AN ATAVISTIC LIE ALGEBRA

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Abstract

An infinite-dimensional Lie Algebra is proposed which includes, in its subalgebras and limits, most Lie Algebras routinely utilized in physics. It relies on the finite oscillator Lie group, and appears applicable to twisted noncommutative QFT and CFT.

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1 Introduction

Consider the familiar forced oscillator

$$H = \alpha^\dagger\ \alpha + \lambda(t)\alpha^\dagger + \lambda^*(t)\alpha\ , \tag{1}$$

driven by an external time-dependent source.

Since $H = \exp(-\lambda(t)\alpha^\dagger + \lambda^*(t)\alpha) \left(\alpha^\dagger\alpha - |\lambda(t)|^2\right) \exp(\lambda(t)\alpha^\dagger - \lambda^*(t)\alpha)$, the integration of the model relies on the coherent states [1] underlain by the noncompact oscillator group $\mathcal{G}$ (or $\mathcal{H}_4$) [2]. This is the solvable, rank 2, dimension 4, Lie group generated by the oscillator creation and annihilation operators, their Heisenberg commutator (which is central), and the occupation number operator. It turns out that this group, $\mathcal{G}$, as well as related Vertex algebras [3], control the structure of an infinite dimensional Lie Algebra brought into play in this letter. This algebra, remarkably, contains as subalgebras or limits of it, most of the Lie algebras routinely encountered in physics.

The infinite dimensional Lie algebra we introduce in this letter and explore below is

$$[J^a_{m_1,m_2}, J^b_{n_1,n_2}] = e^{is(m_1 e^{-a} n_2 - m_2 e^a n_1)} J^{a+b}_{m_1+e^a n_1,\ m_2+e^{-a} n_2} - e^{is(n_1 e^{-b} m_2 - n_2 e^b m_1)} J^{a+b}_{n_1+e^b m_1,\ n_2+e^{-b} m_2}\ , \tag{2}$$

where the upper indices, a, b, the lower ones, $m_1, m_2, ..$ and the parameter s are arbitrary, unless restricted by some further expediency.

It satisfies the Jacobi identity, which is evident from its merely amounting to the antisymmetrization of the associative group product,

$$J^a_{m_1,m_2} J^b_{n_1,n_2} = e^{is(m_1 e^{-a} n_2 - m_2 e^a n_1)} J^{a+b}_{m_1+e^a n_1,\ m_2+e^{-a} n_2}. \tag{3}$$

(Associativity amounts to $(J^a_{m_1,m_2}J^b_{n_1,n_2})J^c_{k_1,k_2} = J^a_{m_1,m_2}(J^b_{n_1,n_2}J^c_{k_1,k_2})$.) Naturally, the symmetrization of this product into an anticommutator further yields a consistent *graded extension* of the Lie algebra.

The algebra (2) contains, as a subalgebra specified by vanishing second subscripts, $m_2 = n_2 = ... = 0$, the Vertex (coherent-state) Algebra [4],

$$[J^a_m, J^b_n] = J^{a+b}_{m+e^a n} - J^{a+b}_{n+e^b m} \ . \tag{4}$$

Another, more familiar and general, subalgebra is the infinite Lie algebra specified by vanishing of all superscripts $a = b = ... = 0$, and by integer subscripts: it is the Sine Algebra [5],

$$[M_{m_1,m_2}, M_{n_1,n_2}] = 2i \sin\left(s(m_1 n_2 - m_2 n_1)\right) \ M_{m_1+n_1,m_2+n_2} \ . \tag{5}$$

This one, in turn, for $s = -\hbar/2$, is recognized as the Moyal Bracket algebra on the basis of the Fourier modes $\exp(im_1 x + im_2 p)$ of a toroidal phase space with unit radii [5]. Up to a normalization, the Moyal Brackets represent the antisymmetrization of Groenewold's celebrated star product [6],

$$\star \equiv e^{\frac{i\hbar}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} \ . \tag{6}$$

Alternatively, for $s = 2\pi/N$, and integer N, the sine algebra is seen to represent $GL(N)$, and to thus include all classical Lie algebras, (A_N, B_N, C_N, D_N) [7]. Furthermore [5], for $s \rightarrow 0$ (or $N \rightarrow \infty$), this algebra goes to the algebra of Poisson Brackets, also realized on a toroidal phase space, SDiff(T^2), cf [8]; and this one contains the Virasoro algebra as a subalgebra, through judicious summation over the first subscripts $m_1, n_1, ...$, [8].

This, then, is the basis of our remark that the Lie Algebra introduced encompasses and frames quite a range of the Lie algebras normally utilized in physics.

In this letter, we explore basic features of this Lie algebra, eqn (2), its applicability, and generalization. We provide a few explicit useful realizations and representations of restrictions thereof, which illuminate its structure.

Throughout, we stress the somewhat untypical correspondence mentioned, which applies both to the algebra (2) itself, as well as the subalgebras we sketched: namely, a *finite-dimensional Lie group* product, which yields the *infinite Lie algebras* when antisymmetrized—and graded (anticommutator) extensions, when symmetrized.

2 The Atavistic Algebra, realizations, and representations

The Atavistic Algebra (2) product (3) might be rewritten more symmetrically, through the redefinition

$$V^a_{m_1,m_2} \equiv J^{2a}_{e^a m_1,\ e^{-a} m_2}, \tag{7}$$

so that the product (3) amounts to

$$V^a_{m_1,m_2} \ V^b_{n_1,n_2} = e^{is(m_1 n_2 e^{-a-b} - m_2 n_1 e^{a+b})} \ V^{a+b}_{e^{-b}m_1 + e^a n_1,\ e^b m_2 + e^{-a} n_2}. \tag{8}$$

Nevertheless, we stick to the original form, to minimize superfluous indices.

The semidirect product nature of eqn (3) and thus of the Atavistic Algebra (2) is readily recognizable by recalling the $\star$-product underlying the sine algebra (5). We first revert to the version of the star product (6) parameterized for our purpose,

$$\star \equiv e^{-is(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} , \tag{9}$$

where x, p range from 0 to 1. The most widely used property of this product is its associativity. Thus, strings of operators of the form $f(x,p)\star$, for any functions $f(x,p)$ on the unit T^2, may be equivalently evaluated indifferently to the grouping of multiplication chosen. Consequently, choosing a Fourier mode basis for integers m_1, m_2, and defining

$$M_{m_1,m_2} \equiv \exp(i(m_1 x + m_2 p)) \ \star \ , \tag{10}$$

the standard product law follows [5],

$$M_{m_1,m_2} M_{n_1,n_2} = \exp\left(is(m_1 n_2 - m_2 n_1)\right) \ M_{m_1+n_1, m_2+n_2} \ , \tag{11}$$

underlying eqn (5) when antisymmetrized. The finite Lie group corresponding to this product is well-known to be the (dimension 3) Heisenberg group [9].

Now consider a phase-space-area-preserving dilation operator $D(a)$, which braids associatively as

$$D(a) \ f(x,p) = f(e^a x, e^{-a} p) \ D(a). \tag{12}$$

A standard realization of this operator is

$$D(a) = \exp\left(a(x \ \overrightarrow{\partial}_x \ - p \ \overrightarrow{\partial}_p) \right) . \tag{13}$$

Moreover, this operator formally commutes with the above star product,

$$D(a) \star = \star \ D(a), \tag{14}$$

as is plain in integral kernel representations of the star product [6]. More directly, this also follows from braiding $D(a)$ past the mode

$$\left(e^{i(m_1 x + m_2 p)} \star e^{i(m_1 x + m_2 p)} \right) = e^{is(m_1 n_2 - m_2 n_1)} \ e^{i((m_1+n_1)x + (m_2+n_2)p)}, \tag{15}$$

as above, eqn (11). Braiding past the simple right-hand side must equal stepwise braiding past each of the factors on the $\star$-product on the left, so it is manifest that the $\star$-product remains invariant. Of course, the reason for the invariance is the area-preservation feature of the dilation operator chosen, since the phase proportioned by s is a two-dimensional cross product, amounting to a phase-space-area element.

Furthermore, evidently,

$$D(a)D(b) = D(a+b), \qquad\qquad D(0) = 1\!\!1. \tag{16}$$

Consequently, it follows directly that the Atavistic Algebra elements $J^a_{m_1,m_2}$ may be constructed out of $M_{m_1,m_2}D(a) = J^0_{m_1,m_2}D(a)$, i.e.,

$$J^a_{m_1,m_2} = \exp(i(m_1x + m_2p)) \;\star\; D(a). \tag{17}$$

Writing this out explicitly,

$$J^a_{m_1,m_2} = e^{i(m_1x+m_2p)}\; e^{s(m_1\partial_p - m_2\partial_x)}\; e^{a(x\partial_x - p\partial_p)}, \tag{18}$$

the reader may recognize that the product (3) and thus the Atavistic Algebra (2) are, indeed, satisfied.

Thus, operation by the $J^a_{m_1,m_2}$ on a function $f(x,p)$ on this phase space consists of sequential rescalings and shifts of its variables and multiplication by a phase,

$$J^a_{m_1,m_2} f(x,p) = e^{i(m_1x+m_2p)}\; f\Big(e^a(x - sm_2), e^{-a}(p + sm_1)\Big). \tag{19}$$

The form (7) corresponds to $V^a_{m_1,m_2} = D(a)J^a_{m_1,m_2}$.

3 Reduction of variables

Now note that the torus variables x, p commute with each other, so that, effectively, the above realization is a direct product of the same type of operator

$$\mathfrak{J}^a_{m_1,m_2} \equiv e^{ism_1m_2/2}\; e^{im_1x}\; e^{sm_2\partial_x}\; e^{ax\partial_x}, \tag{20}$$

acting on variables x, and, disjointly, acting on variables p, which is to say

$$J^a_{m_1,m_2} = \mathfrak{J}^a_{m_1,-m_2} \otimes \mathfrak{J}^{-a}_{m_2,m_1}. \tag{21}$$

In turn, the product of these operators is

$$\mathfrak{J}^a_{m_1,m_2}\mathfrak{J}^b_{n_1,n_2} = e^{-is(m_1e^{-a}n_2 - m_2e^an_1)/2}\; \mathfrak{J}^{a+b}_{m_1+e^an_1,\; m_2+e^{-a}n_2}\;, \tag{22}$$

i.e., *the same as eqn (3), for minus half the value of the parameter s.* It is plain that, in general (irrespective of realization),

$$J^a_{m_1,m_2}(s) = \; J^a_{m_1,-m_2}(s/2) \otimes \; J^{-a}_{m_2,m_1}(s/2)\;. \tag{23}$$

This halving of representations and phases is already apparent in the first of refs [5]: a realization introduced by Hoppe there effectively represents the star product on *half the variables* of phase space. Plainly, this recursive phase-addition phenomenon extends to arbitrarily long direct product strings of Js, as a reflection of the simple coproduct structure of their logarithms[1].

[1] The mathematically inclined reader may recognize the connection of this relation to the Drinfeld twist of quantum deformation coproducts, as implemented by Reshetikhin [10], and exemplified, e.g., by ref [11], eqn (2.13), or ref [12], eqn (2.5), etc. These outrange the scope of the present discussion.

Equivalently, since oscillator operators, $[\alpha, \alpha^\dagger] = 1$, formally parallel the commutation relations of $[\partial_x, x] = 1$, the above realization may also be displayed in a "coherent state" form,

$$\mathfrak{J}^a_{m_1,m_2} = e^{im_1\alpha^\dagger + sm_2\alpha}\ e^{a\ \alpha^\dagger\alpha}. \tag{24}$$

In this form, the general Lie subalgebras specified by all $m_2 \to 0$, yielding (5); and $a \to 0$, yielding (4), are manifest.

One would not expect finite dimensional representations for the generic Atavistic Algebra. If the expression eqn (24) looks familiar, it is because it is. It is, in fact, the parameterization of the oscillator Lie group $\mathcal{G}$ investigated by Miller [2]. He provides, in gratifying detail, the representations of this Lie group on finite vector spaces (by matrices), unfaithfully; and, faithfully, on infinite-dimensional Bargmann space, by differential operators, eqn (19)—for instance, since $\mathfrak{J}^a_{m_1,m_2} f(x) = e^{im_1(x - sm_2/2)}\ f(e^a(x + sm_2))$, representations on associated Laguerre polynomials are specified by this simple action on the generating functions of these polynomials. Representations on Hermite Hilbert space then follow [2].

The four generators of the oscillator group $\mathcal{G}$ are abstractions of the operators $\alpha, \alpha^\dagger$, $\mathbb{1}$, augmented by the occupation number operator, normally $\alpha^\dagger\alpha$. These are now abstracted to

$$[\mathcal{A}, \mathcal{A}^\dagger] = \mathcal{E}, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad [\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \quad [\mathcal{E}, \mathcal{N}] = [\mathcal{E}, \mathcal{A}] = [\mathcal{E}, \mathcal{A}^\dagger] = 0. \tag{25}$$

Faithful representations of this need be infinite dimensional; but unfaithful ones could be finite dimensional, as long as the central generator $\mathcal{E}$ is not the identity. This can be arranged by nilpotent matrices.

For example [2], one might consider 3×3 nonhermitean matrices,

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

These represent eqn (25), although note that $\mathcal{A}^\dagger\mathcal{A} = -\mathcal{E} \neq \mathcal{N}$, $\mathcal{N}^2 = \mathcal{N}$, and $(\mathcal{A}^\dagger)^2 = \mathcal{A}^2 = \mathcal{E}^2 = 0$. In this representation, the proper abstracted analog of (24) reads

$$\mathcal{J}^a_{m_1,m_2} = \begin{pmatrix} e^a & im_1 & -ism_1m_2e^a/2 \\ 0 & 1 & sm_2e^a \\ 0 & 0 & e^a \end{pmatrix}, \tag{27}$$

and satisfies, instead,

$$\mathcal{J}^a_{m_1,m_2}\mathcal{J}^b_{n_1,n_2} = \exp\left(\frac{-is}{2}(m_1e^{-a}n_2 - m_2e^a n_1)\mathcal{E}\right)\ \mathcal{J}^{a+b}_{m_1+e^a n_1,\ m_2+e^{-a}n_2}\ . \tag{28}$$

However,

$$\exp\left(-\frac{is}{2}(m_1e^{-a}n_2 - m_2e^a n_1)\mathcal{E}\right) = \mathbb{1} - \frac{is}{2}(m_1e^{-a}n_2 - m_2e^a n_1)\mathcal{E} \tag{29}$$

is not a pure phase, as the (traceless) center $\mathcal{E}$ is not the identity matrix.

The inadequacy of this construction in representing the pure phase appears generic to all finite dimensional representations sought. Nevertheless, we illustrate explicitly in the next section that finite dimensional matrix representations can be available for suitable *restrictions of the Atavistic Algebra*.

4 Finite matrix representations for restricted cases

Suppose, instead, that we restrict the parameters and indices of the product (3) to $s = -\pi/p$ for an odd prime integer p, so that $\exp(-2is) \equiv \omega = e^{2\pi i/p}$, with $\omega^p = 1$; and take integer subscripts mod p, $m_j = 0, 1, 2, ..., p-1$; and rescaled superscripts to be integer mod $p-1$, $\tilde{a} \equiv a/\ln 2 = 0, 1, 2, ..., p-2$, recalling cyclicity: $2^{p-1} = 1$ mod p, for any odd prime integer p.

The product now reads

$$J^{\tilde{a}}_{m_1,m_2} J^{\tilde{b}}_{n_1,n_2} = \omega^{(2^{\tilde{a}} m_2 n_1 - 2^{p-1-\tilde{a}} m_1 n_2)/2} \; J^{\tilde{a}+\tilde{b}}_{m_1+2^{\tilde{a}} n_1,\; m_2+2^{p-1-\tilde{a}} n_2}. \tag{30}$$

This product (and the corresponding antisymmetrization Lie Algebra) can be represented by Sylvester's celebrated $p \times p$ matrix basis for $GL(p)$ groups [13]. His standard clock and shift unitary unimodular matrices, are

$$Q_{rt} = \omega^r \delta_{r,t} \;, \qquad\qquad P_{rt} = \delta_{r+1,t} \;, \tag{31}$$

for indices r, t defined mod p, $r = 0, 1, 2, ..., p-1$.

Consequently,

$$Q^p = P^p = 1\!\!1, \qquad\qquad PQ = \omega \; QP, \tag{32}$$

the characteristic braiding identity [9, 5, 7].

The complete set of p^2 unitary unimodular $p \times p$ matrices

$$M_{(m_1,m_2)} \equiv \omega^{m_1 m_2/2} \, Q^{m_1} P^{m_2}, \qquad \Longrightarrow \qquad (M_{(m_1,m_2)})_{rt} = \omega^{m_1(r+m_2/2)} \delta_{r+m_2,t} \;, \tag{33}$$

where

$$M^\dagger_{(m_1,m_2)} = M_{(-m_1,-m_2)}, \tag{34}$$

and $\mathrm{Tr} M_{(m_1,m_2)} = p\delta_{m_1,0}\delta_{m_2,0}$, suffice to span the group algebra of $GL(p)$. Since

$$M_{(m_1,m_2)} M_{(n_1,n_2)} = \omega^{(m_2 n_1 - m_1 n_2)/2} \; M_{(m_1+n_1,m_2+n_2)} \;, \tag{35}$$

they further satisfy the Lie algebra of $SU(p)$, [5], a restriction of the Sine Algebra displayed in the Introduction,

$$[M_{(m_1,m_2)}, M_{(n_1,n_2)}] = 2i \; \sin\left(\frac{\pi}{p}(m_2 n_1 - m_1 n_2)\right) M_{(m_1+n_1,m_2+n_2)} \;. \tag{36}$$

Now, in addition, consider the discrete scaling (doubling) matrix [14],

$$R_{rt} \equiv \delta_{2r,t} \;, \qquad\qquad R^{p-1} = 1\!\!1, \qquad\qquad R^\dagger = R^T = R^{p-2}, \tag{37}$$

again for indices r, t defined mod p, $r = 0, 1, 2, ..., p-1$. The cyclic structure holds by virtue of the identity $2^{p-1} = 1$ mod p. (Note that a smaller power of 2 may return to 1 before the power $p-1$, e.g. $2^3 = 1$ mod 7, but this does not compromise the present construction.)

The action of the doubling matrix is

$$RQR^{p-2} = Q^2, \qquad\qquad R^{p-2}PR = P^2. \tag{38}$$

Consequently,

$$R^{\tilde{a}} Q^{m_1} P^{m_2} R^{p-1-\tilde{a}} = Q^{2^{\tilde{a}} m_1} P^{2^{p-1-\tilde{a}} m_2}, \tag{39}$$

and hence the unitary matrices

$$\mathcal{J}^{\tilde{a}}_{m_1,m_2} \equiv M_{(m_1,m_2)} R^{\tilde{a}} \tag{40}$$

satisfy eqn (30). Thus, these $\mathcal{J}$s provide a p-dimensional representation, and hence a representation of the restricted Atavistic Lie Algebra. Through the direct product recursive procedure of the preceding section, they can be augmented to selected higher dimensional ones.

However, since Sylvester's basis is complete, R is representable in terms of the above p^2 Ms—in fact, it is the phased sum of all $p \times p$ matrices M, normalized by p, since, $\forall m_1, m_2$,

$$\text{Tr } M_{(m_1,m_2)} R = \omega^{-3m_1 m_2/2} \ . \tag{41}$$

Thus, since, e.g.,

$$p\mathcal{J}^1_{0,0} - \sum_{m_1,m_2} \omega^{-3m_1m_2/2} \ \mathcal{J}^0_{m_1,m_2} = 0 \ , \tag{42}$$

is represented trivially, the representations displayed are not faithful.

It is not hard to generalize to scaling (squeezing) matrices [14] with analogous effects. For instance, "Fermat's Little Theorem" [15] dictates $n^{p-1} = 1 \bmod p$, for any odd prime p and, e.g., $0 < n < p$. Thus, instead of R, a matrix T scaling by n,

$$T_{rt} \equiv \delta_{nr,t} \ , \qquad\qquad T^{p-1} = \mathbb{1}, \qquad\qquad T^\dagger = T^T = T^{p-2}, \tag{43}$$

acts through log base n, i.e. $a = \tilde{a} \ln n$,

$$T^{\tilde{a}} Q^{m_1} P^{m_2} T^{p-1-\tilde{a}} = Q^{n^{\tilde{a}} m_1} P^{n^{p-1-\tilde{a}} m_2}, \tag{44}$$

yielding analogous representations for the restricted Atavistic Lie Algebra. The question as to whether suitable tensor products of such spaces could yield faithful representations, instead, remains open.

5 Generalizations to linear canonical transforms and discussion

It turns out that $D(a)$ in Section 2 need not be so restricted, if an associative product of the above type is sought. Indeed, any $Sp(2)$ linear symplectic transformation will do, when it comes to leaving the star product invariant [16] (over and above preserving areas and Poisson Brackets—but nonlinear SDiff(T^2) does not, in general). Thus, $D(a)$ may be generalized to a matrix $\mathbf{S}$ with $\det \mathbf{S} = 1$, in

$$\begin{pmatrix} x \\ p \end{pmatrix} \quad \mapsto \quad \mathbf{S} \begin{pmatrix} x \\ p \end{pmatrix} . \tag{45}$$

The above $D(a)$ corresponds to the special case $\mathbf{S}$ =diag(e^a, e^{-a}). The transformation matrix needs transposition, $\mathbf{S}^T$, to act on the Fourier mode coefficients' doublets (m_1, m_2). The Atavistic Algebra then further extends to the associative products of operators

$$J^{\mathbf{S}}_{m_1,m_2} = \exp(i(m_1 x + m_2 p)) \ \star \ \mathbf{S}, \tag{46}$$

where, naturally, the matrix superscripts multiply, and the cross product in the phase is between the untransformed and suitably transformed subscript doublet. This formal structure parallels symplectic squeezing of light [1, 16, 14], where the $Sp(2)$ linear symplectic transformation amounts to a Bogoliubov transformation.

We expect that this product structure, whose antisymmetrization yields the corresponding generalized Atavistic Lie Algebra, is sufficiently general to find applications in a broad variety of physics contexts. That is, the associative product $f(x,p) \star g(\mathbf{S}(x,p))$ is a tractable generalization of the standard $\star$-product, and should be relevant to squeezed state, or lattice/brane deconstruction contexts, in which $f(x,p)$ is defined at one end of a link (brane) and $g(x,p)$ on the other, the symplectic transformation $\mathbf{S}$ providing the link transition function. Moreover, the Drinfeld twist coproduct exemplified in Section 3 bears compelling intuitive connections to applications of this twist in noncommutative QFT [12] and deformation-generalized CFT [17], currently under investigation.

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