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The influence of massive gas clouds on stellar velocity dispersions in galactic discs

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Summary. This paper calculates the evolution of the three components of the velocity dispersion of the stars in a galactic disc due to the influence of massive gas clouds in circular orbits in the disc. We find that there are two phases in this evolution: an initial transient phase in which the shape of the velocity ellipsoid relaxes to a final shape depending only on the ratio Ω/κ of the circular to the radial epicyclic frequencies, followed by a steady heating phase in which for typical disc stars the velocity dispersion σ varies as $d\sigma^2/dt \propto N_c M_c^2 \nu/\sigma^2$, where N_c and M_c are the surface density and mass of the clouds and ν is the vertical epicyclic frequency. We also find that the amount of stellar heating predicted will be comparable with that observed, for young stars at least, if cloud masses are near the upper end of the observationally allowed range, but that the ratio of vertical to horizontal velocity dispersions predicted disagrees with that observed. This may indicate that other disc heating mechanisms are important.

1 Introduction

In this paper we calculate the evolution of the velocity dispersions of the stars in a galactic disc when the orbits of the stars are perturbed by massive gas clouds in circular orbits in the disc.

The influence of gas clouds on stellar velocity dispersions was first considered by Spitzer & Schwarzschild (1951), who calculated the evolution of the stellar distribution function by numerically integrating the Fokker–Planck equation. Their calculation included the velocity dispersion of the gas clouds but ignored galactic rotation. In a second paper, Spitzer & Schwarzschild (1953) argued that their previous neglect of galactic rotation was incorrect, and analytically calculated the evolution when the epicyclic motion of the stars in the background galactic potential was included, in the approximation that the epicyclic velocities of the stars greatly exceeded the random motions of the clouds. Fujimoto (1980) has performed an analytical calculation under the assumption that the clouds are very short-lived, while numerical integrations of star–cloud encounters have been carried out by Woolley & Candy (1968a, b) and Icke (1982).

The calculations of Spitzer & Schwarzschild (1953) and Icke (1982) have the basic shortcoming that they only consider stellar motions in the symmetry plane of the disc. Woolley & Candy (1968b) do include stellar motions perpendicular to the plane but do not derive a velocity dispersion–age relation from their results, while we shall argue that Fujimoto's (1980) assumption of short cloud lifetime is not applicable to galactic discs. This paper fills a major gap in the theory by generalizing the Spitzer & Schwarzschild (1953) calculation to include vertical motions of the stars. This is important both because of the intrinsic interest of predicting the vertical velocity dispersion and because their vertical epicyclic oscillations can take the stars out of the layer of perturbing clouds and so reduce the scattering rates.

There is at present great interest in the possibility that the cloud scattering mechanism may explain the velocity dispersions of disc stars because the giant molecular clouds observed (e.g. Solomon & Sanders 1980; Liszt, Xiang & Burton 1981) have roughly the properties required according to Spitzer & Schwarzschild (1953). Also observations of disc scale-heights in edge-on galaxies (van der Kruit & Searle 1981a, b) now provide information on the radial variation of the vertical component of velocity dispersion, which can be used to put constraints on the functional dependence of the stellar heating (Lacey & Fall 1983). The more complete calculation of the effects of this heating mechanism presented here should make possible a much more reliable assessment of its importance in galactic evolution.

The plan of the remainder of the paper is as follows: Section 2 states the basic assumptions of the calculation and discusses the energy transfer from a general point of view, while Section 3 contains the details of the calculation. Section 4 describes the evolution of the velocity dispersions and contains the main results of the paper. Section 5 considers the effects of modifications to the epicyclic approximation for vertical motions, Section 6 compares the results to those of previous calculations and Section 7 describes the comparison with observations. Section 8 contains the conclusions.

2 Assumptions

The calculation starts from the following assumptions:

(1) The orbits of the stars in the background galactic potential (assumed axisymmetric and plane-symmetric) are described by first-order epicyclic theory.

(2) The clouds are long-lived, much more massive than the stars and move in circular orbits in the symmetry plane.

(3) The clouds are randomly distributed and act independently.

(4) For a typical star–cloud encounter, the effective interaction time is short compared to the epicyclic period, and the velocity difference between the local standards of rest at the star and at the cloud is negligible compared to the non-circular velocity of the star.

(5) The perturbation of the stellar velocities is dominated by the effect of the many distant, weak encounters so that the evolution of the distribution function is given by a diffusion equation.

Assumption (1) is valid if the departures from circular orbits in the symmetry plane are small, which holds for most disc stars except perhaps in the centres of discs. It is a good approximation for virtually all of the disc populations used to determine the velocity dispersion–age relation in the solar neighbourhood. The assumption about cloud masses made in (2) is easily satisfied since they are of order 10^5–10^6 $M_\odot$. We estimate the neglect of the cloud velocity dispersions to be a reasonable approximation provided the stellar epicyclic velocities exceed 2–3 times the 3-D cloud velocity dispersion; for comparison, Liszt & Burton (1981) obtain approximately 6 km s^{-1} for the 3-D velocity dispersion of the most massive clouds, while the velocity dispersions of disc stars in the solar neighbourhood range between about 10 and 80 km s^{-1} according to Wielen (1977). We estimate the effect of finite cloud lifetimes to be unimportant provided these exceed roughly the time for a star to cross a cloud radius, 10^6 yr at most, whereas estimates of cloud lifetimes are in the range 10^7–10^9 yr (Bash, Green & Peters 1977; Solomon & Sanders 1980). Assumption (3) neglects the possibility of the cloud distribution being organized on a large scale, e.g. into a spiral arm. Regarding assumption (4), a typical encounter would be one with impact parameter roughly the geometric mean of the cloud radius (~ 10 pc) and the epicyclic orbital size (~ 1 kpc) for which the encounter time and cloud shear velocity would be roughly 1/10 the epicyclic period and stellar epicyclic velocity respectively. Finally, regarding assumption (5), the physical size of the clouds probably excludes strong encounters.

We now briefly consider energy and angular momentum balance. We use cylindrical polar coordinates (R, θ, z) and define (u, v, w) to be the corresponding components of the non-circular velocity. Let $\Phi(R, z)$ to the potential and $\Omega(R)$ the frequency of a circular orbit at $(R, z = 0)$. Thus:

$$\Omega^2 = (1/R\ \partial\Phi/\partial R)_{z=0}. \tag{1}$$

Then the energy and angular momentum (about the z-axis) per unit mass are:

$$E = \tfrac{1}{2}\{u^2 + [R\Omega(R) + v]^2 + w^2\} + \Phi(R, z), \tag{2}$$

$$J = R\,[R\Omega(R) + v]. \tag{3}$$

In a single star–cloud encounter, as a result of assumptions (2) and (4)

$$\Delta(u^2 + v^2 + w^2) = 0, \tag{4}$$

while (R, θ, z) are unchanged, so

$$\Delta E = R\Omega(R)\,\Delta v = \Omega(R)\,\Delta J. \tag{5}$$

corresponding to conservation of the Jacobi integral $E - \Omega J$. The calculation of Section 3 only follows the evolution of the *stellar* distribution explicitly, but equation (5) shows that overall conservation of energy and angular momentum (of stars + clouds) is consistent with the clouds being on orbits which remain virtually circular but which slowly change radius as they exchange energy and angular momentum with the stars.

We expect the energy in *random* motions of the stars to increase on general thermodynamic principles. This can be accomplished even in a purely stellar disc if there is some mechanism for transferring angular momentum outwards [in a disc in which $\Omega(R)$ declines outwards] (Lynden-Bell & Kalnajs 1972). Our calculation shows that in the presence of clouds such an increase of the energy in random stellar motions does indeed occur, but it does not reveal how much of this increase comes from a decrease in the energy of organized motions of the stars (the clouds playing a purely catalytic role) and how much from a decrease in the orbital energy of the clouds. To determine this, an analysis involving second-order epicyclic theory would be required.

3 Calculation of the evolution of the velocity dispersion

3.1 UNPERTURBED STELLAR ORBITS

In this subsection we set up the equations of motion and define the conserved quantities for the (first-order) epicyclic approximation (e.g. Chandrasekhar 1960). We define the frequencies κ and ν of horizontal and vertical epicyclic motion

$$\kappa^2 = [\partial^2\Phi/\partial R^2 + (3/R)\,\partial\Phi/\partial R]_{z=0}$$
$$= 4\Omega^2 (1 + \tfrac{1}{2} d \ln \Omega / d \ln R), \tag{6}$$

$$\nu^2 = (\partial^2\Phi/\partial z^2)_{z=0}, \tag{7}$$

and the frequency ratio β:

$$\beta = 2\Omega/\kappa. \tag{8}$$

For real discs, β ranges between 1 (solid-body rotation) and 2 (Keplerian). The motion of a star is referred to a radius R_0 (which is not necessarily the guiding centre radius), and J_0 is defined to be the angular momentum for a circular orbit at that radius. We define the phase variables ψ, χ and ϕ

$$\psi = \psi_0 - [(\beta^2 - 1)(J - J_0)/R_0^2]\, t, \tag{9a}$$

$$\chi = \chi_0 + \kappa t, \tag{9b}$$

$$\phi = \phi_0 + \nu t. \tag{9c}$$

in terms of which the position of the star is given by

$$R - R_0 = \beta(J - J_0)/(\kappa R_0) + [(2E_e)^{1/2}/\kappa] \cos\chi, \tag{10a}$$

$$\theta - \Omega t = \psi - \beta[(2E_e)^{1/2}/(\kappa R_0)] \sin\chi, \tag{10b}$$

$$z = [(2E_z)^{1/2}/\nu] \cos\phi, \tag{10c}$$

and its velocity components relative to the local circular velocity by

$$u \equiv \dot{R} = -(2E_e)^{1/2} \sin\chi, \tag{11a}$$

$$v \equiv R[\dot{\theta} - \Omega(R)] = -[(2E_e)^{1/2}/\beta] \cos\chi, \tag{11b}$$

$$w \equiv \dot{z} = -(2E_z)^{1/2} \sin\phi. \tag{11c}$$

In the above equations the frequencies Ω, κ and ν are all evaluated at the reference radius R_0. E_e and E_z are the energies associated with horizontal and vertical epicyclic motions respectively:

$$E_e = \tfrac{1}{2}(u^2 + \beta^2 v^2), \tag{12}$$

$$E_z = \tfrac{1}{2}(w^2 + \nu^2 z^2). \tag{13}$$

We define $E_c(J)$ to be the energy of a star in a circular orbit at $z = 0$ with angular momentum J, so that the total energy is given by

$$E = E_c(J) + E_e + E_z. \tag{14}$$

E_e, E_z and J are all separately conserved in the unperturbed epicyclic motion, but are generally changed by encounters with clouds.

3.2 CHANGES OF THE EPICYCLIC ENERGIES BY ENCOUNTERS

Assumptions (3), (4) and (5) of Section 2 allow us to treat the perturbation of the stellar orbit by the population of clouds using the usual velocity diffusion coefficients

(Chandrasekhar 1960; Hénon 1973) derived for motion with no external forces through a uniform distribution of perturbers. The moments of the velocity change $\Delta\mathbf{V}$ in a time Δt short compared to the orbital time are given by equations (54) of Hénon (1973), where we neglect the stellar mass compared to the cloud mass M_c and also the cloud velocity relative to the stellar velocity $\mathbf{V} = (u, v, w)$ in the local standard of rest at the position of the star. Resolved along three orthogonal axes, one parallel and two perpendicular to $\mathbf{V}$, these are:

$$\langle \Delta V_{\parallel} \rangle = -4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t / V^2, \tag{15a}$$

$$\langle \Delta V_{\perp 1} \rangle = \langle \Delta V_{\perp 2} \rangle = 0, \tag{15b}$$

$$\langle (\Delta V_{\parallel})^2 \rangle = 0, \tag{15c}$$

$$\langle (\Delta V_{\perp 1})^2 \rangle = \langle (\Delta V_{\perp 2})^2 \rangle = 4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t / V, \tag{15d}$$

$$\langle (\Delta V_{\parallel})(\Delta V_{\perp 1}) \rangle = \langle (\Delta V_{\parallel})(\Delta V_{\perp 2}) \rangle = \langle (\Delta V_{\perp 1})(\Delta V_{\perp 2}) \rangle = 0, \tag{15e}$$

where n_c is the number density of clouds at the position of the star and the factor $\ln \Lambda$ which arises from the integration over impact parameters can be approximated by

$$\ln \Lambda \approx \ln (l_{\max}/l_{\min}). \tag{16}$$

Here $l_{\max}$ and $l_{\min}$ are the effective maximum and minimum impact parameters, and for clouds (assumed roughly homogeneous) of physical radius a_c

$$l_{\min} = \max\,[a_c, GM_c/V^2] \tag{17}$$

(see Hénon 1973 for a more detailed discussion). Following the discussion of Hénon (1958) we take for $l_{\max}$ the product of the typical relative velocity with the orbital time, or, equivalently, the size of an epicyclic orbit. Thus

$$l_{\max} \sim (2E_e)^{1/2}/\kappa \quad \text{or} \quad (2E_z)^{1/2}/\nu. \tag{18}$$

From equations (15) we derive the moments of the velocity change in our original coordinate system:

$$\langle \Delta u \rangle = -4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t \;\; u/V^3, \tag{19a}$$

$$\langle \Delta v \rangle = -4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t \;\; v/V^3, \tag{19b}$$

$$\langle \Delta w \rangle = -4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t \;\; w/V^3, \tag{19c}$$

$$\langle (\Delta u)^2 \rangle = 4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t \;\; (v^2 + w^2)/V^3, \tag{20a}$$

$$\langle (\Delta v)^2 \rangle = 4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t \;\; (u^2 + w^2)/V^3, \tag{20b}$$

$$\langle (\Delta w)^2 \rangle = 4\pi G^2 n_c M_c^2 \ln \Lambda \, \Delta t \;\; (u^2 + v^2)/V^3. \tag{20c}$$

Using (12) we find for the expectation value of the change in the epicyclic energy E_e in the same short time interval:

$$\langle \Delta E_e \rangle = u\langle \Delta u \rangle + \tfrac{1}{2}\langle (\Delta u)^2 \rangle + \beta^2 (v\langle \Delta v \rangle + \tfrac{1}{2}\langle (\Delta v)^2 \rangle). \tag{21}$$

We assume the dependence of cloud number density on position to be of the form:

$$n_c(R, z) = [N_c(R)/(2\pi)^{1/2} h_c] \exp(-z^2/2h_c^2), \tag{22}$$

so that N_c is the number of clouds per unit area. The Gaussian z-dependence assumed is consistent with the observations (e.g. Sanders, Solomon & Scoville 1984) and is mathematically convenient. However, the main results are not expected to be sensitive to the detailed z-dependence. We assume that n_c varies only slowly with galactocentric radius, $\partial n_c/\partial R \sim n_c/R$, but the R-dependence is otherwise arbitrary. Then, provided the epicyclic amplitude is small, $(2E_e)^{1/2}/\kappa \ll R$, the rate of change of the epicyclic energies depends only on the local values of N_c and h_c and not on their gradients. Combining equations (19)–(22) we derive the expectation value of the instantaneous rate of change of E_e:

$$\langle dE_e/dt \rangle = (2\pi)^{1/2}(G^2N_cM_c^2 \ln \Lambda/h_c) \exp(-z^2/2h_c^2) \times [(\beta^2-2)u^2+(1-2\beta^2)v^2+(\beta^2+1)w^2]/V^3. \tag{23}$$

Similarly, (13) gives for the change in E_z:

$$\langle \Delta E_z \rangle = w\langle \Delta w \rangle + \tfrac{1}{2}\langle (\Delta w)^2 \rangle, \tag{24}$$

and combining with (19), (20) and (22) gives

$$\langle dE_z/dt \rangle = (2\pi)^{1/2}(G^2N_cM_c^2 \ln \Lambda/h_c) \exp(-z^2/2h_c^2) \times (u^2+v^2-2w^2)/V^3. \tag{25}$$

3.3 RATE OF CHANGE OF THE EPICYCLIC ENERGIES FOR A POPULATION OF STARS

Equations (23) and (25) give the expected rates of change of the epicyclic energies for a single star. To calculate the mean rates of change for the whole population, these expressions must be averaged over the stellar distribution, which we describe by the phase-space density f, defined so that $f d^3x\, d^3v$ = number of stars in the phase-space volume element $d^3x\, d^3v$. We assume that the evolution of the distribution due to encounters is slow compared to the orbital frequencies, so that in its dependence on the phase-space coordinates f always approximates to a steady-state solution of the *collisionless* Boltzmann equation, depending on E_e, E_z and J but not on the phase variables (Jean's theorem). In a fully self-consistent calculation this dependence would have to be derived by solving the orbit-averaged Fokker–Planck equation for f. In this paper, for the purpose of carrying out the averaging, we take the simpler approach of assuming that f is approximately isothermal, i.e. exponential in the epicyclic energies and so Gaussian in the non-circular velocity components. While we do not expect this to be exactly true, it is probably a reasonable approximation at energies comparable to the mean (e.g. see Spitzer & Schwarzschild 1951 for the numerical solution of a similar Fokker–Planck equation). We expect this approach to give the correct general dependence of the population-averaged rates of change on the mean energies etc., though the numerical coefficients may not be exactly right.

We write the phase-space density as

$$f(E_e, E_z, J) = S(J) \exp[-(E_e/\langle E_e \rangle + E_z/\langle E_z \rangle)]. \tag{26}$$

$S(J)$ is related to the radial variation of the surface density and $\langle E_e \rangle$ and $\langle E_z \rangle$ are averages of the epicyclic energies at fixed J. (Henceforth, the brackets $\langle\rangle$ will always denote an average over the stellar distribution rather than over encounters with clouds, except where explicitly stated otherwise.)

Substituting for J, E_e and E_z using equations (3), (12) and (13), we obtain an expression for f as a function of (R, z, u, v, w), where we use the same symbol for the phase-space density as a function of the new variables since no confusion should be caused. To lowest order in the epicyclic amplitude, this is

$$f(R, z, u, v, w) \approx S[\Omega(R)R^2] \exp\left[-\left(\frac{u^2}{2\sigma_u^2}+\frac{v^2}{2\sigma_v^2}+\frac{w^2}{2\sigma_w^2}+\frac{z^2}{2h_s^2}\right)\right]. \tag{27}$$

The dispersions σ_u, σ_v, σ_w and h_s are defined to be averages over all stars at fixed R, and are related to the mean energies by, to lowest order,

$$\sigma_u \equiv \langle (u - \langle u \rangle)^2 \rangle^{1/2} \approx \langle E_e \rangle^{1/2}, \tag{28}$$

$$\sigma_v \equiv \langle (v - \langle v \rangle)^2 \rangle^{1/2} \approx \langle E_e \rangle^{1/2}/\beta, \tag{29}$$

$$\sigma_w \equiv \langle (w - \langle w \rangle)^2 \rangle^{1/2} \approx \langle E_z \rangle^{1/2}, \tag{30}$$

$$h_s \equiv \langle (z - \langle z \rangle)^2 \rangle^{1/2} \approx \langle E_z \rangle^{1/2}/\nu. \tag{31}$$

In the above, $\langle E_e \rangle$ and $\langle E_z \rangle$, which are defined as functions of J, are to be treated as functions of R through the zero-order relation $J = \Omega(R) R^2$. Equations (28)–(31) are true for an arbitrary steady-state distribution function, as can be shown from (10) and (11), though for a non-isothermal distribution the local dispersions (at fixed R, z) will differ from the z-averaged (at fixed R) values which these equations give. The Gaussian z-distribution for the isothermal population (equation 27) depends on our assumption that the z-motion is harmonic, which is equivalent to assuming that the total density of gravitating mass is independent of z. The effects of relaxing this assumption will be considered later.

We need an expression for the number of stars dN in the range $dE_e\, dE_z\, dJ\, d\chi\, d\phi\, d\psi$. This is

$$dN = f(E_e, E_z, J)\, A(E_e, E_z, J)\, dE_e\, dE_z\, dJ\, d\chi\, d\phi\, d\psi, \tag{32}$$

where $A(E_e, E_z, J)\, dE_e\, dE_z\, dJ\, d\chi\, d\phi\, d\psi$ is the phase-space volume corresponding to the range $dE_e\, dE_z\, dJ\, d\chi\, d\phi\, d\psi$. In Appendix A the Jacobian factor $A(E_e, E_z, J)$ is shown to be

$$A(E_e, E_z, J) = 1/\kappa\nu. \tag{33}$$

The rate of change of $\langle E_i \rangle$ ($i = e$ or z) is then given by averaging, over all stars at that J, the expression $\langle dE_i/dt \rangle_{\text{enc}}$ for the expected rate of change for a single star due to encounters, where we add the subscript 'enc' to distinguish the average over encounters with clouds from the average over the stellar distribution:

$$d\langle E_i \rangle/dt = \frac{\iiiint\!\!\int dE_e\, dE_z\, d\chi\, d\phi\, d\psi\, \langle dE_i/dt \rangle_{\text{enc}}\, A(E_e, E_z, J)\, f(E_e, E_z, J)}{\iiiint\!\!\int dE_e\, dE_z\, d\chi\, d\phi\, d\psi\, A(E_e, E_z, J)\, f(E_e, E_z, J)}, \tag{34}$$

or, substituting from (26) and (33),

$$d\langle E_i \rangle/dt = (4\pi^2 \langle E_e \rangle \langle E_z \rangle)^{-1} \int_0^\infty dE_e \int_0^\infty dE_z \int_0^{2\pi} d\chi \int_0^{2\pi} d\phi$$
$$\langle dE_i/dt \rangle_{\text{enc}} \exp\left[-(E_e/\langle E_e \rangle + E_z/\langle E_z \rangle)\right]. \tag{35}$$

To carry out the averaging, we rewrite (23) and (25) for $\langle dE_e/dt \rangle_{\text{enc}}$ and $\langle dE_z/dt \rangle_{\text{enc}}$ in terms of E_e, E_z, χ and ϕ using (10) and (11):

$$\langle dE_e/dt \rangle_{\text{enc}} = \pi^{1/2} (G^2 N_c M_c^2 \ln \Lambda/h_c) \exp\left[-E_z \cos^2 \phi/(\nu^2 h_c^2)\right]$$
$$\times \frac{\{[\beta^2 \sin^2 \chi + (1/\beta^2) \cos^2 \chi - 2]\, E_e + (\beta^2 + 1) \sin^2 \phi\, E_z\},}{\{[\sin^2 \chi + (1/\beta^2) \cos^2 \chi]\, E_e + \sin^2 \phi\, E_z\}^{3/2}} \tag{36}$$

$$\langle dE_z/dt\rangle_{\rm enc} = \pi^{1/2}(G^2N_{\rm c}M_{\rm c}^2 \ln \Lambda/h_{\rm c}) \exp[-E_z \cos^2\phi/(\nu^2 h_{\rm c}^2)] \times \frac{\{[\sin^2\chi + (1/\beta^2)\cos^2\chi]\,E_e - 2\sin^2\phi\,E_z\}}{\{[\sin^2\chi + (1/\beta^2)\cos^2\chi]\,E_e + \sin^2\phi\,E_z\}^{3/2}}. \tag{37}$$

The details of the averaging are given in Appendix B. The final result is

$$d\langle E_e\rangle/dt = CK(\alpha,\beta)/[\langle E_e\rangle^{1/2}(1+q^2)^{1/2}], \tag{38}$$

$$d\langle E_z\rangle/dt = CL(\alpha,\beta)/[\langle E_e\rangle^{1/2}(1+q^2)^{1/2}], \tag{39}$$

where we have defined

$$C = 2G^2N_{\rm c}M_{\rm c}^2 \ln \Lambda/h_{\rm c}, \tag{40}$$

$$\alpha^2 = \langle E_z\rangle/\langle E_e\rangle \approx (\sigma_w/\sigma_u)^2, \tag{41}$$

$$q^2 = \langle E_z\rangle/(\nu^2 h_{\rm c}^2) \approx (h_{\rm s}/h_{\rm c})^2. \tag{42}$$

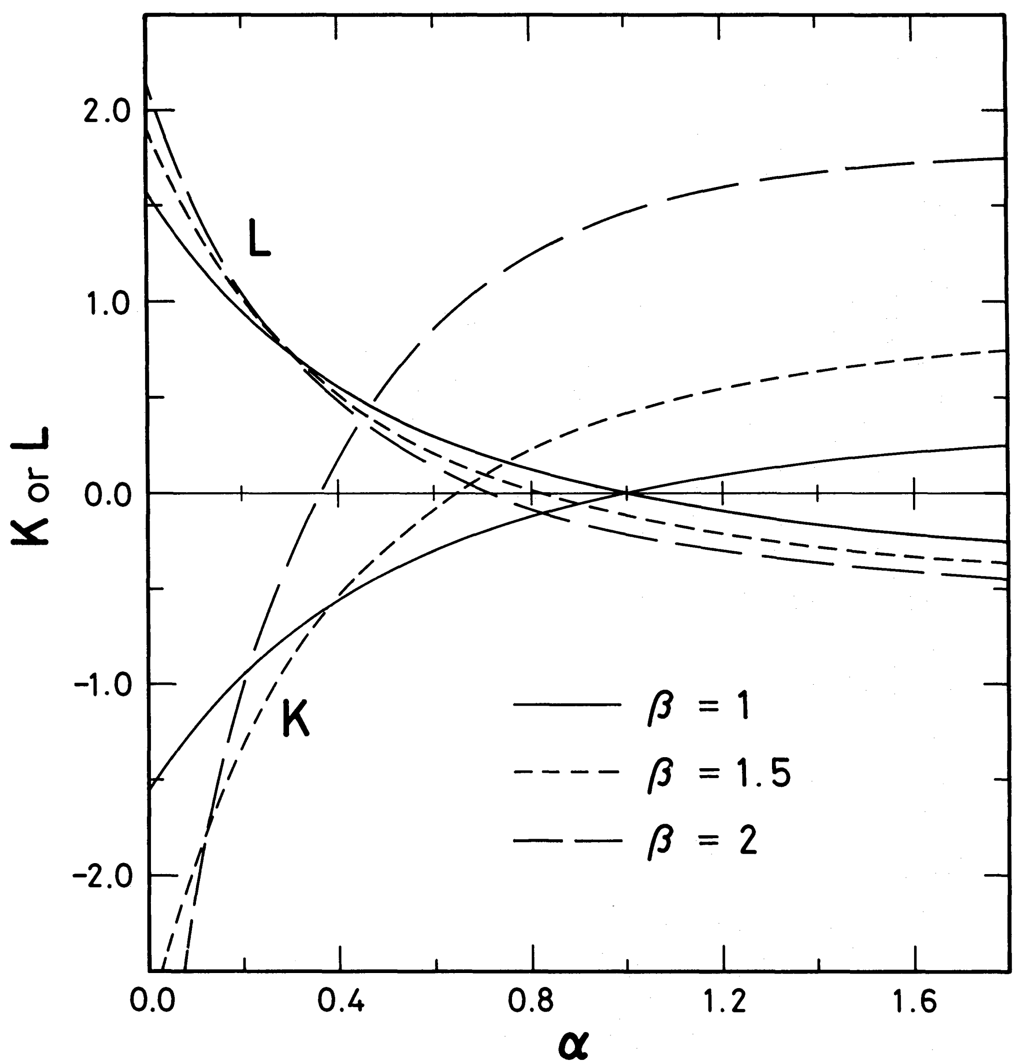

Figure 1. Dependence of rates of change of epicyclic energies on epicyclic energy ratio and rotation curve shape. Plotted are $K(\alpha, \beta)$ and $L(\alpha, \beta)$, proportional to $d\langle E_e\rangle/dt$ and $d\langle E_z\rangle/dt$ respectively (*cf* equations 38, 39), as functions of $\alpha = (\langle E_z\rangle/\langle E_e\rangle)^{1/2}$, for three values of $\beta = 2\Omega/\kappa$.

$K(\alpha, \beta)$ and $L(\alpha, \beta)$ are dimensionless integrals over the epicyclic phase defined in Appendix B (B6 and B7), and are plotted as functions of α for various values of β in Fig. 1. Combining (38), (39) and (41) we find that the ratio of vertical to radial velocity dispersions α evolves according to

$$d(\alpha^2)/dt = C[L(\alpha, \beta) - \alpha^2 K(\alpha, \beta)]/[\langle E_e\rangle^{3/2}(1+q^2)^{1/2}]. \quad (43)$$

4 Evolution of the epicyclic energies

We now use the results of Section 3 to determine the evolution of the epicyclic energies, or equivalently the components of the velocity dispersion, at a fixed radius, for which $\beta = 2\Omega/\kappa$ is a constant. Inspection of (38), (39) and (43) and of Fig. 1 showing the form of the functions K and L reveals that the evolution can be divided into two phases:

4.1 TRANSIENT RELAXATION

There is an initial transient phase in which the ratio α relaxes towards a stable equilibrium value $\alpha_s(\beta)$ defined by

$$L[\alpha_s(\beta), \beta] = \alpha_s^2(\beta)\, K[\alpha_s(\beta), \beta]. \quad (44)$$

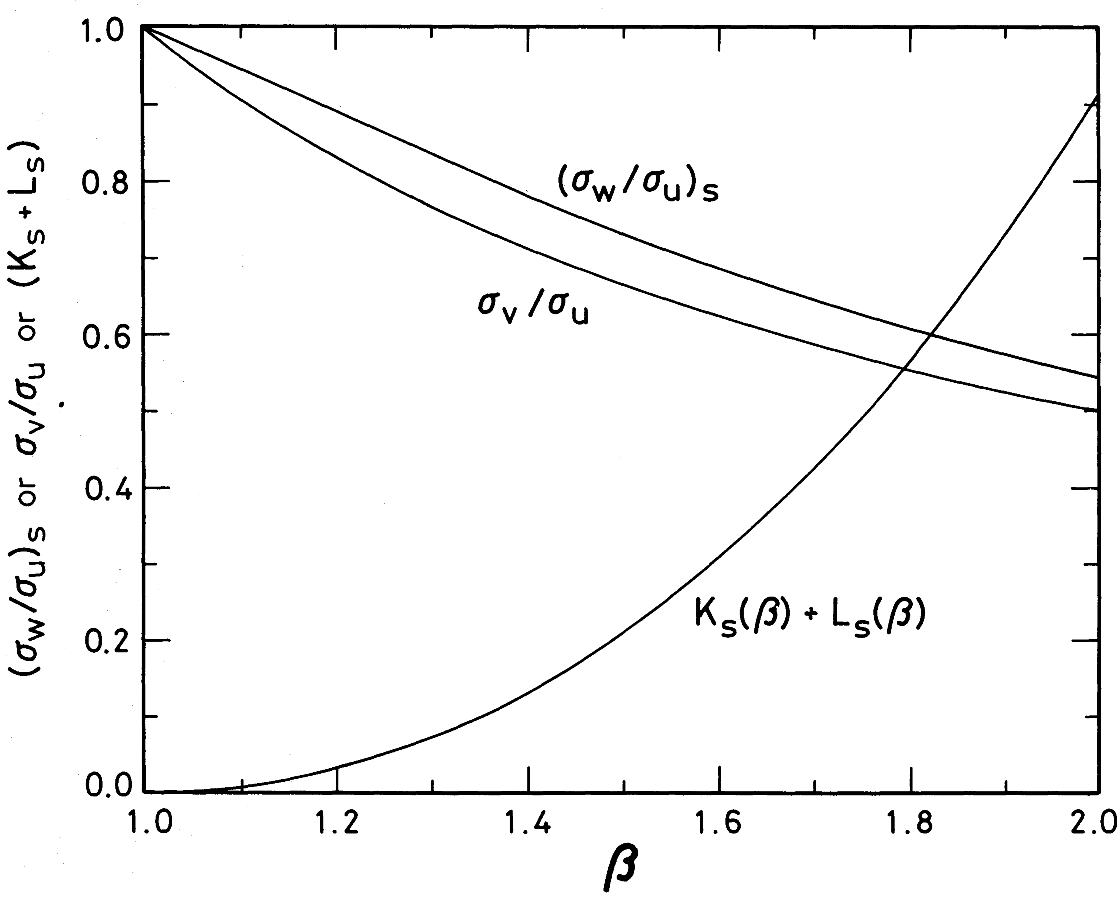

Figure 2. Dependence of velocity dispersion ratios and heating rate on rotation curve shape in steady heating phase. Plotted as functions of $\beta = 2\Omega/\kappa$ are the steady-state values of $\sigma_w/\sigma_u\,[=\alpha_s(\beta)]$ and σ_v/σ_u $(=1/\beta)$ and the function $K_s(\beta) + L_s(\beta)$, which gives the β-dependence of the net heating rate in the steady heating phase (*cf.* 46).

The function $\alpha_s(\beta)$ is plotted in Fig. 2 and numerical values are given in Table 2. The relaxation towards equilibrium occurs on a time-scale

$$T_{\rm rel} \equiv \lim_{\alpha \to \alpha_s} (\alpha_s^2 - \alpha^2)/(d\alpha^2/dt)$$

$$= [\langle E_e \rangle^{3/2} (1+q^2)^{1/2}/C] \{-2\alpha_s \Big/ \frac{d}{d\alpha} [L(\alpha, \beta) - \alpha^2 K(\alpha, \beta)]_{\alpha=\alpha_s}\}. \qquad (45)$$

Numerical values of the β-dependent factor in the brackets are given in Table 1. During this phase either $\langle E_e \rangle$ or $\langle E_z \rangle$ will decrease if α is sufficiently far from its equilibrium value, and the mean total epicyclic energy $\langle E_e + E_z \rangle$ may also decrease. The physical reason for this relaxation is simply that a star whose motion is mainly in the plane is likely to be scattered out of the plane in an encounter, and vice versa.

4.2 STEADY HEATING

Following the relaxation phase, both epicyclic energies increase steadily with time (except

Table 1. Comparison of relaxation and heating time-scales.

β	$T_{\rm rel}/(\langle E_e \rangle^{3/2}(1+q^2)^{\frac{1}{2}}/C)$	$T_{\rm heat}/(\langle E_e \rangle^{3/2}(1+q^2)^{\frac{1}{2}}/C)$
1.0	1.9	∞
1.2	1.4	56
1.4	0.95	12
1.6	0.67	4.8
1.8	0.48	2.4
2.0	0.35	1.4

Table 2. Steady-state values of α and of heating rate coefficients.

β	$\alpha_s(\beta)$	$K_s(\beta)$	$(1+1/\beta^2+\alpha_s^2)^2 K_s(\beta)/\alpha_s$
1.0	1.00	0.0	0.0
1.2	0.89	0.019	0.13
1.4	0.78	0.082	0.47
1.6	0.69	0.21	1.0
1.8	0.61	0.41	1.9
2.0	0.55	0.70	3.1

for the special case $\beta = 1$ – see below) in such a way that the ratio α remains constant at the value $\alpha_s(\beta)$. The total epicyclic energy evolves according to

$$d\langle E_e + E_z\rangle/dt = C[K_s(\beta) + L_s(\beta)]/[\langle E_e\rangle^{1/2}(1+q^2)^{1/2}], \tag{46}$$

where

$$K_s(\beta) = K[\alpha_s(\beta), \beta], \tag{47a}$$

$$L_s(\beta) = L[\alpha_s(\beta), \beta]. \tag{47b}$$

The function $K_s(\beta) + L_s(\beta)$ is plotted in Fig. 2 and numerical values of $K_s(\beta)$ are given in Table 2. The steady heating occurs on a time-scale

$$T_{\text{heat}} \equiv \langle E_e\rangle/(d\langle E_e\rangle/dt) = \langle E_z\rangle/(d\langle E_z\rangle/dt)$$

$$= [\langle E_e\rangle^{3/2}(1+q^2)^{1/2}/C]\,[1/K_s(\beta)], \tag{48}$$

(*cf.* 45). Numerical values of the β-dependent factor are given in Table 1, from which it can be seen that T_{heat} is typically an order of magnitude larger than T_{rel}.

We can restate these results in terms of the velocity dispersions: once the initial relaxation phase is over, the velocity dispersion components maintain constant ratios depending only on β:

$$\sigma_u : \sigma_v : \sigma_w \approx 1 : 1/\beta : \alpha_s(\beta), \tag{49}$$

(by equations 28, 29 and 41) where the ratio σ_v/σ_u depends only on the assumption of a steady state of the *collisionless* Boltzmann equation, while the total velocity dispersion σ, defined by

$$\sigma^2 = \sigma_u^2 + \sigma_v^2 + \sigma_w^2, \tag{50}$$

evolves as

$$d\sigma^2/dt = \{2G^2N_cM_c^2 \ln\Lambda/[\sigma(h_s^2+h_c^2)^{1/2}]\}\,\{[1+1/\beta^2+\alpha_s^2(\beta)]^{3/2}K_s(\beta)\} \tag{51}$$

(combining 46, 49, 40 and 42). The evolution can be divided into two regimes according to whether the stellar scale-height is less or greater than the cloud scale-height. In the former case, $h_s \lesssim h_c$:

$$d\sigma^2/dt \approx D_3/\sigma \qquad (h_s \lesssim h_c), \tag{52}$$

where

$$D_3 = (2G^2N_cM_c^2 \ln\Lambda/h_c)\{[1+1/\beta^2+\alpha_s^2(\beta)]^{3/2}K_s(\beta)\}, \tag{53}$$

so that for N_c and M_c constant,

$$\sigma(t) \approx [\sigma_0^3 + (3/2)D_3 t]^{1/3} \qquad (h_s \lesssim h_c), \tag{54}$$

where σ_0 is the initial velocity dispersion, while in the latter case, $h_s \gtrsim h_c$:

$$d\sigma^2/dt \approx D_4/\sigma^2 \qquad (h_s \gtrsim h_c), \tag{55}$$

where

$$D_4 = (2G^2N_cM_c^2 \ln\Lambda\,\nu)\{[1+1/\beta^2+\alpha_s^2(\beta)]^2K_s(\beta)/\alpha_s(\beta)\} \tag{56}$$

so for N_c and M_c constant

$$\sigma(t) \approx (\sigma_0^4 + 2D_4 t)^{1/4} \qquad (h_s \gtrsim h_c). \tag{57}$$

Numerical values of the β-dependent factor in D_4 are given in Table 2.

The results (52)–(54) for the regime $q = h_s/h_c \lesssim 1$ are given mainly for completeness and for the purpose of comparing with previous calculations. Equations (38) and (39) from which they are derived are formally correct for an arbitrary value of q, but in practice the cloud layer must owe its finite thickness h_c to a finite velocity dispersion, so our approximation of neglecting the cloud velocity dispersion will break down for $q \lesssim 1$ unless the stellar velocity dispersion is very anisotropic.

4.3 SPECIAL CASE OF SOLID-BODY ROTATION

The case when the galactic rotation curve is $\Omega = \text{constant} \Rightarrow \beta = 1$ provides a useful check on the correctness of our results. We note first that $K(\alpha, \beta) = -L(\alpha, \beta)$ in this case $\Rightarrow \langle E_e + E_z \rangle = \text{constant}$. That this must be the case follows from the fact that the Jacobi integral $E - \Omega J$ is now an exact integral when the potential due to the clouds is included in the energy, and that $E - \Omega J \approx E_e + E_z$ except in the vicinity of a cloud. A further consequence of the existence of this exact integral is that $f(E - \Omega J)$ must be a steady-state solution even in the presence of encounters, so that equilibrium must correspond to $\langle E_e \rangle = \langle E_z \rangle$. (I am grateful to D. Lynden-Bell for pointing this out to me.) Our calculation is in agreement with this. Thus for $\beta = 1$, α relaxes to $\alpha_s = 1$, so that $\sigma_u = \sigma_v = \sigma_w$, but no secular increase of velocity dispersions occurs.

5 Effects of modifications to the epicyclic approximation for z-motions

The calculations of Section 3 assume that the unperturbed z-motions of the stars are harmonic with a time-independent frequency ν. In this section we consider the effects of relaxing these assumptions. The results have applications to stars whose velocity dispersions are comparable to or exceed that of the disc as a whole.

5.1 EFFECTS OF DEPARTURES FROM THE HARMONIC APPROXIMATION

For stars whose vertical amplitudes are comparable to or exceed the scale-height h_D of mass in the disc, the z-motion is no longer harmonic and the vertical oscillation frequency is a function of vertical epicyclic energy, $\nu = \nu(E_z)$, where $E_z = \tfrac{1}{2}w^2 + [\Phi(R, z) - \Phi(R, 0)]$. Equations (10c) and (11c) no longer apply, but we can still define a phase ϕ evolving according to (9c), and equation (33) for $A(E_e, E_z, J)$ still applies, with $\nu = \nu(E_z)$ (see Appendix A).

As before, we obtain expressions for the rates of change of $\langle E_e \rangle$ and $\langle E_z \rangle$, defined to be averages at fixed J, by combining (34), (33), (23) and (25). We further take the limit $h_c \ll h_s$, so that the cloud number density (equation 22) becomes

$$N_c [1/(2\pi)^{1/2} h_c] \exp(-z^2/2h_c^2) \to N_c \delta(z)$$
$$= N_c [\nu(E_z)/\sqrt{2E_z}] [\delta(\phi - \pi/2) + \delta(\phi + \pi/2)], \tag{58}$$

where we have used the result $(\partial z/\partial \phi)_{E_z} = w/\nu(E_z)$, which follows using (9c), and chosen the zero-point of ϕ so that $z = 0$ when $\phi = \pi/2, 3\pi/2$. Performing the ϕ-integrations in the averages over the stellar distribution we obtain

$$\frac{d\langle E_e \rangle}{dt} \approx 2G^2 N_c M_c^2 \ln \Lambda \; \frac{\displaystyle\iiint dE_e\, dE_z\, d\chi\, f(E_e, E_z, J) \times \frac{1}{\sqrt{2E_z}} \left[\frac{(\beta^2 - 2) u^2 + (1 - 2\beta^2) v^2 + 2(\beta^2 + 1) E_z}{(u^2 + v^2 + 2E_z)^{3/2}} \right]}{\displaystyle\iiint dE_e\, dE_z\, d\chi \frac{1}{\nu(E_z)} f(E_e, E_z, J)}, \tag{59}$$

$$\frac{d\langle E_z\rangle}{dt} \approx 2G^2 N_c M_c^2 \ln\Lambda \iiint dE_e\, dE_z\, d\chi\, f(E_e, E_z, J)$$

$$\times \frac{\dfrac{1}{\sqrt{2E_z}}\left[\dfrac{u^2+v^2-4E_z}{(u^2+v^2+2E_z)^{3/2}}\right]}{\iiint dE_e\, dE_z\, d\chi \dfrac{1}{\nu(E_z)} f(E_e, E_z, J)}, \tag{60}$$

where u and v are to be expressed in terms of E_e and χ using (11a) and (11b) as before. Note that the mean epicyclic energies are in general no longer related to the velocity dispersions in any simple way. We again assume an isothermal distribution function,

$$f(E_e, E_z, J) = S(J) \exp\{-[E_e/E_e^*(J) + E_z/E_z^*(J)]\}, \tag{61}$$

so that $\sigma_u^2 \approx E_e^*$, $\sigma_v^2 \approx E_e^*/\beta^2$, $\sigma_w^2 \approx E_z^*$, where as before the velocity dispersions are z-averaged. $\langle E_e\rangle = E_e^*$ but $\langle E_z\rangle \neq E_z^*$ in general. The results of the averaging for this case can be straightforwardly derived by noting that $\nu(E_z)$ only appears in the denominators of (59) and (60) and by requiring that for $\nu(E_z) = \text{constant}$ they reduce to the $h_c \to 0$ limits of (38) and (39). We obtain

$$d\langle E_e\rangle/dt \approx 2G^2 N_c M_c^2 \ln\Lambda\, \bar{\nu}(E_z^*)\, K(\alpha,\beta)/(E_e^{*1/2} E_z^{*1/2}) \qquad (h_s \gg h_c), \tag{62}$$

$$d\langle E_z\rangle/dt \approx 2G^2 N_c M_c^2 \ln\Lambda\, \bar{\nu}(E_z^*)\, L(\alpha,\beta)/(E_e^{*1/2} E_z^{*1/2}) \qquad (h_s \gg h_c). \tag{63}$$

where

$$1/\bar{\nu}(E_z^*) = (1/E_z^*) \int_0^\infty dE_z [1/\nu(E_z)] \exp(-E_z/E_z^*). \tag{64}$$

These results reduce further for the case that the stellar scale-height h_s greatly exceeds the disc scale-height h_D (here defined as the ratio of the surface density to twice the central density). Then the stars move in a potential with z-dependence $\Phi(R, z) - \Phi(R, 0) \approx \nu_0^2 h_D z$, where $\nu_0 = \nu(E_z = 0)$, and have frequencies $\nu(E_z) \approx (\pi/2)\nu_0^2 h_D/(2E_z)^{1/2}$. Then $\langle E_z\rangle = (3/2) E_z^*$, and we derive

$$\frac{dE_e^*}{dt} \approx \frac{(2\pi)^{1/2} G^2 N_c M_c^2 \ln\Lambda\, \nu_0^2 h_D}{E_e^{*1/2} E_z^*} \cdot K(\alpha^*, \beta) \qquad (h_s \gg h_D), \tag{65}$$

$$\frac{dE_z^*}{dt} \approx \frac{(2\pi)^{1/2} G^2 N_c M_c^2 \ln\Lambda\, \nu_0^2 h_D}{E_e^{*1/2} E_z^*} \cdot \frac{2}{3} L(\alpha^*, \beta) \qquad (h_s \gg h_D), \tag{66}$$

where we now define $\alpha^* = (E_z^*/E_e^*)^{1/2} \approx \sigma_w/\sigma_u$. The equilibrium ratio of vertical to horizontal velocity dispersions $\alpha_s^*(\beta)$ is the solution of

$$2/3 L[\alpha_s^*(\beta), \beta] = \alpha_s^{*2}(\beta) K[\alpha_s^*(\beta), \beta], \tag{67}$$

(compare equation 44). Over the range $1 \leq \beta \leq 2$, $\alpha_s^*(\beta)$ differs from $\alpha_s(\beta)$ by at most 5 per cent. In the steady heating phase, the total velocity dispersion σ varies as

$$d\sigma^2/dt \approx D_5/\sigma^3 \qquad (h_s \gg h_D), \tag{68}$$

where

$$D_5 = [(2\pi)^{1/2} G^2 N_c M_c^2 \ln\Lambda\, \nu_0^2 h_D]\, \{[1 + 1/\beta^2 + \alpha_s^{*2}(\beta)]^{5/2} K[\alpha_s^*(\beta), \beta]/\alpha_s^{*2}(\beta)\} \tag{69}$$

An equation of the form of (68) was derived by Ostriker (1982, private communication).

From the results of this section we infer that for typical disc stars (i.e. with $h_s \lesssim h_D$) the effects of departures from the harmonic approximation are likely to be small provided the distribution remains approximately isothermal.

5.2 EFFECTS OF A TIME-DEPENDENT VERTICAL FREQUENCY

Another extension of the results of previous sections is to allow the vertical epicyclic frequency ν to be a function of time. If ν varies slowly compared to the orbital frequency, then for stars whose z-motion is harmonic, E_z/ν is adiabatically invariant and there is an extra contribution to the rate of change of the vertical epicyclic energy

$$(d\langle E_z\rangle/dt)_{\text{ad}} = (\dot{\nu}/\nu)\langle E_z\rangle, \tag{70}$$

which must be added to that of (39).

We can apply this to a population which is self-gravitating in the z-direction. If the population is isothermal, its density varies as $\rho(z) = \rho_0 \operatorname{sech}^2(z/h_D)$ with a scale-height

$$h_D = \sigma_w^2/\pi G\mu_D \approx \langle E_z\rangle/\pi G\mu_D, \tag{71}$$

where μ_D is the surface density, and the vertical frequency for small oscillations is

$$\nu_0 = (4\pi G\rho_0)^{1/2} \approx 2^{1/2}\pi G\mu_D/\langle E_z\rangle^{1/2}. \tag{72}$$

The z-motions are treated as being harmonic with frequency ν_0, though this is only a rough approximation as applied to the whole population. Then adiabatic invariance gives

$$(d\langle E_z\rangle/dt)_{\text{ad}} \approx -(1/2)\, d\langle E_z\rangle/dt + (\dot{\mu}_D/\mu_D)\langle E_z\rangle \tag{73}$$

where the $\dot{\mu}_D/\mu_D$ terms allows for the possibility that the disc mass is increased by infall. Combining with (38), (39), (40) and (42) and assuming $h_s \gg h_c$, we derive

$$\frac{d\langle E_e\rangle}{dt} \approx \frac{2^{3/2}\pi G^3\mu_D N_c M_c^2 \ln\Lambda}{\langle E_e\rangle^{1/2}\langle E_z\rangle}\cdot K(\alpha,\beta), \tag{74}$$

$$\frac{d\langle E_z\rangle}{dt} \approx \frac{2^{3/2}\pi G^3\mu_D N_c M_c^2 \ln\Lambda}{\langle E_e\rangle^{1/2}\langle E_z\rangle}\cdot \frac{2}{3}L(\alpha,\beta) + \frac{2}{3}\frac{\dot{\mu}_D}{\mu_D}\langle E_z\rangle. \tag{75}$$

In practice, the $\dot{\mu}_D/\mu_D$ term is likely only to be important for a small number of stars formed very early in the history of the disc (*cf.* Lacey & Fall 1983), so the main effect of making the disc self-consistent is to reduce the heating rate by another power of σ. (The equilibrium value of σ_w/σ_u is also changed, being given by (67) if the $\dot{\mu}_D/\mu_D$ term can be ignored.) For subpopulations with smaller velocity dispersions than that of the disc as a whole, the effects of the variation of ν will be very small.

6 Comparison with previous calculations

The results of this paper are most directly comparable to those of Spitzer & Schwarzschild (1953) since the assumptions made are the same apart from their neglect of vertical motions. Our result for the evolution of the total velocity dispersion σ for stars whose orbits are confined to the cloud layer ($h_s \lesssim h_c$), equations (52)–(53), is identical to their equation (20) apart from numerical factors of order unity. This shows that our results are a generalization of theirs. For stars whose scale-heights exceed that of the cloud layer ($h_s \gtrsim h_c$), we find the heating rate to be suppressed by a factor $h_c/h_s \propto 1/\sigma$ (equations 55–56) relative to what Spitzer & Schwarzschild derive.

It is more difficult to compare our results with those of Icke (1982) since his calculations are all numerical, but in so far as he obtains the Spitzer & Schwarzschild (1953) result $\sigma \sim t^{1/3}$ for epicyclic orbits in the plane, his results are consistent with ours. Icke does not consider vertical motions.

Fujimoto (1980) derives the result $\sigma \sim t^{1/2}$, but this is based on the assumption that the cloud lifetime is much less than the minimum star–cloud encounter time, which does not seem to be the case in practice, as mentioned in Section 2. Fujimoto's calculation does include z-motions, but only appears to be applicable to stars whose vertical motions do not take them outside the cloud layer.

The derivation of the axial ratios of the velocity ellipsoid by Wielen (1977) should also be discussed in order to point out how his calculation differs from this one. Wielen's calculation is not based on a specific physical mechanism, but rather assumes 'isotropic diffusion in velocity space', for which $\langle \Delta u \rangle = \langle \Delta v \rangle = \langle \Delta w \rangle = 0$ and $\langle (\Delta u)^2 \rangle = \langle (\Delta v)^2 \rangle = \langle (\Delta w)^2 \rangle$. It follows that $\Delta \langle E_e \rangle = (1+\beta^2)\, \Delta \langle E_z \rangle$, so that the velocity dispersion ratio approaches $\sigma_w/\sigma_u = (1+\beta^2)^{-1/2}$ for $\sigma \gg \sigma_0$, but there is no relaxation in the sense of Section 4, which depends on the non-vanishing of $\langle \Delta \mathbf{V} \rangle$. The moments of $\Delta \mathbf{V}$ for scattering by clouds (equations 19 and 20) are different from those assumed by Wielen, so the value of σ_w/σ_u derived is different.

After work on this paper was completed I became aware of a recent N-body simulation of the heating of a stellar disc by giant molecular clouds by Villumsen (1983). Villumsen finds different results from those presented here, in particular he obtains the time-dependence $\sigma \sim t^{1/2}$ and obtains the ratio of velocity dispersions $\sigma_w/\sigma_u \approx 0.5$. In his simulation the stars start 'hot' with a velocity dispersion in the plane roughly half the final value, and the clouds have a velocity dispersion in the plane equal to the initial value for the stars. Thus the velocity dispersion of the clouds is always comparable to that of the stars, while the small factor by which the total stellar velocity dispersion increases makes it difficult to determine accurately the asymptotic power-law dependence of velocity dispersion on time. I believe that these two features are the main reasons for the differences in the results obtained.

7 Comparison with observations

A long-standing problem in galactic evolution is to explain why, in the solar neighbourhood at least, the velocity dispersions of disc stars increase with their age (see, e.g. Wielen 1974 for a recent observational analysis). Wielen (1977) has argued persuasively that the origin of this effect must lie in stochastic acceleration of the stars after they are born rather than in any time-dependence of the velocity dispersion of the gas layer from which they form. We will not repeat the arguments here. In this section we use the results of Section 3 to examine whether the giant gas clouds observed in the disc can be the dominant cause of the stochastic acceleration of the stars.

In making this comparison we shall assume that the stars are in the steady heating phase and check whether this is consistent with the observations, since if the stars are still in the relaxation phase there will not yet have been any significant increase of total velocity dispersion. We compare theoretical predictions with observations for three observationally independent aspects: the shape of the velocity ellipsoid, the age-dependence of the total velocity dispersion and the dependence of disc scale-height on radius.

7.1 SHAPE OF THE VELOCITY ELLIPSOID

In the steady heating phase, for $h_s \lesssim h_D$, the ratio of axes $\sigma_u : \sigma_v : \sigma_w$ is given by equation (49) (see also Fig. 2), being independent of time and depending only on the value of $\beta = 2\Omega/\kappa$. Over the whole range $1 < \beta < 2$ the prediction is $\sigma_u > \sigma_w > \sigma_v$, while virtually all observational determinations give $\sigma_u > \sigma_v > \sigma_w$ (Delhaye 1965). To predict the ratios in more detail one needs to know the local value of β, which can be estimated either directly from the rotation curve $V_c(R)$ or from the ratio σ_v/σ_u (by equation 49). According to Knapp (1983), the rotation curve in the solar neighbourhood is flat or slowly rising, $d \ln V_c / d \ln R \approx 0.1 \pm 0.1$, which would be consistent with what is observed in other galaxies (Rubin 1983). This implies $\beta \approx 1.35 \pm 0.05$, for which we predict $\sigma_u : \sigma_v : \sigma_w = 1 : 0.74 \pm 0.03 : 0.81 \pm 0.03$. The observational determinations of the axial ratios are somewhat ambiguous. Two recent determinations are that of Wielen (1974), based on a sample of roughly 300 McCormick K and M dwarfs within 25 pc of the Sun, who finds $\sigma_u : \sigma_v : \sigma_w = 1 : 0.59 \pm 0.05 : 0.51 \pm 0.04$, and that of Woolley *et al.* (1977), based on roughly 700 G and K stars (mainly giants) out to roughly 400 pc, who find $\sigma_u : \sigma_v : \sigma_w = 1 : 0.74 \pm 0.04 : 0.71 \pm 0.04$. (Wielen takes scale-height effects into account empirically to estimate z-averaged velocity dispersions, but finds that this makes no significant difference to the ratios.) Individually both sets of ratios are inconsistent with the theory at the 2σ level when β is determined from σ_v/σ_u, and both in the sense that the observed σ_w/σ_u is less than that predicted. However, the two observational determinations also differ from each other at the 2σ level, and of the σ_v/σ_u values only that of Woolley *et al.* is consistent with the rotation-curve estimate to within the quoted errors.

7.2 AGE-DEPENDENCE OF THE VELOCITY DISPERSION

The scale-height regime relevant for most disc stars is $h_c \lesssim h_s \lesssim h_D$ so that equations (55)–(57) apply. Then for a *constant* diffusion coefficient D_4 the age-dependence for $\sigma^4 \gg \sigma_0^4$ is $\sigma \propto \tau^{1/4}$ for stars all of the same age τ. Observations of velocity dispersion as a function of age (Wielen 1974, 1977) imply a somewhat steeper dependence, $\sigma \propto \tau^{1/3}$ or $\tau^{1/2}$ if a constant star formation rate is assumed. Also Lacey & Fall (1983), fitting the relation $d\sigma_w^2/dt = D\sigma_w^{2-q}$, with D proportional to the star formation rate, to Wielen's data on z-motions alone, found $q \approx 2$ to give the best fit in the context of a set of models for the chemical and kinematical evolution of the solar neighbourhood, while we predict $q = 4$ here. However, there are considerable uncertainties in the observational determination, and consistency between the theory and observations is possible if we allow some time-dependence of the diffusion coefficient D_4 in the sense of it being larger at early times.

Another test is to estimate the present solar neighbourhood value of D_4 from the observed properties of the giant molecular clouds. The clouds actually have a spectrum of masses, so in our formulae we make the replacement

$$N_c M_c^2 \rightarrow \int N_c(M_c)\, M_c^2\, dM_c \equiv \mu_c M_c^*, \tag{76}$$

where μ_c is the surface density of cloud matter and M_c^* is the mass-weighted mean mass:

$$M_c^* = \int N_c(M_c)\, M_c^2\, dM_c \Big/ \int N_c(M_c)\, M_c\, dM_c. \tag{77}$$

The results of Liszt *et al.* (1981) give $\mu_c \approx 2.6\,[n(H_2)/300\ \mathrm{cm}^{-3}]\, M_\odot\,\mathrm{pc}^{-2}$ for the solar neighbourhood and their distribution function for cloud diameters gives $M_c^* \approx 6.6 \times 10^5$

$[n(H_2)/300\ \mathrm{cm}^{-3}]\ M_\odot$, assuming the cloud density to be independent of size (where we have assumed 25 per cent helium by mass and a Gaussian scale-height of 50 pc for the cloud layer). Here $n(H_2)$ is the internal density of the clouds, for which a reasonable range based on Liszt *et al.*'s non-LTE analysis of the CO emission is $150 \lesssim n(H_2) \lesssim 500\ \mathrm{cm}^{-3}$. This implies

$$4\times10^5 \lesssim \mu_c M_c^* \lesssim 5\times10^6\ M_\odot^2\ \mathrm{pc}^{-2}. \tag{78}$$

The results of Sanders *et al.* (1984) and Sanders, Scoville & Solomon (1984), based in part on Virial theorem estimates of cloud masses, give $\mu_c \approx 4\ M_\odot\ \mathrm{pc}^{-2}$ and $M_c^* \approx 1\times10^6\ M_\odot$, which gives $\mu_c M_c^* \approx 4\times10^6\ M_\odot^2\ \mathrm{pc}^{-2}$, at the upper end of the range (78), which is therefore taken as a rough indicator of the range of uncertainty. (However, Sanders & Solomon 1984) find evidence for the clustering of clouds on mass scales of about $10^7\ M_\odot$, which would further increase $\mu_c M_c^*$).

Taking into account both the rotation curve and the velocity ellipsoid data, I estimate the possible range for β to be $1.3 \lesssim \beta \lesssim 1.8$, which gives for the β-dependent factor in D_4

$$0.3 \lesssim [1 + 1/\beta^2 + \alpha_s^2(\beta)]^2\, K_s(\beta)/\alpha_s(\beta) \lesssim 1.9. \tag{79}$$

Oort's (1960) determination of K_z gives $\nu \approx 90\ \mathrm{km\ s^{-1}\ kpc^{-1}}$ and we take $\ln\Lambda \approx 3$ (appropriate for $l_{\min} \approx 20$ pc, $l_{\max} \approx 500$ pc). Then, assuming the ranges (78) and (79), we derive

$$10^3 \lesssim D_4 \lesssim 10^5\ (\mathrm{km\ s^{-1}})^4\ \mathrm{Gy}^{-1}. \tag{80}$$

If D_4 is constant and $\sigma_0 = 10\ \mathrm{km\ s^{-1}}$, then on the basis of (57) we predict that the time $\tau_{1/2}$ for the velocity dispersion to reach twice its initial is $1 \lesssim \tau_{1/2} \lesssim 100$ Gy, compared to the observed value $\tau_{1/2} \sim 1$ Gy (Wielen 1974). The velocity dispersion of the oldest disc stars, with ages of about 12 Gy, is predicted to be in the range $10 \lesssim \sigma \lesssim 40\ \mathrm{km\ s^{-1}}$. (This result is not significantly changed if instead we treat the disc as being self-consistent and use (74) and (75) with a constant $\mu_D \approx 80\ M_\odot\ \mathrm{pc}^{-2}$.) If D_4 is time-dependent then the upper limit should be multiplied by $(\bar{D}_4/D_4)^{1/4}$ where $\bar{D}_4$ is the time-average value. This should be compared with $\sigma \approx 60$–$80\ \mathrm{km\ s^{-1}}$ obtained by Wielen (1974) empirically for the oldest age groups. Thus if the cloud masses are at the upper end of the observationally allowed range then they can account for the increase in velocity dispersions of young stars (in fact this puts an upper limit on D_4), but in order to account for the velocity dispersions of old stars also we require D_4 to have been larger at early times.

7.3 RADIAL DEPENDENCE OF THE DISC SCALE-HEIGHT

If the disc as a whole is approximated as being locally isothermal and is self-gravitating in the z-direction, and if the infall term $\dot{\mu}_D/\mu_D$ is negligible, then according to (74) and (75), the velocity dispersion varies as

$$d\sigma^2/dt \propto \mu_D \mu_c M_c^*/\sigma^3, \tag{81}$$

so that the radial dependence of σ at a fixed time is given by $\sigma \propto (\mu_D \overline{\mu_c M_c^*})^{1/5}$ (for $\sigma \gg \sigma_0$), where we have ignored the β-dependence and the bar denotes a time-average. Then the disc scale-height is given by (71) and varies radially as

$$h_D \propto (\overline{\mu_c M_c^*})^{2/5}/\mu_D^{3/5}. \tag{82}$$

To reduce this further, some assumptions must be made about how μ_c and M_c^* vary with radius. A plausible pair of assumptions is $M_c^* =$ constant, $\overline{\mu_c} \propto \mu_D$ (*cf.* Lacey & Fall 1983), which gives

$$h_D(R) \propto \mu_D(R)^{-1/5} \tag{83}$$

or $h_D \propto \exp(\alpha R/5)$ for an exponential disc with $\mu_D \propto \exp(-\alpha R)$. (This corresponds to $q = 5/2$ in the notation of Lacey & Fall 1983, which differs from the value $q = 4$ for the velocity dispersion–age relation because Lacey & Fall did not include the variation of ν in their treatment.) This is probably consistent with the observations of edge-on disc galaxies by van der Kruit & Searle (1981a, b, 1982) which show that the disc scale-height is approximately independent of radius.

Subsection 7.4 Discussion

We see that the cloud heating mechanism can explain the increase in velocity disperions of young stars if cloud masses are at the upper end of the allowed range, and also the velocity dispersions of old disc stars if sufficient time variation of μ_c and M_c^* is allowed. Given plausible assumptions about the radial variation of μ_c and M_c^* it can also explain the approximate constancy of disc scale-height with radius. However, the predicted σ_w/σ_u is 10–30 per cent too high compared to that observed, and this prediction is independent of cloud properties, depending only on β. But it is not clear how significant the disagreement is, since different measurements of the velocity dispersion ratios are inconsistent with each other and with measurements of the rotation curve. It would be very useful for testing theories of stellar heating if a more consistent set of values could be obtained observationally.

Possible explanations for the discrepancy in the velocity dispersion ratios include distortion of the local stellar kinematics by the potential of a spiral density wave (which might also help to explain the discrepancy between different estimates of β) and departures from isothermality in the stellar distribution function, which might modify the equilibrium ratio of epicyclic energies. Based on the results of Section 5, it does not seem likely that departures from harmonic motion in the z-direction would by themselves significantly change the result. Other effects not taken into account in the present calculation which might modify the results presented here include the clustering of clouds on large scales, and the increase of the effective masses of the clouds by stellar wakes (Julian & Toomre 1966; Julian 1967).

There are other possible mechanisms for heating the disc which might compete with or dominate heating by clouds. These include heating by transient spiral density waves (Barbanis & Woltjer 1967; Carlberg & Sellwood 1984) or by massive black holes in the galactic halo (Lacey, Ostriker & Schmidt, in preparation). It is interesting that heating by a spiral density wave appears to produce too little vertical compared to horizontal heating (Carlberg 1983, private communication), which is the opposite problem to that for the cloud mechanism. It is possible that the resolution of the heating problem lies with a mechanism producing perturbations in the potential on both large and small scales.

8 Conclusions

We have derived equations for the evolution of the three components of velocity dispersion of the stars in a galactic disc when the orbits of these stars are perturbed by massive gas clouds moving in circular orbits in the disc. We find that the evolution can be divided into two phases:

(1) An initial relaxation phase in which the ratios of vertical to horizontal velocity dispersions (σ_w/σ_u and σ_w/σ_v) relax to values depending only on the rotation curve parameter $\beta = 2\Omega/\kappa$.

(2) A steady heating phase (absent for a solid-body rotation curve) in which the velocity dispersions increase with time while the shape of the velocity ellipsoid remains fixed. For a

population of stars whole scale-height exceeds that of the cloud layer but is less than that of the disc, the dispersion varies as $d\sigma^2/dt \propto N_c M_c^2 \nu/\sigma^2$. This result is modified for other scale-height regimes.

Comparing these results with observations, we find that clouds may be important for increasing the velocity dispersions of young stars if the cloud masses are at the upper end of the observationally allowed range (with mass-weighted mean mass $M_c^* \sim 10^6 M_\odot$) but that to account for the velocity dispersions of old stars also, the cloud number densities and/or masses need to have been larger in the past (on the average by a factor ~ 10). If the cloud number density follows the total surface density of the disc then we can explain the approximate constancy of disc scale-height. However, there is a discrepancy between the predicted and observed velocity dispersion ratios, the predicted σ_w/σ_u being 10–30 per cent too large. It may be possible to explain this difference within the context of heating by clouds. Alternatively, some other heating mechanism may be involved.

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Appendix A: derivation of $A(E_e, E_z, J)$

The Jacobian factor $A(E_e, E_z, J)$ is defined by the equation

$$A(E_e, E_z, J)\, dE_e\, dE_z\, dJ\, d\chi\, d\phi\, d\psi = d^2x d^3 v, \tag{A1}$$

where $d^3x\, d^3v$ represents the phase-space volume expressed in terms of Cartesian coordinates and velocities. For the case that the epicyclic oscillations are harmonic, as assumed in Sections 3 and 4, A may be derived directly from (10) and (11). Here we present a more general derivation, valid whenever the potential is axisymmetric and separable in R and z, i.e. $\Phi(R, z) = \Phi_1(R) + \Phi_2(z)$. Then the motion is triply periodic, and we can define action variables (J_R, J, J_z) and angle variables (χ, ψ', ϕ) associated with the (R, θ, z) motions respectively. (For the properties of action-angle variables, see, e.g. Goldstein 1980). For the case that the R- and z-oscillations are harmonic, the definitions of χ and ϕ made here reduce to those of Section 3.1, while $\psi' = \psi + \Omega t$, so that $d\psi' = d\psi$. Action-angle variables are canonically conjugate, so that

$$d^3xd^3v = dJ_R\, dJ\, dJ_z\, d\chi\, d\psi\, d\phi. \tag{A2}$$

Further, our assumption of separability means that (14) is still valid, with $E_e = E_e(J_R, J)$ and $E_z = E_z(J_z)$, so that $\kappa = \partial E(J_R, J, J_z)/\partial J_R = \partial E_e/\partial J_R$ and $\nu = \partial E(J_R, J, J_z)/\partial J_z = dE_z/dJ_z$, where κ and ν are the angular frequencies of the (not necessarily harmonic) R- and z-oscillations. Thus (A2) becomes

$$d^3xd^3v = (1/\kappa\nu)\, dE_e\, dE_z\, dJ\, d\chi\, d\phi\, d\psi, \tag{A3}$$

which, comparing with (A1), proves the result

$$A(E_e, E_z, J) = 1/\kappa\nu. \tag{A4}$$

Appendix B: calculation of population-averaged heating rates

We define

$$a(\chi) = \sin^2\chi + (1/\beta^2)\cos^2\chi, \tag{B1}$$

$$b(\chi) = 2 - (\beta^2\sin^2\chi + (1/\beta^2)\cos^2\chi). \tag{B2}$$

We note that the integrals over E_e and E_z in (35) are Laplace transforms with transform variables $1/\langle E_e\rangle$ and $(1+q^2\cos^2\phi)/\langle E_z\rangle$ respectively. We make use of the result

$$\int_0^\infty dx \int_0^\infty dy \exp[-(sx+ty)]\, x/(ax+by)^{3/2} = \pi^{1/2}/\{a^{1/2}s^{1/2}[(bs)^{1/2}+(at)^{1/2}]^2\} \tag{B3}$$

derived with the help of formulae (4.2.20) and (4.12.10) of Erdelyi *et al.* (1954). Some algebra gives

$$d\langle E_e\rangle/dt = (C/\langle E_e\rangle^{1/2})(2/\pi)\int_0^{\pi/2} d\chi \int_0^{\pi/2} d\phi \times \frac{[(3-b)\,\alpha\sin\phi-(1+q^2\cos^2\phi)^{1/2}\,ba^{1/2}]}{a(1+q^2\cos^2\phi)^{1/2}[(1+q^2\cos^2\phi)^{1/2}a^{1/2}+\alpha\sin\phi]^2}, \quad \text{(B4)}$$

$$d\langle E_z\rangle/dt = (C/\langle E_e\rangle^{1/2})(2/\pi)\int_0^{\pi/2} d\chi \int_0^{\pi/2} d\phi \times \frac{[(1+q^2\cos^2\phi)^{1/2}a^{1/2}-2\alpha\sin\phi]}{(1+q^2\cos^2\phi)^{1/2}[(1+q^2\cos^2\phi)^{1/2}a^{1/2}+\alpha\sin\phi]^2}. \quad \text{(B5)}$$

The integrals over ϕ may be performed by making the substitution $\zeta = \sin\phi/(1+q^2\cos^2\phi)^{1/2}$ but the integrals over χ must be performed numerically. The final answers are equations (38) and (39). The functions $K(\alpha, \beta)$ and $L(\alpha, \beta)$ are defined to be

$$K(\alpha,\beta) = (2/\pi)\int_0^{\pi/2} d\chi\,[3-(bs+3)\tan^{-1}\sqrt{s}/\sqrt{s}]/[\alpha^3 s(s+1)], \quad \text{(B6)}$$

$$L(\alpha,\beta) = (2/\pi)\int_0^{\pi/2} d\chi\,[-3+(s+3)\tan^{-1}\sqrt{s}/\sqrt{s}]/(\alpha s), \quad \text{(B7)}$$

$$s(\chi) = a(\chi)/\alpha^2 - 1. \quad \text{(B8)}$$

(Note that the factor

$$\tan^{-1}\sqrt{s}/\sqrt{s} = \tanh^{-1}\sqrt{-s}/\sqrt{-s}$$

and so is real-valued for any real s).