4\cos^2\left(\frac{|\theta|}{2} + \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)}. \tag{14}$$

A schematic view of $\mathbf{T}_0$ is given in Figure 3, and a realistic view is given in Figure 4. The faces of $\mathbf{T}_0$ are defined as follows.

- The face F_1 of $\mathbf{T}_0$ is its intersection with the meridian given by $\alpha = 0$. Therefore, its points are parametrised by $-\pi/3 \le \theta \le \pi/3$ and

$$0 \le r \le 2\cos\left(\frac{|\theta|}{2} + \frac{\pi}{3}\right).$$

- The face F_2 of $\mathbf{T}_0$ is its intersection with the meridian given by $\alpha = -\pi/3$. Thus its points are parametrised by $-\pi/3 \le \theta \le \pi/3$ and

$$0 \le r \le 2\cos\left(\frac{|\theta|}{2} + \frac{\pi}{3}\right).$$

- The face F_3 of $\mathbf{T}_0$ is its intersection with $S_{-1} = P^{-1}(S_0)$. Therefore, its points are parametrised by $0 \le \theta \le \pi/3$, $-\pi/3 \le \alpha \le 0$, and

$$r = 2\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} + \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)}.$$

- The face F_4 of $\mathbf{T}_0$ is its intersection with $P(S_0)$. Therefore, its points are parametrised by $-\pi/3 \le \theta \le 0$, $-\pi/3 \le \alpha \le 0$, and

$$r = 2\cos\left(\frac{\theta}{2} - \frac{\pi}{6}\right)\cos\left(\alpha + \frac{\pi}{6}\right) - \sqrt{1 - 4\cos^2\left(\frac{\theta}{2} - \frac{\pi}{6}\right)\sin^2\left(\alpha + \frac{\pi}{6}\right)}.$$

It is clear that the edges of $\mathbf{T}_0$ are the geodesic arcs γ_{mn} for distinct $m, n \in \{0, 1, 2, 3\}$ as defined, and its vertices are the points z_0, z_1, z_2, z_3. In particular, we have

$$\gamma_{01} = F_2 \cap F_3, \quad \gamma_{12} = F_3 \cap F_4, \quad \gamma_{02} = F_1 \cap F_3, \quad \gamma_{03} = F_1 \cap F_2,$$
$$\gamma_{13} = F_2 \cap F_4, \quad \gamma_{23} = F_1 \cap F_4, \quad z_0 = F_1 \cap F_2 \cap F_3, \quad z_1 = F_2 \cap F_3 \cap F_4,$$
$$z_2 = F_1 \cap F_3 \cap F_4, \quad z_3 = F_1 \cap F_2 \cap F_4.$$

PROPOSITION 4.6
The involution R maps $\mathbf{T}_0$ to itself. Moreover, $(PQ^{-1})^{-1}(\mathbf{T}_0) \cap \mathbf{T}_0 = F_1$, and PQ^{-1} maps F_1 to F_2; likewise, $P^{-1}(\mathbf{T}_0) \cap \mathbf{T}_0 = F_3$, and P maps F_3 to F_4.

Proof
This follows from the formulae (11) for R, (10) for PQ^{-1}, and (8) for P. □

In Figure 4, we see the edges γ_{mn} using isometric coordinates; that is, we parametrise the S_0 by (z, t), so that $u = \sqrt{4 - t^2} - |z|^2$.

LEMMA 4.7
All points of $\mathbf{T}_0$ satisfy $r \le 2\cos(|\theta|/2 + \pi/3)$ with equality only when $\alpha = 0$ or $-\pi/3$.

Proof
The result follows by examining how inequality (14) varies with α for $-\pi/3 \le \alpha \le 0$. □

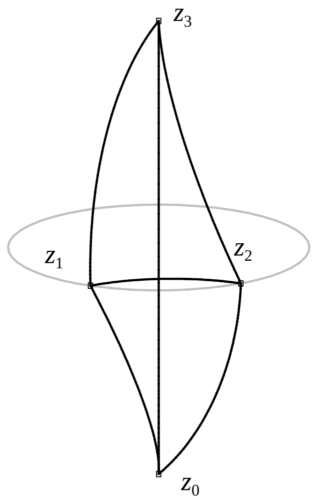

Figure 4. A realistic view of the 1-skeleton of the basic tetrahedron $\mathbf{T}_0$ inside the isometric sphere S_0 of R in adapted coordinates

LEMMA 4.8
All points of $\mathbf{T}_0$ *satisfy* $u \geq 1$ *with equality only at the vertices.*

Proof
From (7), we see that $u = 2\cos(\theta) - r^2$. Using the bound $r \leq 2\cos(|\theta|/2 + \pi/3)$ from Lemma 4.7, we see that

$$\begin{aligned} u &\geq 2\cos(|\theta|) - 4\cos^2\left(\frac{|\theta|}{2} + \frac{\pi}{3}\right) \\ &= 2\cos(|\theta|) - 2\cos\left(|\theta| + \frac{2\pi}{3}\right) - 2 \\ &= 2\sqrt{3}\sin\left(|\theta| + \frac{\pi}{3}\right) - 2 \\ &\geq 1, \end{aligned}$$

where equality in the first line happens only when $\alpha = 0$ or $-\pi/3$ and where equality in the last line is attained for $\theta = 0$ or $\theta = \pm\pi/3$. The result follows. □

LEMMA 4.9
If $(r, \theta, \alpha) \in \mathbf{T}_0$, *then for each* $k = 0, \ldots, 5$,

$$|re^{i(\alpha+\theta/2)} - \sqrt{3}e^{-i(\pi/6+k\pi/3)}| \geq 1.$$

Proof
When $\theta = 0$, we have

$$|re^{i\alpha} - \sqrt{3}e^{-i(\pi/6+k\pi/3)}| \geq |re^{i\alpha} - \sqrt{3}e^{-i\pi/6}| \geq 1$$

by putting $\theta = 0$ in (13).

Fix $0 < \theta \leq \pi/3$, and consider the $re^{i\alpha}$ plane. The intersection of this plane with $\mathbf{T}_0$ is the region

$$\mathbf{T}_0(\theta) = \Big\{re^{i\alpha} : -\frac{\pi}{3} \leq \alpha \leq 0,\ 0 \leq r \leq 2\cos\Big(\frac{\theta}{2} + \frac{\pi}{3}\Big), \\ \Big|re^{i\alpha} - 2\cos\Big(\frac{\theta}{2} + \frac{\pi}{6}\Big)e^{-i\pi/6}\Big| \geq 1\Big\}.$$

(We have used Lemma 4.7.) We need to show that points in $\mathbf{T}_0(\theta)$ satisfy

$$|re^{i\alpha} - \sqrt{3}e^{-i(\theta/2+\pi/6+k\pi/3)}| \geq 1.$$

Let C_k be the circle defined by $\{|re^{i\alpha} - \sqrt{3}e^{-i(\theta/2+\pi/6+k\pi/3)}| = 1\}$. An easy calculation shows that

$$\Big|2\cos\Big(\frac{\theta}{2} + \frac{\pi}{3}\Big) - \sqrt{3}e^{-i(\theta/2+\pi/6+k\pi/3)}\Big| = |e^{i\theta} + e^{2i\pi/3} + i\sqrt{3}e^{-ik\pi/3}| > 1.$$

Since $\pi/6 < \theta/2 + \pi/6 \leq \pi/3$, we see that C_k intersects the disc of radius $2\cos(\theta/2 + \pi/3)$ in the interval where $-(k+1)/3 < \alpha < -k\pi/3$. In particular, for $k = 1, \ldots, 5$, the circle C_k does not intersect the sector where $0 \leq r \leq 2\cos(\theta/2 + \pi/3)$ and $-\pi/3 \leq \alpha \leq 0$ and, hence, does not intersect $\mathbf{T}_0(\theta)$.

We now consider the circle C_0. It intersects the circle $\{|re^{i\alpha} - 2\cos(\theta/2 + \pi/6)e^{-i\pi/6}| = 1\}$ in the points $e^{-i(\theta/2+\pi/3)}$ and $2\cos(\theta/2) + e^{-i(\theta/2+\pi/3)}$. Both points have modulus greater than $2\cos(\theta/2 + \pi/3)$, and therefore, points of C_0 either have $|re^{i\alpha} - 2\cos(\theta/2 + \pi/6)e^{-i\pi/6}| < 1$ or $r > 2\cos(\theta/2 + \pi/3)$. Hence, C_0 does not intersect $\mathbf{T}_0$. This gives the result for each $0 \leq \theta \leq \pi/3$.

When $-\pi/3 \leq \theta < 0$,

$$\mathbf{T}_0(\theta) = \Big\{re^{i\alpha}\ :\ -\frac{\pi}{3} \leq \alpha \leq 0,\ 0 \leq r \leq 2\cos\Big(\frac{\theta}{2} - \frac{\pi}{3}\Big), \\ \Big|re^{i\alpha} - 2\cos\Big(\frac{\theta}{2} - \frac{\pi}{6}\Big)e^{-i\pi/6}\Big| \geq 1\Big\}.$$

The result follows in this case by applying the arguments above but replacing α with $-\alpha - \pi/3$ and θ with $-\theta$. □

LEMMA 4.10
The tetrahedron $\mathbf{T}_0$ *is a three-dimensional simplex embedded in* $\mathbf{H}^2_{\mathbb{C}}$.

Proof

Points of S_0 with distinct geographical coordinates correspond to distinct points of $\mathbf{H}^2_{\mathbb{C}}$. Since $\mathbf{T}_0$ is a three-dimensional simplex in the space of geographical coordinates, the result follows. □

LEMMA 4.11

The only elements of Γ_∞ mapping S_0 to itself are powers of PQ^{-1}.

Proof

If $T \in \Gamma_\infty$ maps S_0 to itself, then T must fix $(0,0,0)$, the centre of S_0. Thus T is diagonal. Using the fact that T is in PU(2, 1) and that the entries of T lie in $\mathbb{Z}[\omega]$, we immediately see that T is a power of PQ^{-1}. □

PROPOSITION 4.12

The interior of $\mathbf{T}_0$ is disjoint from all images of S_0 under $\Gamma_\infty - \langle PQ^{-1}\rangle$.

Proof

Suppose that (z, t, u) lies both on $\mathbf{T}_0$ and on an isometric sphere of radius $\sqrt{2}$ with centre $(z_0, t_0, 0) \neq (0, 0, 0)$. That is,

$$(|z|^2 + u)^2 + t^2 = (|z - z_0|^2 + u)^2 + \big(t - t_0 + 2\,\mathrm{Im}\,(z\bar{z}_0)\big)^2 = 4,$$

or, using geographical coordinates,

$$1 = \left| e^{i\theta} - re^{i(\theta/2+\alpha)}\bar{z}_0 + \frac{(|z_0|^2 + it_0)}{2} \right|.$$

Moreover, z_0 and $(|z_0|^2 + it_0)/2$ must both lie in $\mathbb{Z}[\omega]$.

Since $(|z|^2 + u)^2 + t^2 = 4$ and $u \geq 1$ (from Lemma 4.8), we have $|z| \leq 1$ and $|z|^4 + t^2 \leq 3 - 2|z|^2$. Similarly, $|z - z_0| \leq 1$ and $|z - z_0|^4 + \big(t - t_0 + 2\,\mathrm{Im}(z\bar{z}_0)\big)^2 \leq 3 - 2|z - z_0|^2$. Thus

$$\begin{aligned}
\big||z_0|^2 + it_0\big| &= \big||z - z_0|^2 - it + it_0 - 2i\,\mathrm{Im}(z\bar{z}_0) + |z|^2 + it - 2z(\bar{z} - \bar{z}_0)\big| \\
&\leq \big||z - z_0|^2 - it + it_0 - 2i\,\mathrm{Im}(z\bar{z}_0)\big| + \big||z|^2 + it\big| + 2|z|\,|z - z_0| \\
&\leq \sqrt{3 - 2|z - z_0|^2} + \sqrt{3 - 2|z|^2} + 2|z|\,|z - z_0| \\
&\leq 4
\end{aligned}$$

with equality in the last line if and only if $|z| = |z - z_0| = 1$. Thus we need to investigate the intersection of S_0 with isometric spheres centred at $(z_0, t_0, 0)$, where z_0 and $(|z_0|^2 - it_0)/2$ are both in $\mathbb{Z}[\omega]$ and $||z_0|^2 - it_0| \leq 4$. This immediately implies that $||z_0|^2 - it_0|$ equals 2, $2\sqrt{3}$, or 4.

First, suppose that $||z_0|^2 - it_0| = 2$. Therefore, $(|z_0|^2 - it_0)/2$ is a power of $-\omega$. This implies that $z_0 = (-\omega)^k$ for $k = 0, \ldots, 5$ and $t_0 = \pm\sqrt{3}$. Suppose that (r, θ, α) lies on both S_0 and the image of S_0 with centre $z_0 = (-\omega)^k$, $t_0 = \pm\sqrt{3}$. Then

$$1 = |e^{i\theta} + e^{\pm i\pi/3} - re^{i(\alpha+\theta/2+\pi k/3)}| = \left| re^{i\alpha} - 2\cos\left(\frac{\theta}{2} \mp \frac{\pi}{6}\right) e^{-i(\pi k/3 \mp \pi/6)} \right|.$$

If $(r, \theta, \alpha) \in \mathbf{T}_0$, then we must have

$$1 \le \left| re^{i\alpha} - 2\cos\left(\frac{\theta}{2} \pm \frac{\pi}{6}\right) e^{-i\pi/6} \right|$$

for both choices of sign. Combining these, we see that $re^{i\alpha}$ is at least as close (with respect to the Euclidean metric on $\mathbb{C}$) to $2\cos(\theta/2 \mp \pi/6)e^{-i(\pi k/3 \mp i\pi/6)}$ as to $2\cos(\theta/2 \mp \pi/6)e^{-i\pi/6}$. Since $-\pi/3 \le \alpha \le 0$, we must have $k = 1/2 \pm 1/2$, and so $(z_0, t_0, 0) = (1, -\sqrt{3}, 0) = P^{-1}(0,0,0)$ or $(-\omega, \sqrt{3}, 0) = P(0,0,0)$. Hence (r, θ, α) lies on F_3 or F_4. In particular, it does not lie in the interior of $\mathbf{T}_0$.

Second, suppose that $||z_0|^2 - it_0| = 2\sqrt{3}$. Then either $|z_0| = \sqrt{3}$ and $t_0 = \pm\sqrt{3}$ or else $z_0 = 0$ and $t_0 = \pm 2\sqrt{3}$.

In the former case, $z_0 = (1-\omega)(-\omega)^k = \sqrt{3}e^{-i(\pi/6 + k\pi/3)}$ for some $k = 0, \ldots, 5$. Using Lemmas 4.9 and 4.8, we see that if (z, t, u) lies in $\mathbf{T}_0$, then $|z - z_0| \ge 1$ and $u \ge 1$. In the latter case, we only have equality at the vertices. This implies

$$(|z - z_0|^2 + u)^2 \ge 4$$

with strict inequality except at the vertices. Thus the interiors of the tetrahedra are disjoint.

If $z_0 = 0$ and $t_0 = \pm 2\sqrt{3}$, then we have

$$(|z|^2 + u)^2 + t^2 = (|z|^2 + u)^2 + (t \mp 2\sqrt{3})^2 = 4.$$

The only solutions with $u \ge 1$ are $(0, \pm\sqrt{3}, 1)$, that is, the points z_0 and z_3.

Finally, suppose that $||z_0|^2 - it_0| = 4$. Since z_0 and $(|z_0|^2 + it_0)/2$ are both in $\mathbb{Z}[\omega]$, the only possibility in this case is $|z_0| = 2$, $t_0 = 0$. However, we know that $|z| \le 1$ and $|z - z_0| \le 1$ with equality only when $u = 1$. Using the triangle inequality, we see that $u = 1$, and the interior of $\mathbf{T}_0$ does not intersect this isometric sphere. □

4.3. The four-dimensional simplex

We now define tetrahedra $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$. Each of these is the geodesic cone from q_∞ over the union of faces F_1, F_2, F_3, and F_4 of $\mathbf{T}_0$. To be precise, the tetrahedron $\mathbf{T}_1$ is defined to be the union over all points p of F_1 of the geodesic arc joining p to q_∞, and it is likewise for $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$.

PROPOSITION 4.13
The tetrahedra $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, *and* $\mathbf{T}_4$ *are three-dimensional simplices embedded in* $\mathbf{H}^2_{\mathbb{C}} \cup \{q_\infty\}$.

Proof
It suffices to show that vertical projection Π maps each face of $\mathbf{T}_0$ bijectively onto its image. Equivalently, given a point on $\partial\mathbf{T}_0$ with horospherical coordinates (z, t, u), u is then specified by z and t. Since $\mathbf{T}_0$ is contained in S_0, we have $u = \sqrt{4 - t^2} - |z|^2$. □

By construction, the intersection of $\mathbf{T}_0$ with each of the tetrahedra $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$ is nothing other than the corresponding face of $\mathbf{T}_0$. Similarly, each pair of tetrahedra from $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$ intersects in a two-dimensional subset formed by the geodesic cone from q_∞ of the edges $\gamma_{12}, \ldots, \gamma_{03}$. Finally, each triple of $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$ intersects in the geodesic arc joining the appropriate vertex of $\mathbf{T}_0$ with q_∞.

We define the four-dimensional simplex $\mathbf{D}$ to be the geodesic cone from q_∞ of the tetrahedron $\mathbf{T}_0$. By the same argument given in Proposition 4.13, we see that $\mathbf{D}$ is an embedded 4-simplex in $\mathbf{H}^2_{\mathbb{C}} \cup \{q_\infty\}$. Moreover, $\mathbf{D}$ has five three-dimensional faces, namely, $\mathbf{T}_0$, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$. The goal of this section is to show that $\mathbf{D}$ is a fundamental domain for the Eisenstein-Picard modular group.

PROPOSITION 4.14
The interior of the domain $\mathbf{D}$ *lies outside all isometric spheres of elements of* $\Gamma - \Gamma_\infty$.

Proof
Let $A \in \Gamma - \Gamma_\infty$ be written in the form (2). By definition, the radius of the isometric sphere of A is $\sqrt{2/|g|}$. Since $g \in \mathbb{Z}[\omega]$, we see that $|g|$ is 1, $\sqrt{3}$, or at least 2.

Suppose that (z, t, u) is on an isometric sphere with centre $(z_0, t_0, 0)$ and radius at most 1 (that is, $|g| \geq 2$). Then

$$\left| |z - z_0|^2 + u - it + it_0 - 2i\,\mathrm{Im}(z\bar{z}_0) \right| \leq 1.$$

It is clear that $u \leq 1$, and so (z, t, u) cannot lie in the interior of $\mathbf{T}_0$.

Second, suppose that $A \in \Gamma - \Gamma_\infty$ has isometric sphere of radius $\sqrt{2}$. That is, $|g| = 1$. Then $g = (-\omega)^k$. So as a vector in $\mathbb{C}P^2$,

$$A^{-1}(\infty) = \begin{bmatrix} \bar{j} \\ \bar{h} \\ \bar{g} \end{bmatrix}.$$

We see that j/g and h/g both lie in $\mathbb{Z}[\omega]$. That is, $A^{-1}(\infty)$ lies in the Γ_∞-orbit of $R(q_\infty)$, and so our isometric sphere is the image of S_0 under an element of Γ_∞. Suppose that (z,t,u) lies in the interior of $\mathbf{D}$. Then there exists $u_1 \le u$, so that (z,t,u_1) lies in the interior of $\mathbf{T}_0$. But we know from Proposition 4.12 that $\mathbf{T}_0$ lies outside all Γ_∞-images of S_0 other than S_0. Since $u > u_1$, we see that (z,t,u) lies outside all isometric spheres of radius $\sqrt{2}$.

Finally, suppose that $A \in \Gamma - \Gamma_\infty$ has isometric sphere with radius $\sqrt{2/\sqrt{3}}$ and centre $(z_0, t_0, 0)$ (that is, $|g| = \sqrt{3}$). Again, we write $A^{-1}(\infty)$ as a vector in $\mathbb{C}P^2$, as in the previous equation, and observe that $g = i\sqrt{3}(-\omega)^k$ for some integer k. As A is in $\mathrm{PU}(2,1)$, we have

$$0 = j\overline{g} + |h|^2 + g\overline{j},$$

and so we see that $|h|^2$ is divisible by 3. Thus $h \in i\sqrt{3}\mathbb{Z}[\omega]$. In other words, $h/g \in \mathbb{Z}[\omega]$. Because h and g are both in $i\sqrt{3}\,\mathbb{Z}[\omega]$ and since $|\det(A)| = 1$, we see that $j \pm 1 \in i\sqrt{3}\,\mathbb{Z}[\omega]$. Thus $j/g \mp i/\sqrt{3}$ is in $\mathbb{Z}[\omega]$. Hence $(|z_0|^2 - it_0 \pm 2i/\sqrt{3})/2 \in \mathbb{Z}[\omega]$. In other words, $(z_0, t_0 \mp 2/\sqrt{3})$ is in the Γ_∞-orbit of $R(q_\infty) = (0,0,0)$.

We have

$$(|z - z_0|^2 + u)^2 + \big(t - t_0 + 2\,\mathrm{Im}(z\overline{z}_0)\big)^2 = \frac{4}{3}.$$

If $u > 1$, then

$$\big(t - t_0 + 2\,\mathrm{Im}(z\overline{z}_0)\big)^2 < \frac{4}{3} - 1 = \frac{1}{3}.$$

Therefore,

$$\begin{aligned}
&(|z - z_0|^2 + u)^2 + \left(t - t_0 + 2\,\mathrm{Im}\Big(z\overline{z}_0 \pm \frac{2}{\sqrt{3}}\Big)\right)^2 \\
&\quad = (|z - z_0|^2 + u)^2 + \big(t - t_0 + 2\,\mathrm{Im}(z\overline{z}_0)\big)^2 \pm \frac{4}{\sqrt{3}}\big(t - t_0 + 2\,\mathrm{Im}(z\overline{z}_0)\big) + \frac{4}{3} \\
&\quad < \frac{4}{3} + \frac{4}{3} + \frac{4}{3}.
\end{aligned}$$

Thus (z,t,u) lies inside the isometric sphere of radius $\sqrt{2}$ with centre $(z_0, t_0 \mp 2/\sqrt{3})$, that is, inside the image of S_0 under some element of Γ_∞. Using Proposition 4.12, we see that (z,t,u) is not in $\mathbf{T}_0$. □

THEOREM 4.15
The simplex $\mathbf{D}$ *is a fundamental domain for* $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$.

Proof

The proof follows the standard proof for the standard fundamental domain of PSL$(2, \mathbb{Z})$ (see, e.g., [L, pages 57 – 60]).

First, we show that every orbit has a point inside **D**. Let (z, t, u) be any point in $\mathbf{H}^2_{\mathbb{C}}$. By applying elements of Γ_∞, we may assume that (z, t, u) lies inside the fundamental domain for Γ_∞ obtained by extending the vertical geodesic arcs in **D** to $\partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$. If (z, t, u) also lies outside or on S_0, then it is already in **D**. Otherwise, (z, t, u) lies inside S_0, and applying R gives a point in the orbit of (z, t, u) whose horospherical height is strictly greater than u. We iterate this procedure. Using the proper discontinuity of the action of PU$(2, 1; \mathbb{Z}[\omega])$, we see that this process terminates after finitely many steps. The final point is in the orbit of (z, t, u), lies in a fundamental domain for Γ_∞, and has horospherical height maximal among all points in the orbit of (z, t, u). It must, therefore, lie outside or on S_0 and so be in **D**.

We now show that if two points in **D** differ by an element of A of PU$(2, 1; \mathbb{Z}[\omega])$, then they must lie in ∂**D** and be identified by a side-pairing map. By construction, all points of ∂**D** are the image of a point of ∂**D** under a side-pairing map.

Suppose that (z, t, u) lies in the interior of **D**. Since **D** lies in a fundamental domain for Γ_∞, all images of (z, t, u) under nontrivial elements of Γ_∞ lie outside **D**. From Proposition 4.14, we see that (z, t, u) lies outside all isometric spheres of elements of $\Gamma - \Gamma_\infty$.

Now consider $A(z, t, u) = (z', t', u')$, where $A \in \Gamma - \Gamma_\infty$. We know that A maps the exterior of the isometric sphere of A to the interior of the isometric sphere of A^{-1}. Hence we see that $A(z, t, u)$ cannot lie in the interior of **D**. This gives the result. □

5. Poincaré polyhedra

In this section, we review Poincaré's polyhedron theorem. Since we already know both that the Eisenstein-Picard modular group is discrete and that **D** is a fundamental domain, we do not need the full strength of Poincaré's theorem. In fact, we use it only to establish the connection between the geometry of the action of Γ and the algebra of a presentation for Γ. Specifically, the generators of Γ are the side-pairing maps, and the relations are generated by reflection and cycle relations. However, direct use of Poincaré's theorem yields another proof that Γ is discrete with fundamental domain **D**. We follow the general formulation of Poincaré's polyhedron theorem given in Mostow [M], and we refer to that article for details of the proof. An excellent account of Poincaré's theorem in the case of constant curvature is given in [EP] by Epstein and Petronio.

5.1. Poincaré's polyhedron theorem

A polyhedron is a combinatorial object specified by its vertices, edges, and other faces of higher dimension. We assume that it is a cellular complex homeomorphic to a polytope, possibly with an infinite number of faces. In particular, there exists only one cell of highest-dimension n, and the interior of each cell of codimension two is contained in precisely two codimension-one cells. Its realisation as a cell complex in a manifold X is also referred to as a polyhedron. Let D be the (closed) polyhedron, and let $E_k(D)$ denote the codimension k faces of the polyhedron D. We say a polyhedron is smooth if its faces are smooth.

Definition 5.1
A Poincaré polyhedron is a smooth polyhedron in X with codimension-one faces T_i such that we have following.

- The codimension-one faces are paired by a set Δ of homeomorphisms $A_{ij} : T_i \longrightarrow T_j$ of X called the *side-pairing transformations*, which respect the cell structure. We assume that if $A_{ij} \in \Delta$, then $A_{ij}^{-1} = A_{ji} \in \Delta$.
- For every $A_{ij} \in \Delta$ such that $T_i = A_{ij}(T_j)$, then $A_{ij}(D) \cap D = T_i$.

Remark. If $T_i = T_j$ (that is, a side-pairing maps one side to itself), then we impose the restriction that $A_{ii} : T_i \longrightarrow T_i$ is of order two, and we call it a *reflection*. In this case, the relation ${A_{ii}}^2 = 1$ is called a *reflection relation*.

Let $T_1 \in E_1(D)$ be a codimension-one face, and let $F_1 \in E_2(D)$ be a codimension-two face contained in T_1. Let T_1' be the other codimension-one face containing F_1. Let T_2 be the codimension-one face paired to T_1' by $A_1 \in \Delta$ and $F_2 = A_1(F_1)$. Again, there exists only one other codimension-one face containing F_2, which we call T_2'. We define recursively A_i and F_i, so that $A_{i-1} \circ \cdots \circ A_1(F_1) = F_i$.

Definition 5.2 (Cyclic)
Cyclic is the condition that for each pair (F_1, T_1) (a codimension-two face contained in a codimension-one face), there exists $r \geq 1$ such that, in the construction in the previous paragraph, $A_r \circ \cdots \circ A_1(T_1) = T_1$ and $A_r \circ \cdots \circ A_1$ restricted to F_1 is the identity. Moreover, writing $A = A_r \circ \cdots \circ A_1$, there exists a positive integer m such that $A^m = 1$ and

$$
\begin{aligned}
&{A_1}^{-1}(D) \cup (A_2 \circ A_1)^{-1}(D) \cup (A_3 \circ A_2 \circ A_1)^{-1}(D) \cdots \cup A^{-1}(D) \cup (A_1 \circ A)^{-1}(D) \\
&\qquad \cup (A_2 \circ A_1 \circ A)^{-1}(D) \cdots (A_{r-1} \cdots \circ A_1 \circ A^{m-1})^{-1}(D) \cup (A^m)^{-1}(D)
\end{aligned}
$$

is a cover of a closed neighbourhood of the interior of F_1 by polyhedra with disjoint interiors.

The relation $A^m = (A_r \circ \cdots \circ A_1)^m = 1$ is called a *cycle relation*.

In order to prove Poincaré's theorem, we need a more general version of tiling, which allows, a priori, for ramifications.

Definition 5.3 (Abutted family [M, Section 6.1, page 198])
An abutted family in a topological manifold X is a family of polyhedra $\mathscr{D}$ together with the set of adjacency $\mathscr{N} \subset \mathscr{D} \times \mathscr{D}$ such that

- if $(D, D') \in \mathscr{N}$, then $D \neq D'$ and $(D', D) \in \mathscr{N}$;
- if $(D, D') \in \mathscr{N}$, then $D \cap D' \in E_1(D) \cap E_1(D')$;
- if $(D, D'), (D, D'') \in \mathscr{N}$ and $D \cap D' = D \cap D''$, then $D' = D''$;
- for each $T \in E_1(D)$, there exists D' with $D \cap D' = e$.

Definition 5.4
The joined $\mathscr{D}$-space is the quotient topological space of the subspace of $X \times \mathscr{D}$,

$$\widetilde{Y} = \bigcup_{D \in \mathscr{D}} D \times \{D\},$$

by the equivalence relation

$$(x, D) \equiv (x', D') \quad \text{if and only if } x = x',\ x \in E_1(D) \cap E_1(D').$$

Let Y denote the joined $\mathscr{D}$-space. The projection

$$\pi : Y \to X$$

is continuous. In general, Y may not be a manifold, and even if it is a manifold, π may be branched. The following definition allows us to use induction arguments by intersecting abutted families with spheres.

Definition 5.5
A smooth abutted family is an abutted family such that for each codimension k face $e \in E_k(\mathscr{D})$ and $x \in e$, there exists a tubular neighbourhood of the form $B_k \times B_{n-k}$, where $B_{n-k} \subset e$ is a neighbourhood in e. For each $y \in B_{n-k}$, $B_k \times y$ is transversal to e such that for $S_k \times y$, where $S_k = \partial B_k$, the family $\mathscr{D}$ induces (by intersections) an abutted family $\mathscr{D}_e$ which is combinatorially independent of $y \in B_{n-k}$.

We need the following simple result, which we call the *uniformity condition*.

LEMMA 5.6
If $\pi : Y \to X$ (X complete, connected) is a local isometry and there exists $r > 0$ such that for every $y \in Y$ there exists a neighbourhood homeomorphic under π to a ball of radius r in X, then π is a covering.

Observe that the hypotheses imply that Y is complete.

THEOREM 5.7
Let D be a Poincaré polyhedron with side-pairing transformations $\Delta \subset \mathrm{Isom}(X)$ in a simply connected Riemannian manifold X satisfying the cyclic condition. Let Γ be the group generated by Δ. Then $\mathscr{D} = \Gamma D$ is a smooth abutted family with adjacency defined by the side-pairings. If there exists a positive number r such that every point in the joined space Y has a neighbourhood homeomorphic under π to a ball of radius r, then Γ is a discrete subgroup of $\mathrm{Isom}(X)$ and D is a fundamental domain. A presentation is given by

$$\Gamma = \langle \Delta \mid \textit{reflection relations}, \ \textit{cycle relations} \rangle.$$

Remark

- One first observes that the side-pairings of a Poincaré polyhedron generate a smooth abutted family. The adjacency is given by $\mathscr{N} = \{(\gamma D, \gamma\delta D) \mid \gamma \in \Gamma, \delta \in \Delta\}$. That follows from the smoothness of the polyhedron and the fact that the cycles are finite. The main point is then to prove that the map $\pi : Y \to X$ is a homeomorphism. That is where the *cyclic* condition and the uniformity condition, Lemma 5.6, are used.
- If D is compact, the uniformity condition for the joined space is automatic when the cyclic condition is verified.
- The typical noncompact Poincaré polyhedron that we are interested in is the situation where X is the complex hyperbolic space and D has a cusp. The uniformity condition, Lemma 5.6, has to be verified in this case. One has to prove that the joined space around that cusp contains (the inverse image by π of) a horoball. That amounts to covering a whole horoball by carefully chosen translates of the polyhedron D (see [EP, Figure 12] for an example not satisfying the condition).

Proof (Sketch; see [M], [EP] for more details)
We prove Theorem 5.7 by induction on the dimension. In dimension two, that is the classical Poincaré polyhedron theorem. Suppose that the result is true in dimensions less than n. We want to show first that Y is a manifold. Faces of codimensions one and two are glued nicely by hypothesis. Let $e \in E_k(D)$ for $k > 2$. Consider a small neighbourhood of a point $x \in e$ of the form $B_k \times B_{n-k}$, where $B_{n-k} \subset e$ is a neighbourhood of x in e and where, for each $y \in B_{n-k}$, $B_k \times y$ is transversal to e. Using the side-pairings, we obtain tubular neighbourhoods around each point in the equivalence class of x. At each $S_k = \partial B_k$, we thus obtain an abutted family. By induction, we prove that the family is a tiling of S_k, and by smoothness, we prove that the tiling is the same for each S_k. Therefore, Y is a manifold.

The map $\pi : Y \to X$ is a local isometry. In order to prove that it is a homeomorphism, it suffices to prove that it is a covering map. But that follows from the hypothesis of uniformity. □

5.2. A presentation for Γ

In this section, we use Poincaré's theorem on $\mathbf{D}$ to give a presentation for Γ. We begin by showing that the generators of Γ are side-pairing maps for $\mathbf{D}$.

PROPOSITION 5.8
The following maps are side-pairings of the simplex $\mathbf{D}$:

$$R : \mathbf{T}_0 \longrightarrow \mathbf{T}_0,$$
$$PQ^{-1} : \mathbf{T}_1 \longrightarrow \mathbf{T}_2,$$
$$P : \mathbf{T}_3 \longrightarrow \mathbf{T}_4.$$

Proof
We have already verified that R is a side-pairing map. As PQ^{-1} and P are complex hyperbolic isometries fixing q_∞, it suffices to show that PQ^{-1} maps F_1 to F_2 and that P maps F_3 to F_4. This follows from Proposition 4.6. □

THEOREM 5.9
The simplex $\mathbf{D}$ *is a fundamental domain for the group generated by* R, PQ^{-1}, *and* P. *Moreover, a presentation for this group is given by*

$$\langle P,\ Q,\ R \mid R^2 = (QP^{-1})^6 = PQ^{-1}RQP^{-1}R = P^3Q^{-2} = (RP)^3 = 1\rangle.$$

Since we have already shown that $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$ is generated by P, Q, and R, Theorem 5.9 gives both an alternative proof that $\mathbf{D}$ is a fundamental domain and also a presentation for the Eisenstein-Picard modular group $\mathrm{PU}(2,1;\mathbb{Z}[\omega])$. Other presentations are given in [A] and [H3].

Proof
By the argument of Theorem 3.5, the intersection of the exterior of S_0 with a fundamental domain for $\Gamma_\infty = \langle P,\ Q\rangle$ contains a fundamental domain for $\langle P,\ Q,\ R\rangle = \Gamma$. Let $\hat{\mathbf{D}}$ be the subset of $\mathbf{H}^2_{\mathbb{C}}$ comprising (complete) geodesics with one endpoint q_∞ and passing through $\mathbf{D}$. (Thus $\hat{\mathbf{D}}$ is obtained from $\mathbf{D}$ by extending the geodesic segments used to define $\mathbf{D}$ to meet the boundary.) Then it is clear from Section 3.1 that $\hat{\mathbf{D}}$ is a fundamental domain for Γ_∞. Intersecting $\hat{\mathbf{D}}$ with the exterior of S_0 just gives us $\mathbf{D}$.

For each two-dimensional face F of $\mathbf{D}$, we find the face cycle given by the side-pairing maps.

The faces with one vertex q_∞ are sent to other faces with vertex at q_∞ by maps in $\langle P, Q\rangle = \Gamma_\infty$. Since the simplex **D** and its faces containing q_∞ are cones over $\mathbf{T}_0$ and its edges, the edge cycles are the same as those for $\mathbf{T}_0$ obtained in Section 3.1. By construction, any horoball not intersecting S_0 is covered by the images of **D** under Γ_∞. The face cycles from faces containing q_∞ are the same as the edge cycles from $\mathbf{T}_0$, namely,

$$(PQ^{-1})^6 = 1 \qquad \text{and} \qquad P^3 = Q^2.$$

Similarly, the (one-dimensional) edges of **D** having one vertex at q_∞ each have a neighbourhood covered by images of **D**.

Now consider the face F_1 with vertices the ordered triple (z_2, z_0, z_3). The face cycle is

$$(z_2, z_0, z_3) \xrightarrow{PQ^{-1}} (z_1, z_0, z_3) \xrightarrow{R} (z_1, z_3, z_0) \xrightarrow{(PQ^{-1})^{-1}} (z_2, z_3, z_0) \xrightarrow{R} (z_2, z_0, z_3).$$

Therefore, $R(PQ^{-1})^{-1}RPQ^{-1}$ is the identity on F_1. In fact, $R(PQ^{-1})^{-1}RPQ^{-1}$ is the identity in Γ, as we may easily verify. We must show that **D**, $(PQ^{-1})^{-1}(\mathbf{D})$, $(PQ^{-1})^{-1}R(\mathbf{D}) = R(PQ^{-1})^{-1}(\mathbf{D})$, and $(PQ^{-1})^{-1}RPQ^{-1}(\mathbf{D}) = R(\mathbf{D})$ cover a neighbourhood of F_1. This also shows that a neighbourhood of $PQ^{-1}(F_1) = F_2$.

The map PQ^{-1} maps S_0 to itself. (It is just a rotation of S_0 about its spine.) Therefore, $(PQ^{-1})^{-1}(\mathbf{T}_0)$ is also contained in S_0. The image of **D** under $(PQ^{-1})^{-1}$ is the geodesic cone of $(PQ^{-1})^{-1}(\mathbf{T}_0)$. Hence $\mathbf{D} \cup (PQ^{-1})^{-1}(\mathbf{D})$ covers that part of a neighbourhood of T_α exterior to S_0. Applying R, we see that $\mathbf{D} \cup (PQ^{-1})^{-1}(\mathbf{D}) \cup R(\mathbf{D}) \cup R(PQ^{-1})^{-1}(\mathbf{D})$ covers a neighbourhood of F_1, as claimed.

Next, consider the face F_3 with vertices the ordered triple (z_2, z_0, z_1). The face cycle on this face is

$$(z_2, z_0, z_1) \xrightarrow{P} (z_3, z_1, z_2) \xrightarrow{R} (z_0, z_1, z_2).$$

Therefore, RP maps F_3 to itself but with a rotation of order 3. Hence $(RP)^3$ is the identity on F_3. In fact, $(RP)^3$ is the identity. We must show that **D**, $P^{-1}(\mathbf{D})$, $P^{-1}R(\mathbf{D})$, $P^{-1}RP^{-1}(\mathbf{D})$, $P^{-1}RP^{-1}R(\mathbf{D}) = RP(\mathbf{D})$, and $P^{-1}RP^{-1}RP^{-1}(\mathbf{D}) = R(\mathbf{D})$ cover a neighbourhood of F_3. This also shows that a neighbourhood of $P(F_3) = F_4$.

In order to see this, first observe that the image of S_0 under P^{-1} is S_{-1}; therefore, $\mathbf{D} \cup P^{-1}(\mathbf{D})$ covers a neighbourhood of F_3 exterior to both S_0 and S_{-1}. Now S_0 and S_{-1} are the isometric spheres of $P^{-1}R$ and $(P^{-1}R)^{-1} = RP$. Therefore, the common exterior of S_0 and S_{-1} form a fundamental domain (the Ford domain) for the group $\langle P^{-1}R\rangle$ with three elements. Hence $\mathbf{D} \cup P^{-1}(\mathbf{D})$ and its images under $P^{-1}R$ and RP cover a neighbourhood of F_3.

By Poincaré's theorem, we conclude that the 4-simplex is a fundamental domain, and the presentation is obtained by the reflection and cycle relations. □

Observe that $\Upsilon = \langle P, \ R\rangle \in \mathrm{PU}(2, 1, \mathbb{Z}[\omega])$ is a representation of the triangle group of type $(2, 3, \infty) = \mathrm{PSL}(2, \mathbb{Z})$ (the modular group). P is parabolic, R has order 2, and RP has order 3. But observe that this representation is not faithful. For example, RP^3 has order 6. We see that $\mathrm{PU}(2, 1, \mathbb{Z}[\omega]) = \langle \Upsilon, \ PQ^{-1}\rangle$, where $(PQ^{-1})^6 = 1$. We can also view $\mathrm{PU}(2, 1, \mathbb{Z}[\omega]) = \langle \Upsilon, \ T\rangle$ with relations

$$[T, R] = T^6 = PT^{-1}P^{-1}TP = 1$$

by setting $T = PQ^{-1}$. Thus $\mathrm{PU}(2, 1, \mathbb{Z}[\omega])$ is obtained by adjoining to Υ one elliptic element of order 6 commuting with R. To summarise, we have the following proposition.

PROPOSITION 5.10
The group $\mathrm{PU}(2, 1, \mathbb{Z}[\omega])$ *is obtained from a representation of* $\mathrm{PSL}(2, \mathbb{Z})$ *(discrete but not faithful) in* $\mathrm{PU}(2, 1)$ *by adjoining one elliptic element of order 6.*

Observe that the representation of $\mathrm{PSL}(2, \mathbb{Z})$ in $\mathrm{PU}(2, 1)$ is contained in the family obtained in [FK] and [FP]. It corresponds, in their notation, to the representation $\mathbf{A}(\sqrt{3}/2)$.

5.3. Relation with Mostow's groups

In [M], Mostow constructed a family of groups. Some of his groups are nonarithmetic and, in fact, were the first examples of such groups. In his notation, all of Mostow's examples are generated by three complex reflections, R_1, R_2, and R_3, having orders 3, 4, or 5. Moreover, these groups have an extra cubic symmetry in the sense that there is a map J of order 3, so that $R_{k+1} = JR_kJ^{-1}$, where k is defined mod 3. That map J may not be in the group, and in that case, the group generated by the R_k is a subgroup of index three of the group generated by J and R_1. Mostow used Dirichlet domains to show that those groups were discrete and to obtain presentations. But the combinatorics of those domains are very complicated.

We now show that the Eisenstein-Picard modular group admits a presentation of a similar type. In fact, we show that it is generated by complex reflections of order 6 having a cubic symmetry. We begin by showing that Γ admits a presentation with two generators. Our notation reflects that of Mostow. Other sets of generators and presentations for the Eisenstein-Picard group are investigated in [A] and [H3]. The Eisenstein-Picard modular group fits into a family of lattices first investigated by Livné [Li]. Similar results about their presentations are given in [P2].

PROPOSITION 5.11

The maps $J = RP$ and $R_1 = QP^{-1}$ generate Γ. Moreover, a presentation on these generators is

$$\langle J,\ R_1 \mid J^3 = {R_1}^6 = (J{R_1}^{-1}J)^4 = R_1(J{R_1}^{-1}J)^2{R_1}^{-1}(J{R_1}^{-1}J)^{-2} = 1\rangle.$$

Proof

We begin by showing that the relations involving J and R_1 follow from the relations involving P, Q, and R. First, $J^3 = (RP)^3 = 1$ and ${R_1}^6 = (QP^{-1})^6 = 1$ follow immediately. Also,

$$\begin{aligned}(J{R_1}^{-1}J)^2 &= RPPQ^{-1}(RP)^2PQ^{-1}RP\\ &= RPPQ^{-1}P^{-1}R^{-1}PQ^{-1}RP\\ &= RP^2Q^{-1}P^{-1}PQ^{-1}R^{-1}RP\\ &= RP^2Q^{-2}P\\ &= R,\end{aligned}$$

where we have used the relations $(RP)^3 = 1$, $R^{-1}QP^{-1} = QP^{-1}R^{-1}$, and $Q^2 = P^3$ on the second, third, and fifth lines. Thus $(J{R_1}^{-1}J)^4 = R^2 = 1$ and

$$R_1(J{R_1}^{-1}J)^2{R_1}^{-1}(J{R_1}^{-1}J)^{-2} = (QP^{-1})R(QP^{-1})^{-1}R^{-1} = 1.$$

Using $R = (J{R_1}^{-1}J)^2$, we obtain

$$P = R^{-1}J = J^{-1}R_1JR_1 \qquad \text{and} \qquad Q = R_1P = R_1J^{-1}R_1JR_1.$$

Hence $\langle P,\ Q,\ R\rangle = \langle J,\ R_1\rangle$.

Finally, we show that the relations involving P, Q, and R are a consequence of those involving J and R_1. First, $R^2 = (J{R_1}^{-1}J)^4 = 1$, $(RP)^3 = J^3 = 1$, $(QP^{-1})^6 = {R_1}^6 = 1$, and

$$(QP^{-1})R(QP^{-1})^{-1}R^{-1} = R_1(J{R_1}^{-1}J)^2{R_1}^{-1}(J{R_1}^{-1}J)^{-2} = 1$$

follow immediately. Finally,

$$\begin{aligned}P^3Q^{-2} &= (J^{-1}R_1JR_1)^3({R_1}^{-1}J^{-1}{R_1}^{-1}J{R_1}^{-1})^2\\ &= (J{R_1}^{-1}J)^{-2}R_1(J{R_1}^{-1}J)^2{R_1}^{-1}\\ &= 1.\end{aligned}$$

This completes the proof. □

As in Proposition 5.11, we write $R_1 = QP^{-1}$ and $J = RP$. Define $R_2 = JR_1J^{-1} = RPQ^{-1}P^{-2}R$ and $R_3 = J^{-1}R_1J = P^{-1}Q$. These are all complex reflections of order 6 with a reflection factor $-\overline{\omega} = e^{2i\pi/6}$ (see [M, page 174]). We now show that R_1, R_2, and R_3 generate Γ. We also give relations involving the R_k. The form of these relations is motivated by [M, Theorem 20.1]. We make the connection explicit in Corollary 5.13.

PROPOSITION 5.12
The maps R_1, R_2, and R_3 generate Γ. Moreover, a presentation on these generators is (with indices taken mod 3)

$$\left\langle R_1,\ R_2,\ R_3 \,\middle|\, \begin{array}{l} {R_k}^6 = 1,\ R_kR_{k+1}R_k = R_{k+1}R_kR_{k+1},\ k \in \{1,\ 2,\ 3\}, \\ (R_1R_2R_3)^4 = 1,\ (R_1R_2R_3)^{-2}R_1R_2 = (R_2R_3R_1)^{-2}R_2R_3 \end{array} \right\rangle.$$

Proof
First, observe that $\langle R_1,\ R_2,\ R_3\rangle$ is a subgroup of $\langle J,\ R_1\rangle$. Thus we need to show that J is contained in $\langle R_1,\ R_2,\ R_3\rangle$. We have

$$J = J(J{R_1}^{-1}J)^4 = (J^{-1}{R_1}^{-1}J)(J{R_1}^{-1}J^{-1}){R_1}^{-1}(J^{-1}{R_1}^{-1}J) = (R_1R_2R_3)^{-2}R_1R_2.$$

We now show the equivalence of the presentations. We begin by assuming the relations involving R_1, R_2, and R_3 and showing that these imply the relations involving J and R_1. We already have the fact that ${R_1}^6 = 1$. Moreover, the relation $(R_1R_2R_3)^{-2}R_1R_2 = (R_2R_3R_1)^{-2}R_2R_3$ may be written as $R_3R_1R_2R_3 = R_1R_2R_3R_1$. Thus

$$\begin{aligned} J^{-1} &= \big((R_2R_3R_1)^{-2}R_2R_3\big)^{-1} \\ &= R_1R_2R_3R_1 \\ &= R_1R_2R_1{R_1}^{-1}R_3R_1 \\ &= R_2R_1R_2R_3R_1{R_3}^{-1} \\ &= R_2R_3R_1R_2. \end{aligned}$$

Thus we have

$$J^{-1} = R_1R_2R_3R_1 = R_2R_3R_1R_2 = R_3R_1R_2R_3.$$

Hence $R_2 = JR_1J^{-1}$ and $R_3 = J^{-1}R_1J$. Also, we have

$$J^{-3} = (R_1R_2R_3R_1)(R_2R_3R_1R_2)(R_3R_1R_2R_3) = (R_1R_2R_3)^4 = 1.$$

Observe that

$$(J{R_1}^{-1}J)^{-2} = (R_3R_1R_2R_3R_1R_2R_3R_1R_2)^2 = (R_3R_1R_2)^6 = (R_3R_1R_2)^2.$$

Thus $(JR_1{}^{-1}J)^4 = (R_3R_1R_2)^4 = 1$ and

$$\begin{aligned} R_1(JR_1{}^{-1}J)^{-2} &= R_1R_3R_1R_2R_3R_1R_2 \\ &= R_3R_1R_3R_2R_3R_1R_2 \\ &= R_3R_1R_2R_3R_2R_1R_2 \\ &= R_3R_1R_2R_3R_1R_2R_1 \\ &= (JR_1{}^{-2}J)^{-2}R_1. \end{aligned}$$

Now we assume the relations involving J and R_1. Again, we know $R_k{}^6 = 1$. Next,

$$R_1R_2R_1 = J(JR_1{}^{-1}J)^{-2}R_1 = JR_1(JR_1{}^{-1}J)^{-2} = R_2R_1R_2.$$

As above,

$$(R_3R_1R_2)^2 = (J^{-1}R_1JR_1JR_1J^{-1})^2 = (JR_1{}^{-1}J)^{-6} = (JR_1{}^{-1}J)^{-2},$$

and so we have $(R_3R_1R_2)^4 = 1$ and

$$\begin{aligned} (R_1R_2R_3)^{-2}R_1R_2 &= R_1R_2(R_3R_1R_2)^{-2} \\ &= R_1JR_1J^{-1}(JR_1{}^{-1}J)^2 \\ &= J \\ &= (JR_1{}^{-1}J)^2J^{-1}R_1JR_1 \\ &= (R_3R_1R_2)^{-2}R_3R_1. \end{aligned}$$

□

Following [M], let $\mathbf{e}_k$ be the polar vector of R_k. Then

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ -\omega \\ 1 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} \overline{\omega} \\ -\overline{\omega} \\ 0 \end{bmatrix}.$$

Define φ by (see [M, Section 9.1])

$$\begin{aligned} \varphi &= \exp\Big(i \arg \frac{-\langle \mathbf{e}_1,\ \mathbf{e}_2\rangle\langle \mathbf{e}_2,\ \mathbf{e}_3\rangle\langle \mathbf{e}_3,\ \mathbf{e}_1\rangle}{3}\Big) \\ &= \exp\Big(i \arg \frac{-(-\overline{\omega})(\omega+\overline{\omega})(-\overline{\omega})}{3}\Big) \\ &= \omega^{1/3} \\ &= e^{2i\pi/9}. \end{aligned}$$

COROLLARY 5.13
Using the notation of [M, Theorem 20.1], let $p = 6$ *and* $\varphi = e^{2i\pi/9}$. *Then* Γ *satisfies the relations* $\mathscr{R}'$ *and* $\mathscr{R}''$ *with* $\mu = -1$.

Proof
We have

$$\begin{aligned} \rho &= \operatorname{order} e^{-i\pi/6} i\varphi^3 = \operatorname{order} e^{-i\pi/6+i\pi/2+2i\pi/3} = \operatorname{order} e^{i\pi} = 2, \\ \sigma &= \operatorname{order} e^{-i\pi/6} i\overline{\varphi}^3 = \operatorname{order} e^{-i\pi/6+i\pi/2-2i\pi/3} = \operatorname{order} e^{-i\pi/3} = 6. \end{aligned}$$

Therefore, $r = \rho = 2$ and $s = \sigma/3 = 2$. Then the relations $\mathscr{R}'$ are

$$\left\{ {R_k}^6 = R_k R_{k+1} R_k {R_{k+1}}^{-1} {R_k}^{-1} {R_{k+1}}^{-1} = (R_1 R_2 R_3)^4 = (R_3 R_2 R_1)^4 = 1 : k = 1,\ 2,\ 3 \right\}.$$

Taking $\mu = -1$, the relations $\mathscr{R}''$ become

$$\left\{ (R_1 R_2 R_3)^{-2} R_1 R_2 = (R_2 R_3 R_1)^{-2} R_2 R_3 \right\}.$$

Each of these relations follows from Proposition 5.12 except $(R_3 R_2 R_1)^4 = 1$. We now show that this is a consequence of the other relations.

First, observe that repeated use of the braid relations $R_k R_{k+1} R_k = R_{k+1} R_k R_{k+1}$ gives

$$(R_2 R_1 R_3)^2 R_1 = R_1 (R_2 R_1 R_3)^2.$$

Therefore,

$$\begin{aligned} (R_3 R_2 R_1)^4 &= R_3 {R_1}^6 {R_3}^{-1} (R_3 R_2 R_1)^4 \\ &= (R_3 {R_1}^3 R_2 R_1 R_3 R_2 R_1)^2 \\ &= (R_3 R_1 R_2 R_1 R_2 R_3 R_2 R_3 R_1)^2 \\ &= \left({R_2}^{-1} (R_2 R_3 R_1 R_2) R_1 R_2 R_3 (R_2 R_3 R_1 R_2) {R_2}^{-1} \right)^2 \\ &= \left({R_2}^{-1} (R_3 R_1 R_2 R_3 R_1 R_2 R_3 R_1 R_2 R_3 R_1) {R_2}^{-1} \right)^2 \\ &= {R_2}^{-6} \\ &= 1. \end{aligned}$$

We have made use only of the relations ${R_k}^6 = 1$, $R_k R_{k+1} R_k = R_{k+1} R_k R_{k+1}$, $(R_1 R_2 R_3)^4 = 1$, and

$$R_1 R_2 R_3 R_1 = R_2 R_3 R_1 R_2 = R_3 R_1 R_2 R_3.$$

□

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