# DGP Specteroscopy

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Abstract: We systematically explore the spectrum of gravitational perturbations in codimension-1 DGP braneworlds, and find a $4D$ ghost on the self-accelerating branch of solutions. The ghost appears for any value of the brane tension, although depending on the sign of the tension it is either the helicity-0 component of the lightest localized massive tensor of mass $0 < m^2 < 2H^2$ for positive tension, the scalar 'radion' for negative tension, or their admixture for vanishing tension. Because the ghost is gravitationally coupled to the brane-localized matter, the self-accelerating solutions are not a reliable benchmark for cosmic acceleration driven by gravity modified in the IR. In contrast, the normal branch of solutions is ghost-free, and so these solutions are perturbatively safe at large distance scales. We further find that when the $\mathbb{Z}_2$ orbifold symmetry is broken, new tachyonic instabilities, which are much milder than the ghosts, appear on the self-accelerating branch. Finally, using exact gravitational shock waves we analyze what happens if we relax boundary conditions at infinity. We find that non-normalizable bulk modes, if interpreted as $4D$ phenomena, may open the door to new ghost-like excitations.

Keywords: braneworlds, modified gravity, ghosts.

## Contents

**1. Introduction** **1**

**2. What are DGP braneworlds?** **6**
2.1 Background solutions 8
2.2 How do we obtain $4D$ gravity in the DGP model? 10

**3. The occult sector of DGP** **13**
3.1 Learning to count: mode expansion 14
3.2 Fluctuations around the vacuum 18
3.3 Linearized fields of matter lumps 26
3.4 Forking the ghost: calculating the effective action 28
3.5 Perturbations that break $\mathbb{Z}_2$ symmetry 32

**4. Shocking $4D$ nonlocalities** **34**

**5. Summary** **40**

**6. Proof of tensor decomposition in Eq. (3.8)** **43**

**7. Helicity-0 ghosts in de Sitter space** **45**

# 1. Introduction

Ever since Einstein introduced his famous "biggest blunder", the cosmological constant has been one of the most frustrating, yet intriguing aspects of General Relativity (GR). Ironically, just as Einstein needed a $\Lambda$ to make a static universe, if we take his theory of GR as the description of gravity at the largest scales, we now seem to need a $\Lambda$ to account for the cosmic acceleration observed at redshifts $z \lesssim 1.7$ [1, 2, 3]. Unfortunately, manufacturing a sufficiently small, positive cosmological constant from a consistent theory is not entirely straightforward, to say the least. The methods of effective field theory have so far failed to yield a satisfactory microscopic theory of the cosmological constant [4, 5]. Moreover, while the mystery of the cosmological constant is usually posed as a problem for the field theory of matter, one may even wonder if in fact it might really be related to our formulation of gravity and inertia. Our hands-on experimental knowledge of gravity conforms with GR at distances between $\sim 0.1$ mm [6, 7] and, say, $\sim 10 - 100$ MPc. At these large scales we enter the domain of dark matter, a necessary component of the standard cosmological model needed to explain galactic rotation curves, which cannot be accounted for with GR and baryonic matter alone. At the moment, dark matter still needs to be completely explained by particle physics despite a plethora of reasonable candidates. A popular common theme in recent research is that perhaps it is not matter that is needed, but a modification of Newton's law and/or gravity at large scales. This idea is not new: ever since galactic rotation curves were found to be inconsistent with the luminous matter, such alternatives have been pursued [8].

While it is natural to hope that modifying gravity could be an interesting alternative to dark matter, why might one hope that it could help with the cosmological constant? To illustrate this, we offer the following simple, heuristic argument. It is clear that in the Einstein-Hilbert action, the cosmological constant term appears as the Legendre transform of the field variable $\sqrt{-g} = \sqrt{|\det(g_{\mu\nu})|}$:

$$S_{EH} = M_4^2 \int d^4x \sqrt{-g} R - 2\sqrt{-g}\Lambda + \dots \, . \tag{1.1}$$

From the canonical field theory rules, this means that this term trades the independent field variable $\sqrt{|\det(g_{\mu\nu})|}$ for another *independent* variable $\Lambda$. This is exactly the same as in quantum field theory, where one defines the generating function of the theory by shifting the Lagrangian by a 'coupling' $\int \phi(x) J(x)$. This trades the independent variable $\phi$ for another independent variable $J$. After this transformation, the variable $J$ is not *calculable*; it is an *external* parameter that must be fixed by hand at the end of the calculation, by a choice of boundary conditions. Once $J$ is fixed to some value, $\phi$ is calculable in terms of it. The only difference between the usual field theory Legendre transform and the cosmological constant term arises because of gauge symmetries of GR, which render $\sqrt{|\det(g_{\mu\nu})|}$ non-propagating. It is a pure gauge variable that can always be set to a constant number by a change of coordinates. Therefore the Legendre transformation (1.1) loses information about only one number, which must be fixed externally: namely, by the value of $\Lambda$ itself. As a result, in GR the cosmological constant is a boundary condition rather than a calculable quantity. One may then hope that by changing gravitational dynamics one could render $\sqrt{|\det(g_{\mu\nu})|}$ propagating, so that, in turn, $\Lambda$ is also rendered dynamical. This could provide us with new avenues for relaxing the value of $\Lambda$. Such hopes have been already expressed before on a few occasions [9, 10, 11]. However, analyzing modifications of gravity systematically, to check if they remain compliant with the tests of GR, hasn't been easy.

On the other hand, in recent years the *braneworld paradigm* has emerged as a compelling alternative to standard Kaluza-Klein (KK) methods of hiding extra dimensions and a new framework for solving the hierarchy problem. In this approach our universe is realized as a slice, or submanifold, of a higher dimensional spacetime. Unlike in KK compactifications, where the extra dimensions are small and compact, in the braneworld approach they can be relatively large [12, 13], or infinite [14, 15]. We do not directly see them since we are confined to our braneworld, rather, their presence is felt via corrections to Newton's law. Many of the more fascinating phenomenological features of these braneworld scenarios arise in models of warped compactification. In warped compactifications the scale factor of a four-dimensional brane universe actually varies throughout the extra dimensions, providing us with a new way of making a higher dimensional world appear four-dimensional. In general, one can conceal extra dimensions from low energy probes by either 1) making the degrees of freedom which propagate through the extra dimensions very massive so as to cut the corrections to Newton's law off at long distances, or 2) suppressing the couplings of the higher dimensional modes to ordinary matter so that the $4D$ gravitational couplings dominate, ensuring that the corrections to Newton's law are very small at long distances. The latter case is naturally realized in warped models, so that even infinite extra dimensions may be hidden to currently available probes.

Braneworlds provide a natural relativistic framework for exploring means of modifying gravity. It was quickly realized that by using free negative tension branes, one could alter Newton's constant at large scales [16]. More dramatically, Gregory, Rubakov and Sibiryakov (GRS) [17] noticed that by combining negative tension branes with infinite extra dimensions, it was possible to "open-up" extra dimensions at very large scales, making gravity effectively higher-dimensional very far away. However, it was soon discovered by the authors that this model contained ghosts [18]. This was unfortunate since the metastable graviton had many desirable gravitational properties, but from a particle physics point of view the existence of a ghost is disastrous. Soon after, a radically new braneworld model was put forward, the DGP (Dvali-Gabadadze-Porrati) model [19], with graviton kinetic terms on the brane as well as in the bulk. The simpler versions of this theory are described by the action

$$\begin{aligned} S_{\rm DGP} = M_5^{4+n} \int_{(4+n)D \text{ bulk}} \sqrt{-g} R(g) + M_4^2 \int_{\rm brane} \sqrt{-\gamma} R(\gamma) \\ + \text{extrinsic curvature terms} + \int_{\rm brane} \mathcal{L}_{\rm matter} \,. \end{aligned} \tag{1.2}$$

In general, there may be additional terms in the bulk. The key new ingredient here is the induced curvature on the brane. It could be generated, as it was claimed initially, by quantum corrections from matter loops on the brane [20], or again in a purely classical picture of a finite width domain wall[1] as corrections to the pure tension Dirac-Nambu-Goto brane action [22, 23]. Furthermore, it is also intriguing to note that induced curvature terms appear quite generically in junction conditions of higher codimension branes when considering natural generalizations of Einstein gravity [24] as well as in string theory compactifications [25]. Using holographic renormalization group arguments [26], DGP was shown to be equivalent in the infrared to GRS, however, crucially, it appeared to be ghost-free, corroborating the perturbative analysis of [27]. This made it seem a real candidate for a new gravitational phenomenology at large distances. The induced curvature term yields a particularly interesting new phenomenon. In the case of a brane in $5D$ Minkowski bulk it allows for a *self-accelerating* cosmological solution [28], for which the vacuum brane is de Sitter space, with constant Hubble parameter $H = 2M_5^3/M_4^2$, even though the brane tension vanishes.

Clearly, the possibility of a fully consistent explanation of large scale acceleration is extremely exciting. It has generated a great deal of activity and investigation into the DGP set-up [29] (for a recent review, see [30]), with an astrophysical emphasis on black hole solutions [31], solar system tests [32], shock wave limits [33], and of course, whether DGP can truly explain dark energy [34]. Although many cosmologists have already embraced the DGP model, it has been found to suffer from various problems. There is the issue of strong coupling [35, 36, 37, 38], related to the feature that the graviton interactions go nonlinear at intermediate scales. More importantly, various investigations pointed out that there are ghosts on the self-accelerating branch [37, 38, 39], however this debate still persists.

Our aim here is to explore this issue in full detail. Since most of the explicit work on DGP has been done for the simplest case of a brane in flat $5D$ bulk, with the dynamics given by (1.2), we will work in the same environment, and start with a review of this case. We will next consider the spectrum of small perturbations of the cosmological vacua of DGP, which describe a $4D$ de Sitter geometry. One of the confusing aspects of the literature on these braneworld perturbations (as opposed to braneworld *cosmological* perturbations!) is the alternate approaches of direct 'hands-on' calculations, which analyze the curved space wave operator for the gravitational perturbation directly [14, 40, 41] and the "effective action" approach, which was used to particular effect to confirm the ghost [18] of the GRS model via a radion mode analysis [42]. Naturally these approaches should be entirely equivalent, and we will

[1] Harking back to the early manifestations of braneworlds [21].

indeed see that. The technical complications in the identification of the spectrum of DGP gravity arise from the *mixed* boundary conditions for perturbations that may obscure the computation of the norm.

The relevant modes in the spectrum of perturbations for addressing the concerns about stability are the tensors and the scalars. By going to a unitary gauge, we will see that the tensors are generically organized as a gapped continuum of transverse-traceless tensor modes, with 5 polarizations per mass level, and an isolated localized normalizable tensor, which lies below the gap. On the normal branch, this localized tensor is massless, implying that it has only two helicity-2 polarizations; on the self-accelerating branch it is massive, with $0 < m^2 < 2H^2$ for positive tension, and has 5 polarizations. When the brane tension is positive, the helicity-0 mode is the ghost, precisely because its mass sits in the region prohibited by unitarity, explored in [43, 44, 45, 46]. Furthermore, the propagating scalar mode, or the 'radion', is tachyonic. This tachyonic instability of scalar perturbations is very generic, and by and large benign (see section 3.2). Moreover, the tachyonic scalar completely decouples on the normal branch in the limiting case where the bulk ends on the horizon[2]. On the self-accelerating branch, the scalar mode remains tachyonic but mostly harmless for positive tension branes, but as the tension vanishes it mixes with the helicity-0 tensor, and prevents the ghost from decoupling even in the vacuum by breaking the accidental symmetry of the massive tensor theory in de Sitter space in the limit $m^2 = 2H^2$, studied in partially massless theories [43, 45, 46]. This mode becomes a pure and unadulterated ghost when the brane tension is negative, because it contains the brane Goldstone mode which does not decouple in a way similar to the GRS model [17, 18], consistent with the claim of [37]. *Thus the self-accelerating solutions always have a ghost, and therefore do not represent a reliable benchmark for an accelerating universe in their present form. On the other hand the normal solutions are ghost-free, and thus may be useful as a model of gravitational modifications during cosmic acceleration.*

Our calculations further allow us to extend the analysis to perturbations which are not $\mathbb{Z}_2$-symmetric around DGP branes. This symmetry can be relaxed for braneworld models[3], and actually this may be a more natural setting for the DGP setup which is more closely analogous to finite width defects or quantum corrected walls. In fact, in general braneworld models, when the requirement of $\mathbb{Z}_2$-symmetry is dropped one can get a whole range of interesting gravitational phenomenology, including self-acceleration, without appealing to induced gravity [48, 49]. We will show here that

[2]We remind the reader that the situation here is similar to the single positive tension brane in the RS model where the radion also decouples.

[3]The Randall-Sundrum model [13, 14] was $\mathbb{Z}_2$-symmetric by construction, enabling to interpret it as a dual AdS/CFT with a UV cut-off and coupling to gravity [47].

if $\mathbb{Z}_2$-antisymmetric modes are allowed, then in addition to the ghost, there is an extra excitation which corresponds to the free motion of the DGP brane. This mode is tachyonic, and while it decouples on the normal branch in the single brane limit, it persists on the self-accelerating branch of DGP solutions. Nevertheless it still remains tame, since the scale of instability is controlled by the Hubble parameter, and so the instability may remain very slow.

The presence of the ghost in the $4D$ description of the self-accelerating solutions of DGP indicates that the instability originates from the 'reduction' of the theory, and may not really represent a fundamental problem of the bulk set-up. A different prescription for boundary conditions might be able to circumvent the contributions from the brane localized ghost. However this requires rather special boundary conditions very far from the brane mass, that would not normally arise dynamically in a local theory. They allow the leakage of energy to, or from, infinity. Worse yet, an explicit exploration of potentials of relativistic sources shows that in this case other modes behave like ghosts, if interpreted in the $4D$ language. We can see this directly from the gravitational shock wave solutions which include the contributions from the modes that are not localized on the brane.

The paper is organized as follows. In the next section we will review some of the salient features of the DGP model, describing its two branches of background solutions, the *normal* branch and the *self-accelerating* branch. In section 3, we will discuss the perturbation theory around the $4D$ cosmological vacua of DGP, and identify its occult sector by an explicit calculation. In section 4, using gravitational shock waves, we will consider what happens when we include the contributions from non-normalizable bulk modes to the long range gravitational potential of brane masses. We will summarize in section 5.

## 2. What are DGP braneworlds?

We will work with the simplest and most explicit incarnation of DGP, where our universe is a single 3-brane embedded in a 5 dimensional bulk spacetime. The bulk is locally Minkowski and the brane carries the curvature of the induced metric as well as the brane localized matter. The induced curvature terms will generically arise from the finite brane width corrections. The brane may be viewed as a $\delta$-function source in the bulk Einstein equations, whose dynamics ensues from the total stress-energy conservation that follows from the covariance of the theory. Alternatively the brane may be treated as a common boundary of two distinct regions, $\mathcal{M}^+$ and $\mathcal{M}^-$ in the bulk $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$, which are on the different sides of the brane $\Sigma = \partial\mathcal{M}^+ = \partial\mathcal{M}^-$. The boundary conditions at the brane are given by the Israel equations [50], which correspond precisely to the brane equations of motion. These two approaches are physically completely equivalent because the theory is completed with the inclusion of the Gibbons-Hawking boundary terms [51], which properly covariantize the bulk Einstein-Hilbert action in the presence of a boundary. As a result, varying with respect to the metric gives the correct boundary equations as well as the correct bulk. The simpler $\delta$-function form of the field equations then corresponds to the unitary gauge, realized by going to brane Gaussian-normal coordinates, which essentially describe the brane's rest frame in the bulk, and then gauge fixing residual gauge invariance.

The dynamics of the model can therefore be derived from the action

$$S = M_5^3 \int_{\mathcal{M}} d^5x\sqrt{-g}R + 2M_5^3 \int_{\Sigma} d^4x\sqrt{-\gamma}\Delta K + \int_{\Sigma} d^4x\sqrt{-\gamma}(M_4^2\mathcal{R} - \sigma + \mathcal{L}_{matter}) \quad (2.1)$$

Here $g_{ab}$ is the bulk metric with the corresponding Ricci tensor, $R_{ab}$ (in $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$). The induced metric on the brane is given by $\gamma_{ab}$ and its Ricci tensor is $\mathcal{R}_{ab}$, while $\sigma$ is the brane tension. The extrinsic curvature of the brane is given by $K_{ab} = -\frac{1}{2}\mathcal{L}_n\gamma_{ab}$, where $\mathcal{L}_n$ is the Lie derivative of the induced metric with respect to unit normal, $n^a$, oriented from $\mathcal{M}^-$ into $\mathcal{M}^+$; $\Delta K_{ab} = K_{ab}^+ - K_{ab}^-$ is the jump of $K_{ab}$ from $\mathcal{M}^-$ to $\mathcal{M}^+$, and $\mathcal{L}_{matter}$ is the Lagrangian of brane localized matter fields, with vanishing vacuum expectation value, because the brane vacuum energy was explicitly extracted as tension.

In what follows we will use different gauges for the bulk geometry, because the brane Gaussian-Normal gauge is very convenient for counting up the modes in the spectrum of the theory, while other gauges may be easier to compute the effective actions for particular modes. Thus, thinking of the solutions geometrically as a bulk in which the brane moves, we will write the field equations which follow from (2.1) as separate bulk and brane equations of motion respectively. These are valid in an arbitrary gauge, and may be thought of as a breakdown of the full set of field equations on a space with a boundary, where the boundary conditions describe a codimension-1 brane. The bulk equations of motion are simply the vacuum Einstein equations,

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 0\,, \quad (2.2)$$

whereas the brane equations of motion are given by the Israel junction conditions [50],

$$\Theta_{ab} = M_5^3\Delta\left[K_{ab} - K\gamma_{ab}\right] + M_4^2(\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}\gamma_{ab}) + \frac{\sigma}{2}\gamma_{ab} = \frac{1}{2}T_{ab}\,. \quad (2.3)$$

where

$$T_{ab} = -\frac{2}{\sqrt{-\gamma}}\frac{\partial(\sqrt{-\gamma}\mathcal{L}_{matter})}{\partial\gamma^{ab}}, \quad (2.4)$$

explicitly does not include the brane energy-momentum, $\sigma\gamma_{ab}$.

In most of what follows we will impose $\mathbb{Z}_2$ orbifold symmetry about the brane (section 3.5 will deal with general perturbations). In other words, we will identify $\mathcal{M}^+$ and $\mathcal{M}^-$, restricting the dynamics to the $\mathbb{Z}_2$ symmetric action given by

$$S = 2M_5^3 \int_{\mathcal{M}} d^5x\sqrt{-g}R + 4M_5^3 \int_{\Sigma} d^4x\sqrt{-\gamma}K + \int_{\Sigma} d^4x\sqrt{-\gamma}(M_4^2\mathcal{R} - \sigma + \mathcal{L}_{matter}) \quad (2.5)$$

where $K_{ab} = K_{ab}^+ = -K_{ab}^-$. The bulk equations of motion (2.2) are of course unchanged, while the brane equations of motion simplify to

$$\Theta_{ab} = 2M_5^3 \left[K_{ab} - K\gamma_{ab}\right] + M_4^2(\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}\gamma_{ab}) + \frac{\sigma}{2}\gamma_{ab} = \frac{1}{2}T_{ab}\,. \quad (2.6)$$

## 2.1 Background solutions

Cosmological DGP vacua describe tensional branes in $5D$ locally Minkowski patches glued together such that the jump in extrinsic curvature matches the tension and the intrinsic Ricci curvature contributions as in Eq. (2.6). The solutions can be easily constructed by taking a bulk geometry which solves the sourceless bulk equations (2.2), and slicing it along a trajectory $(t(\tau), R(\tau))$ which solves (2.3). Then $R$ becomes the cosmological scale factor and $\tau$ the comoving time coordinate. Such techniques have been used before in the RS2 framework [52, 53]. When the brane only carries nonzero tension, its worldvolume is precisely a $4D$ de Sitter hyperboloid representing the $4D$ de Sitter embedding in a $5D$ Minkowski space as required by the symmetries of the problem [54]. This solution generalizes the geometry of Vilenkin-Ipser-Sikivie inflating domain walls in $4D$ [55], and was in fact also found in [56] in the context of finite thickness domain walls.

In conformal coordinates $x^a = (x^\mu, y)$, the full background metric is given by

$$ds^2 = \bar{g}_{ab}dx^a dx^b = a^2(y)\left(dy^2 + \bar{\gamma}_{\mu\nu}dx^\mu dx^\nu\right)\,, \quad (2.7)$$

where

$$\bar{\gamma}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + e^{2Ht}\,d\vec{x}^2\,. \quad (2.8)$$

and

$$a(y) = \exp(\epsilon H y), \qquad \epsilon = \pm 1. \quad (2.9)$$

The bulk spacetime, $\mathcal{M}$, is the image of the line $0 < y < \infty$, with the brane positioned at $y = 0$. In DGP brane induced gravity theory there exist *two* distinct branches of bulk solutions, labelled by $\epsilon = \pm 1$. The solution with $\epsilon = -1$ is commonly referred to as the *normal* branch whereas the solution with $\epsilon = +1$ is referred to as the *self-accelerating* branch, a terminology which will become transparent shortly. The brane metric in (2.8) represents the $4D$ de Sitter geometry in spatially flat coordinates, which cover only one half of the $4D$ de Sitter hyperboloid. The complete cover with global coordinates involves the metric $ds^2 = -d\tau^2 + \frac{1}{H^2}\cosh^2(H\tau)d\Omega_3$ describing a sequence of spatial spheres $S^3$, of radius $\frac{1}{H}\cosh(Ht)$ and spatial line element $\mathrm{d}\Omega_3$, which initially shrink from infinite radius to radius $1/H$, and then re-expand back to infinity.

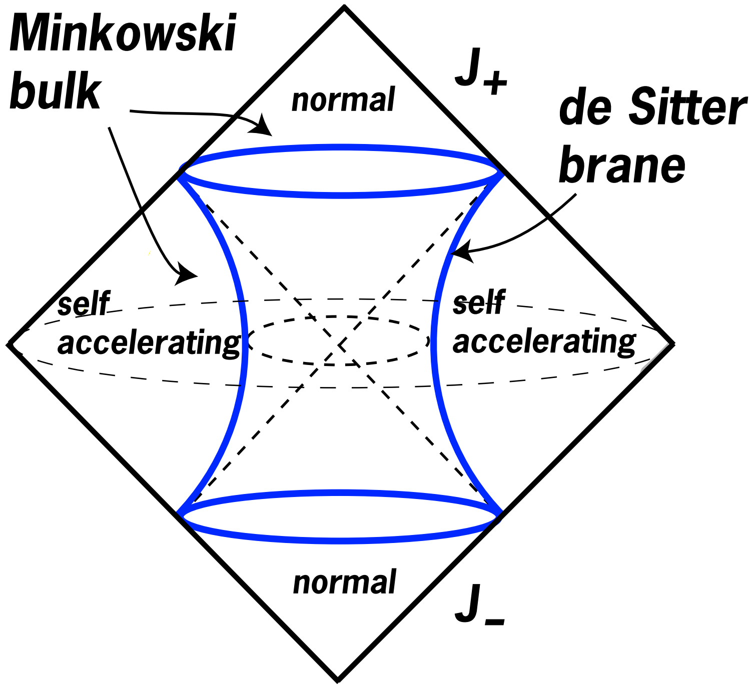


**Figure 1:** Embedding of a de Sitter brane in a flat 5D bulk. The brane world volume is the hyperboloid in the Minkowski bulk. The *normal* branch ($\epsilon = -1$) corresponds to keeping the interior of the hyperboloid, and its mirror image around the brane. In contrast, for the *self-accelerating* branch ($\epsilon = +1$), we keep the exterior, and its reflection. The latter scenario includes the inflating tensionless brane solution.

The intrinsic curvature $H$ is given by the tension, as dictated by the brane equations of motion (a.k.a. brane junction equations) (2.3) at $y = 0$,

$$3M_4^2H^2 - 6\epsilon M_5^3 H = \frac{\sigma}{2}\,. \tag{2.10}$$

The solutions are

$$H = \frac{\epsilon M_5^3}{M_4^2}\left[1 \pm \sqrt{1 + \frac{M_4^2\sigma}{6M_5^6}}\right]\,. \tag{2.11}$$

This equation suggests that there are in fact *four* possible values of the intrinsic curvature. However this is not the case. It is easy to see that only *two* of these solutions are independent. Indeed, note that a bulk reflection $z \to -z$ and a time reversal $t \to -t$ map two of the solutions (2.7,2.8) with $H < 0$ onto the solutions with $H > 0$ simultaneously reversing the sign of $\epsilon$. Thus without any loss of generality we fix the signs by requiring that a positive tension corresponds to positive intrinsic curvature $H$, so that

$$H = \frac{M_5^3}{M_4^2} \left[ \epsilon + \sqrt{1 + \frac{M_4^2 \sigma}{6 M_5^6}} \right] . \tag{2.12}$$

We can embed these solutions in the bulk as in figure 1 [33]. For $\epsilon = 1$, or "self-accelerating" branch, we retain the exterior of the hyperboloid. For $\epsilon = -1$, or the "normal branch", we keep the interior of the hyperboloid. It is now clear whence the terminology: on the self-accelerating branch, even when the tension vanishes, $\sigma = 0$, the geometry describes an accelerating universe, with a non-vanishing curvature ($H = 2M_5^3/M_4^2$) produced solely by the modification of gravity. The scale of the curvature needs to be specially tuned to the present horizon scale of $\sim 10^{-33}$ eV, which corresponds to about $M_5 \sim 40$ MeV [28, 29], but once this is done one may hope to explain the current bout of cosmic acceleration even without any Standard Model vacuum energy. The self-accelerating branch of solutions are a distinct new feature of the DGP model, they do not exist on $\mathbb{Z}_2$-symmetric brane without the induced gravity terms on the brane [54, 28]. However, related solutions may arise in theories with asymmetric bulk truncations [48, 49].

### 2.2 How do we obtain $4D$ gravity in the DGP model?

A crucial question is: given the cosmological DGP vacua reviewed above, how could there be a low energy $4D$ gravitational force between masses inhabiting them? Unlike in RS2, for solutions given by (2.8) and (2.10), the 'apparent' warping of the bulk cannot play a significant role in manufacturing $4D$ gravity at large distances. In RS2 bulk gravitational effects pull the KK gravitons away from the brane, strongly suppressing their couplings to brane localized matter. As a result, the extra dimension is hidden. That does not happen here because the bulk in (2.8) is locally flat. Moreover, on the self-accelerating backgrounds the bulk volume is infinite, and so the $4D$ graviton zero mode is decoupled: it is not perturbatively normalizable, and the mass scale which governs its coupling diverges. Although the bulk volume for the normal branch solutions is finite for finite $1/H$, and there is a normalizable graviton mode, its coupling[4] is $g_0 \propto H/M_5^3$, and so it also decouples in the limit $H \to 0$ [33]. In fact, from the general embedding of a $4D$ de Sitter hyperboloid in $5D$ Minkowski space (2.8) we see that the

[4]This formula is precisely the analogue of the Gauss law relation between bulk and effective $4D$ Newton's constant in models with large extra dimensions [12].

$H \to 0$ limit corresponds to taking the radius of extrinsic curvature of the hyperboloid on the normal branch to infinity, de facto pushing it to the spatial infinity of Minkowski space. In this limit the bulk volume between the brane and the horizon diverges, which is why the zero mode graviton decouples. This agrees with the perturbative analysis of the $H = 0$ case of [19, 27] where the zero mode graviton was completely absent. Hence $4D$ gravity ought to emerge from the exchange of bulk graviton modes.

Suppose first that the graviton kinetic terms reside only in the bulk. In an infinite bulk, a typical bulk graviton sourced by a mass on the brane will not venture too far from the brane because it would cost it too much energy. Nonetheless if kinetic terms reside only in the bulk, a typical bulk graviton would still peel away from the brane and explore the region of the bulk around the brane out to distances comparable to the distance $r$ between the source and a probe on the brane. The momentum transfer by each such virtual graviton to the brane probe would be $\sim 1/p$, where $p$ is the $4D$ momentum along the brane, as dictated by the $5D$ graviton propagator and brane couplings. Thus the gravitational potential would scale as $1/r^2$, and the resulting force as $1/r^3$. Such force-distance dependence would reveal the presence of the extra dimension. This would remain true even when $H \neq 0$ on the normal branch. Although a zero mode is present in this case, it cannot conceal the extra dimension because it would still be too weakly coupled to provide the dominant contribution to the long range force at sub-horizon scales.

The induced curvature terms on the brane change this in DGP. In order for this trick to work, one needs $M_4$ to be *big*. In this case, the brane localized kinetic terms effectively pull the zero mode gravitons closer to the brane, making their exploration of the bulk at distances shorter than $r_c \sim M_4^2/2M_5^3$ energetically costly [19, 27]. This alters the scaling of the momentum transfer to $1/p^2$ for momenta $p > M_5^3/M_4^2$, which in turn produces a force which scales as $1/r^2$. This is manifest from the explicit form of the graviton propagator projected on to the Minkowski brane (i.e. the $H = 0$ limit of the normal branch solutions of (2.8) and (2.10)) [19, 27]:

$$G(p)|_{z=0} = \frac{1}{M_4^2 p^2 + 2M_5^3 p}\Big(\frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta} + \frac{1}{2}\eta^{\mu\beta}\eta^{\nu\alpha} - \frac{1}{3}\eta^{\mu\nu}\eta^{\alpha\beta}\Big)\,. \tag{2.13}$$

From the $4D$ point of view, the graviton resonance which is exchanged is composed of massive tensor modes, and so it will contain admixtures of longitudinal gravitons. This is encoded in the propagator (2.13) in the coefficient $1/3$ of the last term of the spin projector, as opposed to $1/2$ which appears in the linearized limit of standard $4D$ GR. This difference is an example of the venerated Iwasaki-van Dam-Veltman-Zakharov (IvDVZ) discontinuity of modified gravity [57], and signifies the persistence of a scalar component of gravity in the theory, that could conflict with the known tests of GR.

However, it has been argued for massive gravity [58] and similarly for the DGP model [59] that the extra scalar may be tamed by nonlinearities once the correct background field of the source is included. The idea is that the perturbative treatment of the scalar graviton breaks down at a distance scale $r_V$ first elucidated by Vainshtein [58]. For DGP, for a source of mass $m$, this new scale is given by $r_V \sim (mr_c^2/M_4^2)^{1/3}$ [60, 61]. Below that distance, one can't trust linearized perturbation theory and must re-sum the background corrections, which should presumably decouple the scalar graviton mode. Similar weakening of the scalar graviton coupling may occur at cosmological scales if the universe is curved.

This scale dependence of the scalar graviton couplings has very interesting and important implications for the DGP setup. It has been pointed out [35, 36, 37, 38] that the effective field theory description of DGP gravity will suffer from a loss of predictivity due to the problems with strong couplings at distances $r_{\rm strong} \sim (r_c^2/M_4)^{1/3}$, which could be much larger than the naive UV cutoff. The most recent analysis of this issue [38] however suggests that the brane nonlinearities may push the scale of strong coupling down, to about $\tilde{r}_{\rm strong} \sim r_{\rm strong}/\sqrt{M_{\rm earth}/M_4} \sim 1$cm on the surface of the Earth, possibly making DGP marginally consistent with current table top experimental bounds [6, 7]. In what follows we will assume this claim [38] and imagine that we work in the perturbative regime of DGP, although we feel that this issue deserves further attention. We note that the exploration of DGP with gravitational shock waves [33] shows that the scalar graviton decouples from the background of relativistic sources, indicating that the coupling is effectively suppressed by the ratio of $\sqrt{(T^\mu{}_\mu)^2/T^{\mu\nu}T_{\mu\nu}}$ of the source. Note, that this is not enough to ascertain that a theory is phenomenologically *safe*. For example, a Brans-Dicke theory will admit identical shock waves as ordinary GR for any value of the Brans-Dicke parameter $\omega$, while the observations require that $\omega \geq 5000$. Thus one still needs to study the model for slowly moving sources to check if the predictions agree with observations. However one may hope that the strong coupling problems might be resolved in a satisfactory fashion. After all, the shock waves [33] remain valid down to arbitrarily short distances from the source, behaving much better than they are entitled to given the concern about the strong coupling problems.

In what follows we will focus on uncovering the ghosts (and/or other instabilities) on the self-accelerating branch, and a discussion of their implications for DGP. Before we turn to this, we should stress that there is no technical inconsistency between our results and the earlier claims that there are ghost-free regimes in DGP [19, 27]. Indeed: starting with the backgrounds of the family (2.8,2.10) and fixing $M_5$ and $M_4$, the only way to consistently take the limit to $H \to 0$ is to pick the normal branch solutions and dial the brane tension $\sigma$ to zero. In this way one reproduces the $H = 0$ brane backgrounds with fixed $M_5$, $M_4$ that were studied in [19, 27]. Moreover, ghosts may also be absent if the brane geometry is anti de Sitter, as opposed to dS [62]. Thus the results of the perturbative analysis of [19, 27], implying the absence of ghosts and other instabilities on $H = 0$ branes, applies *only* to the normal branch backgrounds of DGP (2.8), (2.10). In fact, our results will confirm this for the general $H \neq 0$ backgrounds of the normal branch, showing that they are ghost-free. However the analysis of [19, 27] has nothing to say about the self-accelerating branch solutions, and specifically about the $\sigma = 0$ limit, that describes a universe where cosmic acceleration arises from modification of gravity alone. In what follows we will confirm that in all those cases there are ghosts, which invalidate the self-accelerating branch solutions in their present form as realistic cosmological vacua.

## 3. The occult sector of DGP

We now turn to the exploration of the spectrum of light modes in the gravitational sector of DGP, around the cosmological vacua (2.8), (2.10). We will confirm that there are ghosts in the $4D$ effective field theory description on the self-accelerating branch of DGP solutions. More specifically: in the $4D$ effective field theory which describes the perturbative regime of self-accelerating branch of DGP backgrounds (2.8), (2.10) between the Vainshtein scale $r_V$ and the scale of modification of gravity $r_c = m_{pl}^2/2M_5^3$ there are scalar fields with negative, or vanishing, kinetic term around the vacuum, which couple to the brane-localized matter with at least gravitational strength. Now, this may appear surprising at the first glance: there are no ghosts in the action (2.1) of the full $5D$ bulk theory. Indeed, the full bulk Lagrangian in (2.1) does not appear to contain any instabilities. However, the background solutions (2.8), (2.10) of (2.1) involve an end-of-the-world brane, which is a dynamical object, whose world-volume is determined by (2.3). The problems arise because the brane will curl up and wiggle when burdened with a localized mass, in a way that alters the gravitational fields of the source mass, spoiling the $4D$ guise of the theory. Thus the perturbative ghost encountered in $4D$ theory is really a diagnostic of the failure of the $4D$ perturbation theory to describe the dynamics of the long range gravity on the self-accelerating solutions. Thus although the applications [28, 29, 63] of the self-accelerating solutions to $4D$ cosmology are interesting and tempting, the presence of the ghost renders them unreliable at the present stage of understanding of the theory, and hence *de facto* inadequate as a method of accommodating the present stage of cosmic acceleration.

In the following subsections we will identify the independent degrees of freedom describing small perturbations around DGP vacua in both branches, derive their linearized equations of motion and solve them. We will then compute the four dimensional effective action, isolate the ghost of the $4D$ theory, and discuss its consequences.

The physical interpretation of these solutions is based on the mathematical analysis of a differential operator derived by considering perturbations of Einstein's equations: the Lichnerowicz operator $\Delta_L h_{ab}$. This operator acts on a five-dimensional spacetime with a timelike boundary (the brane). We can solve these perturbation equations in whatever gauge we like, however, in order to get a braneworld interpretation of the results, the cleanest procedure we can follow is to separate this problem (operator plus space on which it acts) into a direct sum of a purely four-dimensional operator acting on a four-dimensional spacetime, and a self-adjoint ordinary differential operator acting on the semi-infinite real line. Obviously this latter operator acts on the space perpendicular to the brane, and hence to really benefit from this factorization, in these coordinates the brane should be held at a fixed coordinate position. Once we have made this decomposition, we will be able to identify the physical states and their norms from the braneworld point of view. To this end, we should write the perturbation in its irreducible components with respect to the braneworld, correctly identify the degrees of freedom corresponding to "motion" of the brane, and reduce our perturbation equations to a self-adjoint form.

### 3.1 Learning to count: mode expansion

We turn to the linearized perturbations $h_{ab}(x, y)$ about the background metric (2.7), (2.8), (2.10), defined by the general formula

$$ds^2 = a^2(y)\Big(\hat{\gamma}_{ab} + a(y)^{-3/2} h_{ab}(x, y)\Big)dx^a dx^b \,, \tag{3.1}$$

where we use the shorthand $\hat{\gamma}_{ab}dx^a dx^b = dy^2 + \bar{\gamma}_{\mu\nu}dx^\mu dx^\nu$. Note that $a(y) = \exp(\epsilon H y)$ as specified in (2.7), (2.8), (2.10). From now on, we will raise and lower $4D$ indices $(\mu, \nu, \ldots)$ with respect to the $4D$ de Sitter metric $\bar{\gamma}_{\mu\nu}$, and designate $4D$ de Sitter covariant derivatives by $D_\mu$. Our normalization convention for the perturbations in (3.1) reflects after-the-fact wisdom, in that it simplifies the bulk mode equations to a Schrödinger form, as we will see later on.

Since the spacetime ends on the brane, if we fix the gauge in the unperturbed solution (2.7), (2.8), (2.10) such that the brane resides at $y = 0$, a general perturbation of the system will also allow the brane itself to flutter, moving to

$$y = F(x^\mu) \,. \tag{3.2}$$

The explicit expressions for the perturbations $h_{ab}, F$ are obviously gauge-dependent. Now, to consider their transformation properties under diffeomorphisms

$$\begin{aligned} y &\to y' = y + \zeta(x, y) \,, \\ x^\mu &\to x'^\mu = x^\mu + \chi^\mu(x, y) \,, \end{aligned} \tag{3.3}$$

we should first classify them according to different representations of the $4D$ diffeomorphism group as

$$\text{perturbations} = \begin{cases} h_{\mu\nu}\,, & \text{a tower of 4D tensors ;} \\ h_{y\mu}\,, & \text{a tower of 4D vectors ;} \\ h_{yy}\,, & \text{a tower of 4D scalars ;} \\ F\,, & \text{a single 4D scalar.} \end{cases} \tag{3.4}$$

comprising in total 10 tensor + 4 vector + 1 scalar towers = 15 towers of degrees of freedom plus one more $4D$ scalar, i.e. precisely the number of independent fluctuations of a symmetric $5\times5$ bulk metric and the brane location. Clearly, not all of these degrees of freedom are physical: some can be undone by diffeomorphisms (3.3). Indeed, we can easily derive the explicit infinitesimal transformation rules,

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} - a^{3/2}\Big(D_\mu\chi_\nu + D_\nu\chi_\mu + 2\epsilon H\zeta\bar{\gamma}_{\mu\nu}\Big)\,, \\ h'_{y\mu} &= h_{y\mu} - a^{3/2}\Big(D_\mu\zeta + \partial_y\chi_\mu\Big)\,, \\ h'_{yy} &= h_{yy} - 2a^{3/2}\Big(\partial_y\zeta + \epsilon H\zeta\Big)\,, \\ F' &= F + \zeta_{y=0}\,, \end{aligned} \tag{3.5}$$

where we have used that $\partial_y a = \epsilon H\, a$.

In order to have a clear braneworld interpretation of variables, we find it convenient to work in the Gaussian-normal (GN) gauge (see e.g. [64]), in which any orthogonal component of the metric perturbation vanishes. Given any perturbation (3.4), we can transform to a GN gauge by picking the gauge parameters $\zeta, \chi^\mu$

$$\begin{aligned} \zeta &= \frac{1}{2a}\int_0^y dy a^{-1/2} h_{yy}\,, \\ \chi_\mu &= \int dy\, a^{-3/2} h_{y\mu} - D_\mu \int dy\,\zeta\,, \end{aligned} \tag{3.6}$$

which set $h'_{y\nu}$ and $h'_{yy}$ to zero. This still leaves us with 10 components of $h_{\mu\nu}$ and the brane location $F$ (omitting the primes), accompanied by 5 residual gauge transformations

$$\begin{aligned} \zeta &= \frac{f(x)}{a}\,, \\ \chi^\mu &= \chi_0^\mu(x) + \frac{1}{\epsilon H a} D^\mu f(x)\,, \end{aligned} \tag{3.7}$$

which can remove several more mass multiplets from the perturbations. However these could only be zero modes of some of the bulk fields, because of the restricted nature of the bulk variation of (3.7). Rather than completely gauge fix the perturbations now, it is more useful to resort to dynamics to find out which of the modes $h_{\mu\nu}$, $F$ are propagating and which are merely Lagrange multipliers. To this end we can first decompose the tensor $h_{\mu\nu}$ in terms of irreducible representations of the diffeomorphism group. This yields (for a proof, see Appendix (6))

$$h_{\mu\nu} = h^{\mathtt{TT}}_{\mu\nu} + D_\mu A_\nu + D_\nu A_\mu + D_\mu D_\nu \phi - \frac{1}{4}\bar{\gamma}_{\mu\nu} D^2\phi + \frac{h}{4}\bar{\gamma}_{\mu\nu} \, , \tag{3.8}$$

where $h^{\mathtt{TT}}_{\mu\nu}$ is a transverse traceless tensor, $D_\mu h^{\mathtt{TT}\ \mu}{}_\nu = h^{\mathtt{TT}\ \mu}{}_\mu = 0$, with 5 components, $A_\mu$ is a Lorentz-gauge vector, $D_\mu A^\mu = 0$, with 3 components, and $\phi$ and $h = h^\mu{}_\mu$ are two scalar fields (such that they correctly add up to the total of 10 degrees of freedom).

To get some feel of the dynamics before looking directly at the field equations, we can check how these modes transform under the residual gauge transformations (3.7). Substituting the residual gauge transformation (3.7) into (3.5), we find that the surviving, symmetric, tensor mode in the GN gauge and the brane location field transform as

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} - a^{3/2}\Big(D_\mu \chi_{0\,\nu} + D_\nu \chi_{0\,\mu}\Big) - 2\frac{a^{1/2}}{\epsilon H}\mathcal{O}_{\mu\nu} f \, , \\ F' &= F + f \, , \end{aligned} \tag{3.9}$$

where $\mathcal{O}_{\mu\nu} = D_\mu D_\nu + H^2\bar{\gamma}_{\mu\nu}$, and we have used that $a(0) = 1$. If we further split up the gauge transformation parameter $\chi_{0\,\mu} = \mathcal{E}_\mu + \frac{1}{2}D_\mu\omega$, where $D_\mu\mathcal{E}^\mu = 0$, and apply the decomposition (3.4) of $h_{\mu\nu}$ into the irreducible representations $h^{\mathtt{TT}}_{\mu\nu}, A_\mu, \phi$ and $h$ to (3.9), after a straightforward computation we find that the irreducible representations transform according to

$$\begin{aligned} h'^{\mathtt{TT}}_{\mu\nu} &= h^{\mathtt{TT}}_{\mu\nu} \, , \\ A'_\mu &= A_\mu - a^{3/2}\mathcal{E}_\mu \, , \\ \phi' &= \phi - a^{3/2}\omega - \frac{2a^{1/2}}{\epsilon H} f \, , \\ h' &= h - a^{3/2}D^2\omega - \frac{2a^{1/2}}{\epsilon H} D^2 f - 8\epsilon H a^{1/2} f \, , \\ F' &= F + f \, . \end{aligned} \tag{3.10}$$

Note that while the decomposition (3.8) of a general $h_{\mu\nu}$ into irreducible representations of the diffeomorphism group is kinematically unique, implying the breakdown of the residual gauge transformations as in (3.10), it does not - in general - guarantee that different modes won't mix dynamically. Indeed, in writing (3.8) we are implicitly assuming that different irreducible transformations live on different mass shells, and hence cannot mix dynamically at the quadratic level. This can be glimpsed at, for example, by noting that while the symmetries of the problem allow us to write the couplings like $h^{\mathtt{TT}}_{\mu\nu}(g_1 D^\mu D^\nu + g_2\bar\gamma^{\mu\nu})\phi$ etc, the $\mathtt{TT}$ conditions for $h^{\mathtt{TT}}_{\mu\nu}$ would imply that such couplings are pure boundary terms for non-singular couplings $g_1, g_2$. While this is true in general, the situation is considerably subtler when the representations become degenerate. In this instance the decomposition (3.8) requires more care. New accidental symmetries mixing different representations, notably tensor and scalar, may arise, modifying (3.10) and dynamically mixing the modes. This occurs in the vanishing brane tension limit on the self-accelerating branch of DGP. We will revisit this limit in more detail later on.

Keeping with the general situation for now, the transverse-traceless tensor $h^{\mathtt{TT}}_{\mu\nu}$ is gauge invariant, while the vector and the scalars are gauge dependent – we can gauge away the zero mode of the vector and one of the scalars. In addition, we see how the motion of the brane can be gauged away, choosing $f = -F$ to set the location of the brane to $y = 0$. By doing this, we are explicitly choosing coordinates which are brane-based, and the metric perturbation (3.8) has an explicit $\mathcal{O}_{\mu\nu}F$ term describing the brane fluctuation. In the brane-based approach, we have completely and rigorously separated the Lichnerowicz operator into brane parallel and transverse parts. However: once we have taken these coordinates, we do not have the liberty of making residual gauge transformations parameterized by $f$ in (3.10), because they would move the brane from the coordinate origin. In effect, the brane is tied to the dynamical fields $\phi$ and $h$ in the bulk, but its fluctuation $F$ turns into a Goldstone boson of the system. We emphasize that this is a *gauge choice*. We are choosing the brane-GN gauge to make the separation of the Lichnerowicz operator mathematically clean. However, one can also choose to allow the brane to fluctuate freely (and indeed the effective action computation is better done this way) by having a *bulk*-GN gauge, with the $f$-gauge freedom in (3.10), and the brane sitting at $y = F$. In this case, there are no fixed terms in the perturbation, and the brane motion enters into the boundary condition via evaluation of the *background* solution at $y = F$. The actual equations of motion and boundary terms in both gauges are identical, giving the same physical results and the same dynamical scalar fields. Thus, explicitly in brane-GN gauge:

$$h_{\mu\nu} = h^{\mathtt{TT}}_{\mu\nu} + D_\mu A_\nu + D_\nu A_\mu + \left(\mathcal{O}_{\mu\nu} - \tfrac{1}{4}\mathcal{O}^\lambda{}_\lambda\bar\gamma_{\mu\nu}\right)\phi + \frac{2a^{1/2}}{\epsilon H}\mathcal{O}_{\mu\nu}F + \tfrac{1}{4}h\bar\gamma_{\mu\nu}\,. \qquad (3.11)$$

To proceed with setting up the problem, we derive the field equations for the irreducible modes. Having restricted to the family of brane-GN gauge perturbations (3.11), we can substitute them in the field equations (2.2), (2.3) and after straightforward algebra write the linearized field equations in the bulk,

$$\delta G_{ab} = 0\,, \tag{3.12}$$

where

$$a^{3/2}\delta G_{\mu\nu} = X_{\mu\nu}(h) - \tfrac{1}{2}\left[\tfrac{\partial^2}{\partial y^2} - \tfrac{9H^2}{4}\right](h_{\mu\nu} - h\bar{\gamma}_{\mu\nu})\ , \tag{3.13}$$

$$a^{3/2}\delta G_{\mu y} = \frac{1}{2}\left[\frac{\partial}{\partial y} - \frac{3\epsilon H}{2}\right] D^\nu\left(h_{\mu\nu} - h\bar{\gamma}_{\mu\nu}\right)\ , \tag{3.14}$$

$$a^{3/2}\delta G_{yy} = \frac{3\epsilon H}{2}\left[\frac{\partial}{\partial y} - \frac{3\epsilon H}{2}\right] h - \frac{1}{2}\left[D^\mu D^\nu - \bar{\gamma}^{\mu\nu}(D^2 + 3H^2)\right] h_{\mu\nu}\,, \tag{3.15}$$

and

$$X_{\mu\nu}(h) = -\frac{1}{2}\left(D^2 - 2H^2\right) h_{\mu\nu} + D_{(\mu}D^\alpha h_{\nu)\alpha} - \tfrac{1}{2}D_\mu D_\nu h - \tfrac{1}{2}\bar{\gamma}_{\mu\nu}\left[D^\alpha D^\beta h_{\alpha\beta} - (D^2 + H^2)\, h\right]\ , \tag{3.16}$$

and on the brane,

$$\delta\Theta_{\mu\nu} = \left\{M_4^2 X_{\mu\nu}(h) - M_5^3\left[\frac{\partial}{\partial y} - \frac{3\epsilon H}{2}\right](h_{\mu\nu} - h\bar{\gamma}_{\mu\nu})\right\}_{y=0} = \tfrac{1}{2}T_{\mu\nu}\,. \tag{3.17}$$

Before we proceed with the details of the mode decomposition of this system by direct substitution of (3.8), we note that the Lorentz-gauge vector $A_\mu$ turns out to be a free field in the linearized theory in flat bulk. Thus the solutions for $A_\mu$ decouple from the brane-localized matter in the leading order. They are irrelevant for the stability analysis which is our purpose here. Hence we will set $A_\mu = 0$ from now on, assuming we have arranged for bulk boundary conditions which guarantee this in the linear order.

### 3.2 Fluctuations around the vacuum

First note that independently of matter on the brane, the $yy$ and $y\mu$ equations must be identically satisfied. In conjunction with the trace of the $\mu\nu$ equation, this can be seen to imply that a gauge can be chosen in which $h = 0$, and $\mathcal{O}^\lambda{}_\lambda\phi = 0$. If in addition we have no matter on the brane, then we see that

$$\mathcal{O}^\lambda{}_\lambda F = (D^2 + 4H^2)F = 0\,, \tag{3.18}$$

and so the metric perturbation (3.11) is completely transverse and traceless.

The remaining $\mu\nu$ bulk equations then simplify considerably to give

$$\left[D^2 - 2H^2 + \frac{\partial^2}{\partial y^2} - \frac{9H^2}{4}\right] h_{\mu\nu}(x,y) = 0\,. \tag{3.19}$$

with the boundary condition

$$\left[M_4^2\left(D^2-2H^2\right)h_{\mu\nu}+2M_5^3\left(\frac{\partial}{\partial y}-\frac{3\epsilon H}{2}\right)h_{\mu\nu}\right]_{y=0}=0 \tag{3.20}$$

Now, to solve this equation we should carefully decompose the tensor into orthogonal modes, which in general do not mix at the linearized level. Those are exactly the irreducible representations we discussed previously. Thus using linear superposition, we can expand the general metric fluctuation in $h^{\mathtt{TT}}_{\mu\nu}$ and $\phi$, the latter of which couples to the field $F$ through the boundary condition (3.20), leaving the TT-tensors with entirely homogeneous boundary conditions. We therefore write

$$h_{\mu\nu}=\sum_m u_m(y)\chi^{(m)}_{\mu\nu}(x)+h^{(\phi)}_{\mu\nu}+\frac{2a^{1/2}}{\epsilon H}\mathcal{O}_{\mu\nu}F \tag{3.21}$$

where we have performed the mode expansion $h^{\mathtt{TT}}_{\mu\nu}(x,y)=\sum_m u_m(y)\chi^{(m)}_{\mu\nu}(x)$, in terms of the $4D$ modes $\chi^{(m)}_{\mu\nu}(x)$ which satisfy $(D^2-2H^2)\chi^{(m)}_{\mu\nu}=m^2\chi^{(m)}_{\mu\nu}$. We have also defined the scalar mode $h^{(\phi)}_{\mu\nu}=\mathcal{O}_{\mu\nu}\phi$, and separated variables in the scalar field by setting $\phi=W(y)\hat{\phi}(x)$, where $\hat{\phi}$ is a general $4D$ tachyonic field obeying

$$(D^2+4H^2)\hat{\phi}=0\,. \tag{3.22}$$

This tachyonic mode is present whenever we compactify the theory on an interval with de Sitter boundary branes. It can be traced back to the repulsive nature of inflating domain walls [55]. Here, it is simply an indication that a multi-de Sitter brane configuration requires a special stabilizing potential, as is familiar already in the context of RS2 braneworld models [65, 66, 67]. This kind of an instability is generically much slower and hence less dangerous than the ghost, as it is governed by the scale $\tau\sim 1/m_{\rm tachyon}\sim H^{-1}$ that is as long as the age of the universe. When tension is positive, this mode therefore remains largely harmless for the phenomenological applications of the theory. However, on the self-accelerating branch it mixes with the ghost in the vanishing tension limit on the self-accelerating branch, and becomes the ghost itself for negative tension, as we will see later.

We now turn to the analysis of the TT perturbations which form the main part of the propagator, and determine the norm on the transverse $y$-space. The bulk field equation (3.19) and the boundary condition (3.20) reduce to the boundary value problem

$$\begin{aligned}&u_m''(y)+\left(m^2-\frac{9H^2}{4}\right)u_m(y)=0\,,\\&M_5^3\left[u_m'(0)-\frac{3\epsilon H}{2}u_m(0)\right]+\tfrac{1}{2}m^2M_4^2u_m(0)=0\,,\end{aligned} \tag{3.23}$$

which is self-adjoint with respect to the inner product

$$\langle u|v\rangle = \int_{-\infty}^{\infty} dy\, \Big(M_5^3 + M_4^2\delta(y)\Big)u(y)v(y) = 2M_5^3\int_0^{\infty} dy\, u(y)v(y) + M_4^2 u(0)v(0)\,. \quad (3.24)$$

The eigenmodes $u_m$ with different eigenvalues $m$ are orthogonal. We choose the normalization such that the discrete modes, if any, satisfy $\langle u_m|u_n\rangle = \delta_{mn}$, while the continuum modes satisfy $\langle u_m|u_n\rangle = \delta(m-n)$. This is simply a reflection of the fact that far from the brane the bulk modes behave just like bulk plane waves, and the $4D$ mass is precisely the $p_y$-component of the $5D$ momentum.

To determine the spectrum of the boundary value problem (3.23), (3.24) we rewrite the boundary value problem (3.23) as a Schrödinger equation

$$u_m'' + \Big[m^2 - \frac{9H^2}{4} + (\frac{M_4^2}{M_5^3}m^2 - 3\epsilon H)\delta(y)\Big]u_m = 0\,. \quad (3.25)$$

It is now clear that the solutions of (3.23) must fall into two categories: $(i)$ one discrete mode for each branch, localized to the $\delta$-function potential, if it is normalizable according to (3.24), and $(ii)$ a continuum of 'free' modes, gapped by $m \geq \frac{3}{2}H$.

- $m^2 < \frac{9H^2}{4}$: the normalizable solution of (3.23) in the bulk, representing a single, light, localized graviton on each branch, with a mass

$$m_d^2 = \frac{M_5^3}{M_4^2}\left[3H - \frac{2M_5^3}{M_4^2}\right](1+\epsilon)\ , \quad (3.26)$$

fixed by the boundary conditions (3.23), and wave function

$$u_m(y) = \alpha_m \exp(-\lambda_m y)\,, \qquad \alpha_m = \frac{1}{M_4}\left[\frac{3M_4^2H - 2M_5^3(1+\epsilon)}{3M_4^2H - 2M_5^3\epsilon}\right]^{\frac{1}{2}}\,. \quad (3.27)$$

where $\lambda_m = \sqrt{\frac{9H^2}{4} - m^2}$ and $\langle u|u\rangle = 1$. On the normal branch ($\epsilon = -1$),

$$m_d = 0\,. \quad (3.28)$$

On the self accelerating branch ($\epsilon = +1$),

$$\begin{aligned} 0 < m_d^2 < 2H^2\,, &\qquad \text{for } \sigma > 0\,; \\ m_d^2 > 2H^2\,, &\qquad \text{for } \sigma < 0\,. \end{aligned} \quad (3.29)$$

Herein is our first glimpse of the tensor ghost: for positive tension, the localized light graviton mode on the self accelerating branch lies in the forbidden mass range $0 < m^2 < 2H^2$ discussed in [44, 45, 46]. Its helicity-0 component is the ghost, as we will review later on (see Appendix (7)).

- $m^2 \geq \frac{9H^2}{4}$: the $\delta$-function normalizable modes are

$$u_m(y) = \alpha_m \sin(\omega_m y + \delta_m)\,, \qquad \alpha_m = \sqrt{\frac{m}{\pi M_5^3 \omega_m}}\,, \tag{3.30}$$

where $\omega_m = \sqrt{m^2 - \frac{9H^2}{4}}$ and $\langle u_m | u_{\bar{m}} \rangle = \delta(m - \bar{m})$. The integration constant $\delta_m$ which solves the boundary condition (3.23) is

$$\tan \delta_m = \frac{2M_5^3 \omega_m}{3M_5^3 \epsilon H - m^2 M_4^2} \tag{3.31}$$

Turning now to the scalar component $h^{(\phi)}_{\mu\nu}(x, y) = W(y)\mathcal{O}_{\mu\nu}\hat{\phi}(x)$, it is not difficult to see that it obeys

$$\left(D^2 - 2H^2\right) h^{(\phi)}_{\mu\nu} = 2H^2 h^{(\phi)}_{\mu\nu}\,, \tag{3.32}$$

(equivalent to a $4D$ mass $m^2 = 2H^2$). The bulk equation (3.19) then yields the wave equation for $W$,

$$W''(y) - \frac{H^2}{4} W(y) = 0\,. \tag{3.33}$$

The boundary condition (3.20) enforces a relation between $\hat{\phi}$ and $F$:

$$\left( W'(0) - \Big(\frac{3}{2}\epsilon H - \frac{M_4^2 H^2}{M_5^3}\Big) W(0) \right) \hat{\phi} = 2\Big(1 - \epsilon \frac{H M_4^2}{M_5^3}\Big) F\,. \tag{3.34}$$

The wave function solutions for either of the DGP branches are

$$W(y) = \alpha \exp\left(-\frac{H}{2} y\right) + \beta \exp\left(\frac{H}{2} y\right)\,. \tag{3.35}$$

From (3.24), the norm is determined by $\int_0^\infty dy\, W^2(y)$, where the lower limit of integration accounts for the unperturbed location of the brane at $y = 0$, around which we impose the $\mathbb{Z}_2$ symmetry. Thus the $\alpha$-mode is normalizable but the $\beta$-mode is not. We therefore set $\beta = 0$. This choice, at least in principle, corresponds to prescribing boundary conditions at infinity, which ensure the brane is an isolated system. Thus setting $\beta = 0$, and separating the variables by setting $\hat{\phi} = F$ in (3.34) we find

$$-\Big(\frac{(1 + 3\epsilon)H}{2} - \frac{M_4^2 H^2}{M_5^3}\Big)\, \alpha = 2\Big(1 - \frac{M_4^2 \epsilon H}{M_5^3}\Big)\,. \tag{3.36}$$

However, we must be mindful of this choice because of the possible interplay with the brane bending term (3.10), as we will see next.

Now: on the normal branch ($\epsilon = -1$), it follows from (3.10) that the normalizable $\alpha$-mode is gauge-dependent: in fact, it is of the same form as the brane-bending mode since it is proportional to $a^{1/2} = \exp\left(-\frac{H}{2}y\right)$. On the other hand the non-normalizable $\beta$-mode is gauge-invariant by itself, and so setting it equal to zero is straightforward. Then (3.36) gives $\alpha = 2/H$, which means that the brane boundary condition (3.20) in fact precisely sets the normalizable gauge-invariant mode $\alpha\hat{\phi} - 2F/H$ to zero. Hence

$$h_{\mu\nu} \equiv h^{\mathtt{TT}}_{\mu\nu} \tag{3.37}$$

Thus the net effect of the $\alpha$-mode is to undo the brane bending. This is because the translational invariance of the brane-bulk system, which yields the residual gauge symmetry (3.10) is linearly realized in the presence of the brane, which imposes *gauge-invariant* boundary condition, so that the normalizable bulk mode and the brane bending completely compensate each other. Put another way, the only consistent matter-free solution for the normal branch DGP brane is where the brane does not move from $y = 0$, and only TT GN perturbations in the metric are allowed. This, of course, should have been expected all along, as it is just the statement that the radion field decouples in the case of single UV brane with $4D$ Minkowski or de Sitter geometry embedded in the standard way in $5D$ Minkowski or $AdS$ space. Here we see explicitly how gauge invariance and normalizability enter this subtle conspiracy to remove this mode, essentially allowing that any scalar bulk perturbation localized to the brane can be bent away.

On the self-accelerating branch ($\epsilon = +1$), the situation is very different: now, the normalizable scalar mode is gauge-invariant by itself. The non-normalizable $\beta$-mode is not, and so imposing boundary conditions which require $\beta = 0$ breaks the residual gauge invariance (3.10). The brane bending mode $F$ is the Goldstone field of the broken symmetry, and the brane boundary condition (3.20) for a generic value of $H$ (i.e. for non-zero tension) yields

$$\alpha = -\frac{2}{H}\left[\frac{M_5^3 - M_4^2 H}{2M_5^3 - M_4^2 H}\right] , \tag{3.38}$$

which pins the Goldstone $F$ to the normalizable gauge-invariant scalar perturbation $\phi$:

$$h^{(\phi)}_{\mu\nu} = -\frac{2}{H}\left[\frac{M_5^3 - M_4^2 H}{2M_5^3 - M_4^2 H}\right] e^{-Hy/2}\mathcal{O}_{\mu\nu}F\,. \tag{3.39}$$

This perturbation represents a genuine radion, or physical motion of the brane with respect to infinity. Although our choice of brane-GN gauge fixes the brane to the coordinate position $y = 0$, it does so at the cost of, this time, breaking the residual gauge symmetry generated by $f$ in (3.10) and introducing the explicit "book-keeping" $\mathcal{O}_{\mu\nu}F$ term in $h_{\mu\nu}$, which is the remnant of the translational zero mode of the brane. Had we instead allowed the brane position to be arbitrary, at $y = F$, (without the $\mathcal{O}_{\mu\nu}F$ term in (3.11)), the boundary conditions at $y = F$ would still have had the same form, since the $F$-terms would have entered when evaluating the background at nonzero $y$. Both approaches are completely equivalent, the former being more suitable to a brane based observer and the latter to an asymptotic observer. The gauge transformation between these is a $y$-translation, which therefore corresponds to real motion of the brane, just as in the 2-brane RS case [41]. The absence of this mode on the normal branch reflects the fact that there is no distinguishable motion of an individual $\mathbb{Z}_2$ symmetric brane.

When the tension is different from zero, the solutions $\chi^{(m)}_{\mu\nu}$ are precisely the TT-tensors $h^{\mathtt{TT}}_{\mu\nu}$ of (3.4) from the previous subsection. The scalar mode $h^{(\phi)}_{\mu\nu}$ has eigenvalue $m^2 = 2H^2$, as seen from (3.32), and the eigenvalues of the eigenmodes $\chi^{(m)}_{\mu\nu}$ are all different from $2H^2$ when $\sigma \neq 0$. Thus the scalar mode $\phi$, disguised as the tensor $h^{(\phi)}_{\mu\nu}$, is orthogonal to all $\chi^{(m)}_{\mu\nu}$. Hence $\chi^{(m)}_{\mu\nu}$ coincide with the TT tensors $h^{\mathtt{TT}}_{\mu\nu}$, and so when there is no matter on the brane, the solutions are given by

$$\begin{aligned} h_{\mu\nu}(x,y) = \quad & \alpha_{m_d} e^{-\lambda_{m_d} y} \chi^{(m_d)}_{\mu\nu}(x) + \int_{\frac{3H}{2}}^{\infty} dm\ u_m(y) \chi^{(m)}_{\mu\nu}(x) \\ & + \frac{(1+\epsilon)}{H} \left\{ a^{1/2} \mathcal{O}_{\mu\nu} F - \left[ \frac{M_5^3 - M_4^2 H}{2M_5^3 - M_4^2 H} \right] a^{-1/2} \mathcal{O}_{\mu\nu} F \right\} . \end{aligned} \tag{3.40}$$

This solution clearly remains valid on the normal branch even in the limit of vanishing tension, $\sigma \to 0$, and for the full range of $\sigma < 0$, because when $\epsilon = -1$ the potentially dangerous $\mathcal{O}_{\mu\nu} F$ terms vanish identically.

However on the self-accelerating branch where $\epsilon = +1$ the solution (3.40) – as it stands – fails when the tension vanishes, $\sigma = 0$, because of the pole in $\phi$, or $\alpha$, (3.38), (3.39). Indeed, (2.12) implies that when $\sigma \to 0$, $H \to 2M_5^3/M_4^2$, and so the parameter $\alpha$ in (3.38) diverges. Thus the mode $\phi$ as given by (3.39) is ill-defined in this limit. At a glance, noting that the coefficient of $\hat{\phi}$ in (3.34) vanishes, one may interpret equation (3.38) as implying $F = 0$, thus fixing the brane rigidly at $y = 0$, and allowing $\hat{\phi}$ to fluctuate independently of $F$. However, in light or the residual gauge transformations (3.10) and our gauge-fixing $\beta = 0$, that removed the non-normalizable gauge-dependent bulk scalar, setting $F = 0$ also would completely break the residual gauge symmetry group. This is dangerous, since it may miss physical degrees of freedom, which warns us against such a quick conclusion. To see what is really going on we must tread carefully.

What's going on when the tension vanishes is that the mass of the localized tensor mode on the self-accelerating branch approaches $m_d^2 = 2H^2$, as is clear from (3.26). Further, the bulk wave function of the lightest localized tensor (3.27) converges to $\exp(-\lambda_{m_d} y) = \exp(-Hy/2)$, i.e. it becomes identical to the bulk wave function of the gauge-invariant scalar mode $h^{(\phi)}_{\mu\nu}$. Thus the lightest tensor, $h^{\mathtt{TT}(\mathtt{m_d})}_{\mu\nu}$, and the scalar

$h^{(\phi)}_{\mu\nu} = \mathcal{O}_{\mu\nu}\phi$ become dynamically *degenerate*, and can mix[5] together: they both solve the $4D$ field equations $(D^2 - 4H^2)\chi_{\mu\nu} = 0$ and have formally the same tensor structure. Now, it has been noted by Deser and Nepomechie [43] that in the special case when the mass of the massive $4D$ Pauli-Fierz theory in de Sitter space equals $2H^2$, the theory develops an accidental symmetry[43, 44, 45, 46]. The tensor dynamics becomes invariant under the transformation $\chi^{(\sqrt{2}H)}_{\mu\nu} \to \chi^{(\sqrt{2}H)}_{\mu\nu} + \mathcal{O}_{\mu\nu}\vartheta$, where $\vartheta$ is any solution of the equation $(D^2 + 4H^2)\vartheta = 0$, affecting only the helicity-0 component of $\chi^{(\sqrt{2}H)}_{\mu\nu}$.

Lifting this symmetry to the present case is considerably more intricate because of the degenerate scalar $h^{(\phi)}_{\mu\nu}$. Noting first that the wave profile of the lightest localized tensor is now $e^{-Hy/2} = 1/\sqrt{a}$, the accidental symmetry of [43] shifts the bulk $\mathtt{TT}$-tensor by

$$h'^{\mathtt{TT}}_{\mu\nu} = h^{\mathtt{TT}}_{\mu\nu} + a^{-1/2}\mathcal{O}_{\mu\nu}\vartheta\,. \tag{3.41}$$

However given the higher-dimensional origin of the perturbations $h_{\mu\nu}$ we cannot arbitrarily shift these modes around. The only gauge generators available to us, that could in principle generate such shifts, are the residual gauge transformation rules of (3.9). However as is clear from (3.9), none of the residual gauge transformations have the correct bulk wave profile to yield (3.41). Thus the transformation (3.41) must be understood as the Stückelberg symmetry of the problem: shifting $\chi^{(\sqrt{2}H)}_{\mu\nu}$ by a $\vartheta$ piece *must* be compensated by shifting another field in the decomposition (3.8) to keep the total metric perturbation $h_{\mu\nu}$ invariant. The only available mode with the correct wave profile, and the correct tensor structure so as not to break the diffeomorphism invariance, is the normalizable scalar that is invariant under (3.10). Thus the scalar must now be promoted into a Stückelberg field for $\vartheta$. Note that this is completely analogous to rewriting the massive $U(1)$ gauge theory in the Stückelberg form, formally giving up on the Lorentz gauge for $A_\mu$ by introducing the Stückelberg scalar field.

So to properly account for the accidental symmetry on the self-accelerating brane in the vanishing tension limit generated by $\vartheta$, we must *enhance* the residual gauge transformation group (3.10) by also including in it

$$h'^{\mathtt{TT}}_{\mu\nu} = h^{\mathtt{TT}}_{\mu\nu} + a^{-1/2}\mathcal{O}_{\mu\nu}\vartheta\,, \qquad \phi' = \phi - a^{-1/2}\vartheta\,. \tag{3.42}$$

Hence once we insert $\sigma = 0$ explicitly in the field equations (2.2), (3.20), we can separate the field equations for the scalar and the lightest tensor from each other only *after* explicitly gauge-fixing the Stückelberg symmetry generated by $\vartheta$. The full field equations are merely *covariant* under it because of the scalar field $\phi$. Once we have

[5]This mixing has been noticed as the resonance instability in the shock wave analysis of [33], and discussed at length in [68].

gauge-fixed the brane at $y = 0$, the boundary conditions at the brane will really relate $F$ to a *linear combination* of the helicity-0 tensor and the gauge-invariant normalizable scalar $\hat{\phi}$. A simple way to think about the boundary conditions is to fix the $\vartheta$ gauge by completely removing the helicity-0 mode from the tensor and absorbing it into $\hat{\phi}$. Then the brane boundary condition (3.38) just states that this $\vartheta$-gauge fixed field $\hat{\phi}$ is fluctuating freely - but it does not disappear from the spectrum. Indeed, we can go to a different $\vartheta$-gauge, fixing it now such that the $\hat{\phi}$ is completely eaten by the tensor $\chi_{\mu\nu}^{(\sqrt{2}H)}$, which regains its helicity-0 component. This of course is completely equivalent to the unitary gauge of a theory with the Stückelberg fields, where the gauge fields eat Stückelberg and gain mass. This is *crucial* for the failure of the ghost to decouple in the tensionless brane limit, and is a simple way to understand the analysis of [68].

However, neither of these gauges is convenient for the computation of the effective action, to be pursued later on. Instead, we can pick another $\vartheta$ gauge by taking the general solution (3.40) and defining a $\vartheta$ which removes the pole in (3.40) as $\sigma \to 0$ and produces a smooth limit [39, 68]. We can do this by a shift

$$\alpha_{m_d} \chi_{\mu\nu}^{(m_d)}(x) = \mathcal{H}_{\mu\nu}(x) - \alpha\, \mathcal{O}_{\mu\nu} F\,. \tag{3.43}$$

Substituting this in (3.40) yields

$$\begin{aligned} h_{\mu\nu}(x,y) = {} & e^{-\lambda_{m_d} y} \mathcal{H}_{\mu\nu}(x) + \int_{\frac{3H}{2}}^{\infty} dm\; u_m(y) \chi_{\mu\nu}^{(m)}(x) \\ & + \alpha \left\{ e^{-Hy/2} - e^{-\lambda_{m_d} y} \right\} \mathcal{O}_{\mu\nu} F + \frac{2}{H} e^{Hy/2} \mathcal{O}_{\mu\nu} F\,. \end{aligned} \tag{3.44}$$

Then carefully taking the limit $\sigma \to 0$ (noting that $\alpha \propto 1/\sigma$, and $\lambda_{m_d} = H/2 + O(\sigma)$) yields

$$h_{\mu\nu}(x,y) = e^{-\frac{H}{2} y} \Big( \mathcal{H}_{\mu\nu}(x) - y\, \mathcal{O}_{\mu\nu} F \Big) + \int_{\frac{3H}{2}}^{\infty} dm\; u_m(y) \chi_{\mu\nu}^{(m)}(x) + \frac{2}{H} e^{Hy/2} \mathcal{O}_{\mu\nu} F\,. \tag{3.45}$$

where the $4D$ tensor $\mathcal{H}_{\mu\nu}$ satisfies [6]

$$(D^2 - 4H^2) \mathcal{H}_{\mu\nu} = -H \mathcal{O}_{\mu\nu} F\,. \tag{3.46}$$

Note that the trace of this equation yields $\mathcal{O}^{\lambda}{}_{\lambda} F = (D^2 + 4H^2) F = 0$. In this gauge, a way to think about Eq. (3.46) is to view the field $F$ as the independent degree of freedom, and the helicity-0 component of the graviton $\mathcal{H}_{\mu\nu}$ as being completely

[6] This follows from the $\sigma \to 0$ limit of $(D^2 - m_d^2 - 2H^2)\mathcal{H}_{\mu\nu}$ from (3.43), or can be readily derived from the equations of motion (3.19, 3.20).

determined by the source $F$. Yet, the brane localized matter can only feel its influence through the couplings to the helicity-0 component of $\mathcal{H}_{\mu\nu}$, as can be seen from Eq. (3.45), which shows that on the brane at $y = 0$ the $\propto \mathcal{O}_{\mu\nu} F$ terms vanish. In this way, the tensor $\mathcal{H}_{\mu\nu}(x)$ retains *five* physical, gauge-invariant degrees of freedom precisely because of this mixing with $F$, inherited from the bulk scalar $\phi$. In effect what happened in the limiting procedure is that the pole of (3.40) was a pure gauge term of the Stückelberg gauge symmetry, and was absorbed away by the choice of $\vartheta$, leaving in its wake the smooth function (3.45).

### 3.3 Linearized fields of matter lumps

Here we include the contributions from localized stress-energy lumps on the brane. In the linearized theory, the general solution is a linear combination of the homogeneous solution, given by (3.40) or (3.45), describing propagating graviton modes, and a particular solution comprising of a `TT` piece and the relevant brane bending term which describe the response of the fields to the source. Thus we write:

$$h_{\mu\nu}(x,y) = h^{(hom)}_{\mu\nu}(x,y) + \tilde{\chi}_{\mu\nu}(x,y) + \frac{2a^{1/2}}{\epsilon H} \mathcal{O}_{\mu\nu} f(x)\,, \tag{3.47}$$

where the fields $\tilde{\chi}_{\mu\nu}(x)$ and $f(x)$ are the sought-after particular solutions that include the effects of the brane sources. They must be the solution of the boundary value problem,

$$\begin{aligned} &\left[D^2 - 2H^2 + \frac{\partial^2}{\partial y^2} - \frac{9H^2}{4}\right] \tilde{\chi}_{\mu\nu}(x,y) = 0\,, \\ &\Big[\tfrac{1}{2} M_4^2 \left(D^2 - 2H^2\right) \tilde{\chi}_{\mu\nu} + M_5^3 \left(\tfrac{\partial}{\partial y} - \tfrac{3\epsilon H}{2}\right) \tilde{\chi}_{\mu\nu}\Big]_{y=0} \\ &\qquad\qquad -2\left(M_5^3 - M_4^2 \epsilon H\right)\left(\mathcal{O}_{\mu\nu} - \mathcal{O}^{\lambda}{}_{\lambda}\bar{\gamma}_{\mu\nu}\right) f = -\tfrac{1}{2} T_{\mu\nu}(x)\,, \end{aligned} \tag{3.48}$$

Tracing the boundary condition immediately gives

$$\mathcal{O}^{\lambda}{}_{\lambda} f = \left[D^2 + 4H^2\right] f(x) = -\left[\frac{T}{12(M_5^3 - M_4^2 \epsilon H)}\right]. \tag{3.49}$$

This fixes $f(x)$ completely, as any homogeneous brane-bending term is accounted for in $h^{(hom)}_{\mu\nu}$.

We can write the particular solution as a spectral expansion, using the properties of the $4D$ mass eigenmodes determined in section 3.2. Formally, we consider the $4D$ differential operator $D^2 - 2H^2$ whose tensor spectrum is given by `TT`-tensors $\chi^{(p)}_{\mu\nu}$ obeying $(D^2 - 2H^2)\chi^{(p)}_{\mu\nu} = p^2 \chi^{(p)}_{\mu\nu}$, and expand the solutions and the sources as

$$\tilde{\chi}_{\mu\nu}(x,y) = \int_p v_p(y)\, \chi^{(p)}_{\mu\nu}(x)\,, \qquad\qquad \tau_{\mu\nu}(x) = \int_p \tau_p\, \chi^{(p)}_{\mu\nu}(x)\,. \tag{3.50}$$

where

$$\tau_{\mu\nu}(x) = T_{\mu\nu} - 4(M_5^3 - M_4^2\epsilon H)\left[D_\mu D_\nu - \bar\gamma_{\mu\nu}(D^2+3H^2)\right] f(x)\,. \qquad (3.51)$$

The $\chi^{(p)}_{\mu\nu}$ tensors are an orthonormal basis of the spectrum of $D^2 - 2H^2$, whose eigenvalues $p^2 \neq m^2$ are taken to be off mass shell as is usual in the inhomogeneous problem. Here, $\int_p$ is a generalized sum, accounting for the integration over the continuum part of the spectrum and the summation over the discrete, localized modes. Then the boundary value problem (3.48) reduces to the system

$$\begin{aligned} &v_p''(y) + \left(p^2 - \frac{9H^2}{4}\right) v_p(y) = 0\,, \\ &M_5^3\left[v_p'(0) - \frac{3\epsilon H}{2} v_p(0)\right] + \tfrac{1}{2}p^2 M_4^2 v_p(0) = -\tfrac{1}{2}\tau_p\,, \end{aligned} \qquad (3.52)$$

extending (3.23) of section 3.2 with a source term $\tau_p$. We can write the solutions $v_p$'s in terms of the on-shell eigenfunctions $u_m(y)$. First, rewrite (3.52) as

$$v_p'' + \left[p^2 - \frac{9H^2}{4} + (\frac{M_4^2}{M_5^3}p^2 - 3\epsilon H)\delta(y)\right] v_p = -\frac{\tau_p}{M_5^3}\delta(y)\,. \qquad (3.53)$$

Then expanding as $v_p(y) = v_{p\,m_d} u_{m_d}(y) + \int_{3H/2}^\infty dm\, v_{p\,m} u_m(y)$, substituting in (3.53) and comparing with (3.25), and finally using the orthonormality of the eigenmodes $u_m$ with respect to the inner product (3.24), we find

$$v_p(y) = -\left[\frac{u_{m_d}(y)u_{m_d}(0)}{p^2 - m_d^2} + \int_{\frac{3H}{2}}^\infty dm\, \frac{u_m(y)u_m(0)}{p^2-m^2}\right]\tau_p\,. \qquad (3.54)$$

Hence we can rewrite the linearized field of the matter on the brane as

$$\tilde\chi_{\mu\nu}(x,y) = \int d^4x'\sqrt{-\bar\gamma}\; G_{\mu\nu}{}^{\alpha\beta}(x,y;x',0)\tau_{\alpha\beta}(x')\,, \qquad (3.55)$$

where the Green's function is given by the eigenmode expansion

$$G_{\mu\nu}{}^{\alpha\beta}(x,y;x',0) = -\int_p \left[\frac{u_{m_d}(y)u_{m_d}(0)}{p^2 - m_d^2} + \int_{\frac{3H}{2}}^\infty dm\, \frac{u_m(y)u_m(0)}{p^2-m^2}\right]\chi^{(p)}_{\mu\nu}(x)\chi^{*\,(p)\alpha\beta}(x')\,, \qquad (3.56)$$

where the asterisk denotes complex conjugation. Thus in the case when the only source of perturbation of the vacuum is matter on the brane, and the propagating geometric modes are not excited, the brane geometry is perturbed by

$$\delta\gamma_{\mu\nu} = \tilde\chi_{\mu\nu}(x,0) + \frac{2\epsilon}{H}\mathcal{O}_{\mu\nu} f \qquad (3.57)$$

where $f$ is given by (3.49) and $\tilde{\chi}_{\mu\nu}(x,0)$ by (3.55). Note however that one must treat the Green's function (3.56) with care, because the summation $\sum_p$ over the continuum has a branch cut at $m^2 = 9H^2/4$, which can be seen from the form of the continuum eigenfunctions presented in section 3.2.

The ghost is hidden in the localized mode contribution to $G_{\mu\nu}{}^{\alpha\beta}(x,y;x',0)$, (i.e. the $\frac{u_{m_d}(y)u_{m_d}(0)}{p^2-m_d^2}$ term), specifically, it resides in the helicity-0 component. We could divine the ghost by computing the residues at the pole $p^2 = m_d^2$ of the propagator. Alternatively, as we will do in the next section, we can simply compute the effective action for small metric fluctuations and unveil the ghost-like behavior from the negative contributions to it.

### 3.4 Forking the ghost: calculating the effective action

Let us now fork[7] the ghost: we compute the effective $4D$ action of normalizable small perturbations considered in the previous section, that will serve as a straightforward diagnostic of the ghost. We start with the general case of non-vanishing tension, $\sigma \neq 0$, and consider the limit $\sigma = 0$ separately. Let us consider the general metric perturbation in bulk GN gauge. Starting from (3.8), considering the $\mu y$ and $yy$ Einstein equations (3.14), (3.15), and the mode decomposition discussed in the previous sections we can write,

$$\begin{aligned} \delta g_{\mu\nu}(x,y) &= \sqrt{a(y)}\Big(\mathcal{O}_{\mu\nu}\phi + u_{m_d}(y)h^{(m_d)}_{\mu\nu}(x) + \int_{\frac{3H}{2}}^{\infty} dm\; u_m(y)h^{(m)}_{\mu\nu}(x)\Big)\,, \\ \delta g_{yy} &= \delta g_{\mu y} = 0\,, \end{aligned} \tag{3.58}$$

where $\phi(x,y) = \exp(-Hy/2)\hat{\phi}(x)$ as before, and the $\mathcal{O}^{\lambda}{}_{\lambda}\phi$ and $h$ terms automatically cancel each other in bulk GN gauge. In order to calculate the effective action, it is convenient to fix the brane position so that it lies at $y = 0$, whilst maintaining the bulk GN gauge near infinity. This can be done with a carefully chosen $y$-dependent gauge transformation:

$$x^\mu \to x^\mu - \xi^\mu\,, \qquad y \to y - \xi^y\,, \tag{3.59}$$

where

$$\xi^\mu(x,y) = \begin{cases} \frac{1}{\epsilon H a}D^\mu(\frac{\hat{\phi}}{\alpha} + f) & \text{for } y \ll R \\ 0 & \text{for } y \gg R \end{cases}\,, \qquad \xi^y(x,y) = \begin{cases} \frac{1}{a}(\frac{\hat{\phi}}{\alpha} + f) & \text{for } y \ll R \\ 0 & \text{for } y \gg R \end{cases}\,, \tag{3.60}$$

where $R > 0$ is some arbitrary finite radius, and $\alpha$ is given in Eq. (3.36). The gauge transformation (3.60) should be viewed as the limiting form of a smooth interpolating

[7] "Forking", or "dowsing", is a practice which sometimes reveals an occult presence by means of a two-pronged fork, whose role in our case is assumed by the second order effective action.

family of test functions that continuously vary in the bulk. The new gauge is *not* Gaussian Normal everywhere, interpolating instead between a brane-GN gauge at the brane positioned at $y = 0$, and a bulk-GN gauge near infinity. The bulk metric perturbation in this gauge is

$$\delta g_{ab} \to \delta g_{ab}(x,y) + \bar{\nabla}_a \xi_b + \bar{\nabla}_a \xi_b \,, \tag{3.61}$$

where $\bar{\nabla}$ is the covariant derivative for $\bar{g}_{ab}$. The brane metric perturbation in turn is

$$\delta\gamma_{\mu\nu} = h_{\mu\nu}(x,0) + \frac{2\epsilon}{H}\mathcal{O}_{\mu\nu}(\frac{\hat{\phi}}{\alpha} + f) \,, \tag{3.62}$$

where $f$ is the gauge transformation of section 3.3, needed to keep the tensor perturbation transverse-traceless in the presence of matter perturbations. The second-order perturbation of the total action is

$$S = M_5^3 \int_{\mathcal{M}} d^5x \, \sqrt{-\bar{g}} \delta g^{ab} \delta G_{ab} + \frac{1}{2} \int_{\Sigma} d^4x \, \sqrt{-\bar{\gamma}} \delta\gamma^{\mu\nu} \left( \delta\Theta_{\mu\nu} - T_{\mu\nu} \right) \,, \tag{3.63}$$

and to get the $4D$ effective action we should integrate out the bulk, substituting the mode expansion for the radial coordinate $y$, while keeping all the $4D$, $x$-dependent modes off-shell. This means that in the explicit evaluation of the terms in the action (3.63) we do not require that $(D^2 + 4H^2)\hat{\phi} = 0$, or $(D^2 - m^2 - 2H^2)\chi^{(m)}_{\mu\nu} = 0$. In fact, once we have used our Ansatz for the perturbations (3.58), which respects the TT conditions, as a means for properly identifying the propagating degrees of freedom in the theory about the backgrounds (2.7), we can relax these conditions when working out the effective action by evaluating (3.63) on (3.58). The TT conditions for $\chi^{(m)}_{\mu\nu}$ will nevertheless still emerge from the $4D$ field equations obtained by varying the effective action, just like in massive $U(1)$ gauge theory (see also [68]). We stress that we could have used the gauge-fixed action from the start, enforcing TT constraints directly in the effective action. We won't do so for the sake of simplicity, because the results are completely equivalent at the classical level.

Using (3.61), (3.62) and the Bianchi identity $\bar{\nabla}^a \delta G_{ab} = 0$, we find that to the quadratic order in perturbations the action is

$$\begin{aligned} S = & -M_5^3 \int d^4x \sqrt{-\bar{\gamma}} \int_0^\infty dy \; a\sqrt{a} h^{\mu\nu} \delta G_{\mu\nu} - 2M_5^2 \int d^4x \sqrt{-\bar{\gamma}} \; \xi^a(x,0) \delta G_{ay}|_{y=0} \\ & -\frac{1}{2} \int d^4x \sqrt{-\bar{\gamma}} \left[ h^{\mu\nu}(x,0) + \frac{2\epsilon}{H}\mathcal{O}_{\mu\nu}(\frac{\hat{\phi}}{\alpha} + f) \right] \left( \delta\Theta_{\mu\nu} - T_{\mu\nu} \right) \,. \end{aligned} \tag{3.64}$$

From the metric (3.61) and Eqs. (3.13) - (3.15) it then follows that the variations of the Einstein tensor obey

$$a\sqrt{a}\delta G_{\mu\nu} = u_{m_d}(y)X^{(m_d)}_{\mu\nu} + \int_{\frac{3H}{2}}^{\infty} dm\ u_m(y)X^{(m)}_{\mu\nu}\,, \tag{3.65}$$

$$\begin{aligned}\delta G_{\mu y}|_{y=0} = & -\frac{M_4^2}{4M_5^3}\Big[m_d^2 u_{m_d}(0)D^\nu\left(h^{(m_d)}_{\mu\nu} - h^{(m_d)}\bar\gamma_{\mu\nu}\right) \\ & + \int_{\frac{3H}{2}}^{\infty} dm\ m^2 u_m(0)D^\nu\left(h^{(m)}_{\mu\nu} - h^{(m)}\bar\gamma_{\mu\nu}\right)\Big]\,,\end{aligned} \tag{3.66}$$

$$\begin{aligned}\delta G_{yy}|_{y=0} = & -\frac{1}{2}\left[D^\mu D^\nu - \bar\gamma^{\mu\nu}(D^2+3H^2)\right]\left[u_{m_d}(0)h^{(m_d)}_{\mu\nu} + \int_{\frac{3H}{2}}^{\infty} dm\ u_m(0)h^{(m)}_{\mu\nu}\right] \\ & -\frac{3\epsilon H}{2}\left(\frac{M_4^2}{2M_5^3}\right)\left[m_d^2 u_{m_d}(0)h^{(m_d)} + \int_{\frac{3H}{2}}^{\infty} dm\ m^2 u_m(0)h^{(m)}\right] \\ & -\frac{3}{4}H^2(1+\epsilon)(D^2+4H^2)\hat\phi\,.\end{aligned} \tag{3.67}$$

In these equations we have been using the tensorial operator $X^{(m)}_{\mu\nu}$, defined by

$$X^{(m)}_{\mu\nu} = X_{\mu\nu}(h^{(m)}) + \tfrac{1}{2}m^2\left(h^{(m)}_{\mu\nu} - h^{(m)}\bar\gamma_{\mu\nu}\right)\,. \tag{3.68}$$

where $X_{\mu\nu}(h^{(m)})$ is given by (3.16). We further use the formula for the variation of the brane stress-energy, which is

$$\begin{aligned}\delta\Theta_{\mu\nu} = & M_4^2\left[u_{m_d}(0)X^{(m_d)}_{\mu\nu} + \int_{\frac{3H}{2}}^{\infty} dm\ u_m(0)X^{(m)}_{\mu\nu}\right] \\ & +2\left(M_5^3 - M_4^2\epsilon H\right)\left[D_\mu D_\nu - \bar\gamma_{\mu\nu}(D^2+3H^2)\right]f\,.\end{aligned} \tag{3.69}$$

A useful identity which follows from Bianchi identities and stress-energy conservation $D^\mu X_{\mu\nu}(h) = D^\mu T_{\mu\nu} = 0$ is $D^\mu\left(\delta\Theta_{\mu\nu} - T_{\mu\nu}\right) = -2M_5^3\delta G_{\mu y}|_{y=0}$. Using it, the orthogonality of mode functions $u_m$, the projections

$$\begin{aligned}(m_d^2 - 2H^2)\Big\langle u_{m_d}\Big|\alpha\exp\left(-\frac{H}{2}y\right)\Big\rangle &= -4(M_5^3 - M_4^2\epsilon H)u_{m_d}(0)\,, \\ (m^2 - 2H^2)\Big\langle u_m\Big|\alpha\exp\left(-\frac{H}{2}y\right)\Big\rangle &= -4(M_5^3 - M_4^2\epsilon H)u_m(0)\,,\end{aligned} \tag{3.70}$$

and the defining equation (3.49) of the gauge parameter $f$, after a straightforward albeit tedious calculation we finally determine the $4D$ effective action, $S_{\rm eff} = \int d^4x\sqrt{-\bar\gamma}\ \mathcal{L}_{\rm eff}$, where the Lagrangian density is

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & -\tfrac{1}{2}h^{(m_d)\mu\nu}X^{(m_d)}_{\mu\nu} + \tfrac{1}{2}u_{m_d}(0)h^{(m_d)\mu\nu}\tau_{\mu\nu} \\ & + \int_{\frac{3H}{2}}^{\infty} dm \left[-\tfrac{1}{2}\, h^{(m)\mu\nu}X^{(m)}_{\mu\nu} + \tfrac{1}{2}u_m(0)h^{(m)\mu\nu}\tau_{\mu\nu}\right] \\ & - \frac{3(1+\epsilon)}{2}M_5^3 H^2\, \frac{\hat{\phi}}{\alpha}(D^2+4H^2)\hat{\phi}\,, \end{aligned} \tag{3.71}$$

and where we use the gauge invariant brane stress-energy perturbation

$$\tau_{\mu\nu} = T_{\mu\nu} - 4\left(M_5^3 - M_4^2\epsilon H\right)\left[D_\mu D_\nu - \bar{\gamma}_{\mu\nu}(D^2+3H^2)\right] f\,. \tag{3.72}$$

Varying this action reproduces the correct field equations for $h^{(m)}_{\mu\nu}$ and $\hat{\phi}$. The scalar field $\hat{\phi}$ does not have direct matter couplings at the level of quadratic action, and in fact drops out altogether on the normal branch ($\epsilon = -1$), reflecting the fact that the normalizable scalar mode in this case is pure gauge. On the self-accelerating branch ($\epsilon = +1$), this mode is gauge-invariant, and since it sees the metric it should couple at least to gravity at higher orders in perturbative expansion. This mode is itself a ghost when $\alpha > 0$, which occurs when brane tension is negative, as can be readily seen by using (2.10). The tensors, $h^{(m)}_{\mu\nu}$, are described mode-by-mode by the standard Pauli-Fierz Lagrangian for massive gravity. They couple to matter with the coupling strength given by the bulk wave function overlap with the brane $u_m(0)$. For the continuum modes, this coupling is of the order of $M_5^{-\frac{3}{2}}$. For the discrete mode, it is

$$u_{m_d}(0) = \frac{1}{M_4}\left[\frac{3M_4^2H - 2M_5^3(1+\epsilon)}{3M_4^2H - 2M_5^3\epsilon}\right]^{\frac{1}{2}} \tag{3.73}$$

On the normal branch, this coupling vanishes as $\sigma \to 0$. This is simply the consequence that the normalizable zero mode on the normal branch decouples in the limit of infinite bulk volume, as is well known [19, 27].

In contrast, on the self accelerating branch, the coupling does not vanish, but remains of the order of $1/M_4$. As we have already discussed above, for all positive values of the tension, the helicity-0 component of this mode is a ghost, which therefore remains coupled to matter on the brane with the gravitational strength. It does not decouple even when in the vanishing tension limit where the accidental symmetry of [43] for the tensor of mass $m^2 = 2H^2$ appears, because the symmetry is now realized as a Stückelberg symmetry which mixes the lightest tensor and the normalizable scalar mode $\phi$ because they are degenerate. This has also been discussed in the recent work [68]. Thus the dynamical degrees of freedom of the tensionless self-accelerating solution are given by the combination (3.45), which we repeat here for completeness,

$$h_{\mu\nu} = e^{-\frac{H}{2}y}\Big(\mathcal{H}_{\mu\nu}(x) - y\,\mathcal{O}_{\mu\nu}F\Big) + \int_{\frac{3H}{2}}^{\infty} dm\; u_m(y)\chi^{(m)}_{\mu\nu}(x) + \frac{2}{H}e^{Hy/2}\mathcal{O}_{\mu\nu}F\,,$$

where

$$\mathcal{H}_{\mu\nu}(x) = \lim_{\sigma\to 0}\Big(\alpha_{m_d}\chi^{(m_d)}_{\mu\nu}(x) + \alpha\,\mathcal{O}_{\mu\nu}F\Big)\,.$$

The $4D$ tensor $\mathcal{H}_{\mu\nu}$ obeys $(D^2 - 4H^2)\mathcal{H}_{\mu\nu} = -H\mathcal{O}_{\mu\nu}F$. The mixing of the lightest tensor with the brane Goldstone $F$ in effect really just promotes the particular, non-dynamical, gauge function $f$ that enforces the `TT` gauge conditions into a full-fledged dynamical mode $\xi = F + f$ which mixes with the field $\mathcal{H}_{\mu\nu}$. Indeed, in this case we can rewrite the boundary condition for $\mathcal{H}_{\mu\nu}(x)$ by subtracting the degenerate eigenmode $\mathcal{O}_{\mu\nu}F$ to (3.48), which will modify the stress-energy source for this mode to

$$\tau^{(2H^2)}_{\mu\nu} = T_{\mu\nu} + 4M_5^3\left[D_\mu D_\nu - \bar{\gamma}_{\mu\nu}(D^2 + 3H^2)\right]\xi\,, \tag{3.74}$$

where we have used that in the limit of vanishing tension on the self-accelerating branch, $H \to 2M_5^3/M_4^2$ and $m_d^2 \to 2H^2$. Substituting this in the action, and renormalizing $\mathcal{H}_{\mu\nu} \to \frac{1}{\sqrt{2}M_4}\mathcal{H}_{\mu\nu}$ we finally find the zero tension effective lagrangian

$$\begin{aligned}\mathcal{L}_{\rm eff} = &-\tfrac{1}{2}\mathcal{H}^{\mu\nu}X_{\mu\nu}(\mathcal{H}) - \tfrac{1}{2}H^2\left(\mathcal{H}^{\mu\nu}\mathcal{H}_{\mu\nu} - \mathcal{H}^2\right) + \tfrac{1}{2\sqrt{2}M_4}\mathcal{H}^{\mu\nu}\tau^{(2H^2)}_{\mu\nu}\\ &+ \int_{\frac{3H}{2}}^{\infty} dm\left[-\tfrac{1}{2}\,h^{(m)\mu\nu}X^{(m)}_{\mu\nu} + \tfrac{1}{2}u_m(0)h^{(m)\mu\nu}\tau_{\mu\nu}\right]\,,\end{aligned} \tag{3.75}$$

where $\tau_{\mu\nu}$ is given by (3.72), and $\tau^{(2H^2)}_{\mu\nu}$ by (3.74). In this case the scalar $\xi$ does not have a normal kinetic term, and only enters the action through mixing with the discrete mode $\mathcal{H}_{\mu\nu}$. This occurs because $1/\alpha$ vanishes as $\sigma \to 0$, as per Eq. (3.36). Yet this is sufficient to ensure that the ghost survives the vanishing tension limit from the point of view of the effective action.

### 3.5 Perturbations that break $\mathbb{Z}_2$ symmetry

So far we have been mostly concerned with $\mathbb{Z}_2$ symmetric perturbations about the $\mathbb{Z}_2$ symmetric background (2.7), (2.9), following the conventional analysis of the stability of DGP backgrounds. However, even with a $\mathbb{Z}_2$ symmetric background, if we regard the brane as a domain wall rather than an orbifold, which may be well-motivated for DGP branes, there is no reason why perturbations about that background should respect the $\mathbb{Z}_2$ symmetry. Indeed, given the notion that induced curvature is a finite width correction for domain walls, one may argue that non-$\mathbb{Z}_2$ perturbations are in principle just as important as their $\mathbb{Z}_2$ symmetric counterparts.

To extend the perturbations to non-$\mathbb{Z}_2$ symmetric configurations, we take the warp factor to still be

$$a(y) = \exp(\epsilon H|y|)\,, \tag{3.76}$$

but imagine that the bulk spacetime describes two separate half-intervals, parameterized explicitly by $-\infty < y < 0$ and $0 < y < \infty$, with the brane positioned at $y = 0$. We now consider non-$\mathbb{Z}_2$ symmetric perturbations about this background for the case where $T_{\mu\nu} = 0$. These perturbations must satisfy the bulk equations of motion (2.2) and the non-$\mathbb{Z}_2$ symmetric Israel equations (2.3) to linear order. In addition we demand continuity of the metric across the brane. This comes for free by construction for $\mathbb{Z}_2$ symmetric perturbations, but not otherwise. If we choose a GN gauge ($\delta g_{yy} = \delta g_{\mu y} = 0$) with brane fixed at $y = 0$, we note that the following perturbation satisfies (2.2) and (2.3), to linear order, and is continuous at $y = 0$,

$$\delta g_{\mu\nu}(x, y) = \frac{2(1+\epsilon)}{H}\sqrt{a(y)}\sinh\left(\frac{Hy}{2}\right)\mathcal{O}_{\mu\nu}\tilde{\phi}(x) + \sqrt{a(y)}\int_{\frac{3H}{2}}^{\infty} dm\ \sin(\omega_m y)\chi^{(m)}_{\mu\nu}(x) \tag{3.77}$$

where $\chi^{(m)}_{\mu\nu}$ is transverse-traceless, and

$$(D^2 - 2H^2)\chi^{(m)}_{\mu\nu} = m^2\chi^{(m)}_{\mu\nu}, \qquad (D^2 + 4H^2)\tilde{\phi} = 0 \tag{3.78}$$

This solution is clearly non-$\mathbb{Z}_2$ symmetric since $\delta g_{ab}(x, -y) = -\delta g_{ab}(x, y)$. Note that the tachyonic scalar, $\tilde{\phi}$, is not present for the normal branch ($\epsilon = -1$), but is present on the self-accelerating branch ($\epsilon = +1$). It represents yet another instability for the self accelerating background. Indeed, for $\epsilon = +1$, $\tilde{\phi}$ corresponds to a crinkling up of the brane. We can see this by transforming to bulk-GN gauge, which is the appropriate gauge for an observer at infinity. To transform to this gauge, we take

$$x^\mu \to x^\mu + \frac{D^\mu\tilde{\phi}}{Ha}\mathrm{sgn}(y), \qquad y \to y + \frac{\tilde{\phi}}{a} \tag{3.79}$$

The bulk metric is now given by $\delta g_{yy} = \delta g_{\mu y} = 0$, and

$$\delta g_{\mu\nu}(x, y) = -\frac{2}{H}\mathrm{sgn}(y)\mathcal{O}_{\mu\nu}\tilde{\phi}(x) + \sqrt{a(y)}\int_{\frac{3H}{2}}^{\infty} dm\ \sin(\omega_m y)\chi^{(m)}_{\mu\nu}(x) \tag{3.80}$$

Since the brane resides $y = \tilde{\phi}(x)$ we can immediately interpret the tachyon $\tilde{\phi}$ as a crinkling up of the brane. However, we stress that this instability is phenomenologically very mild when compared to the ghost, since it is controlled by the time scale given by the inverse mass of $\tilde{\phi}$, $\tau_{instability} \sim H^{-1}$.

## 4. Shocking $4D$ nonlocalities

Up until now we have only considered perturbations of the backgrounds (2.7) which are normalizable in the bulk. They admit to an effective $4D$ description mode-by-mode, insofar as one is interested in computing their couplings and propagators as measured by brane-localized processes. This does not imply that the full picture is $4D$ over all the relevant scales in the infra-red. The additional helicities of massive gravitons spoil the $4D$ description, albeit at very long distances $\propto r_c$, by altering momentum transfer at very low momenta. As obscure as this may seem in the $4D$ effective action, it becomes transparent in the shock wave analysis of [33]. On the other hand, once one restricts only to the normalizable modes localized to the brane, as we have seen above one inevitably encounters the lightest graviton on the self-accelerating branch, with mass in the unitarity violating window of [44, 45, 46], and so with a helicity-0 ghost. This signals an instability which renders the perturbation theory within the effective $4D$ description essentially meaningless. Indeed one does not know how to define the ground state of the theory on top of which to perturb, and has no clear description of the evolutionary end points to which the perturbative ghost may lead. Thus before trusting $4D$ perturbative description one must find ways to neutralize the ghost.

Since the ghost comes on board with the localized massive graviton, one might try changing perturbative definition of the theory, for example by changing prescriptions for boundary conditions at infinity, to avoid this mode[8]. This possibility seems natural since after all DGP is really a higher-dimensional theory, disguising as $4D$ at best over a finite range of scales. Its ghost arises only *after* one 'reduces' the theory on the bulk-brane background and restricts to the normalizable bulk modes, and so the ghost may represent merely an intrinsic instability of this reduction. A brane laden with matter may want to move around the bulk in ways which require reintroduction of genuine bulk modes, that are not normalizable and hence remain completely outside of the scope of the usual $4D$ effective action analysis. This is supported by the observation of [33] that the singular behavior of a shock wave sourced by a photon on the tensionless brane may be smoothed by reintroducing a genuine bulk mode, which then resonates with the brane. A brane carrying a photon pulse behaves like an antenna, emitting bulk gravitons.

However in this approach, energy will leak from the brane into the bulk as time goes on, revealing the fifth dimension. This leakage may eventually strongly deform the bulk far away. Alternatively, one could imagine a time-reversal of this process, where the description of a particle moving on the brane requires an incoming wave in the bulk, with the phase precisely tuned to cancel the ghost divergence, pouring

[8]We thank G. Gabadadze for useful discussions about this approach.

energy into the brane. One might try cutting the bulk infinity out of the picture, seeking boundary conditions that ought to keep the brane stable and self-accelerating. It however remains difficult to imagine how this could ever insulate the brane physics from distant bulk in the full nonlinear covariant theory, and simultaneously retain the guise of a local, causal, $4D$ description. Every time something happens on the brane, one would think that one needs to change the bulk far away, which either requires unacceptable external interference, appearing as nonlocalities, or the cross-bulk causal transfer of signals that would violate $4D$ description. Furthermore, while individual non-normalizable modes may be treated separately in the linearized theory, they will in general couple to each other at higher orders because they will have non-vanishing overlaps when nonlinearities are included. Thus once any one non-normalizable mode at a fixed $4D$ mass level is brought back, it should pull alongside it modes at other mass levels. These modes may introduce new dangers.

A full bulk perturbation theory including all non-normalizable modes is beyond the scope of our work, requiring first setting up the precise perturbative formulation of the problem, defining the set of new orthogonal modes et cetera. However to shed some light on the problem we can completely circumvent all those issues by going directly to a special limit where we can solve field equations *exactly*. We shall solve exactly the field equations describing the gravitational field of a photon moving on the brane, including non-normalizable modes. Such methods have been used in the context of braneworlds in [33, 69]. The result of this calculation shows that this time the exchange of the new, lightest non-normalizable tensor modes also generates *repulsive* contributions to the potential. This points that light non-normalizable tensor modes behave like $4D$ ghosts, since their contribution to the potential is precisely the off-shell scattering amplitude for the single-particle exchange between the brane source and a probe, which in the $4D$ language would be the propagator, that would need to have its sign flipped to account for repulsive force.

To see this, let us revisit the calculation of [33], describing the shocked background geometry (2.7) with the metric

$$ds_5^2 = e^{2\epsilon H|y|}\Big\{\frac{4dudv}{(1+H^2uv)^2} - \frac{4\delta(u)\Phi du^2}{(1+H^2uv)^2} + (\frac{1-H^2uv}{1+H^2uv})^2\,\frac{d\Omega_2}{H^2} + dy^2\Big\}\,. \tag{4.1}$$

The induced metric on the brane is

$$ds_4^2 = \frac{4dudv}{(1+H^2uv)^2} - \frac{4\delta(u)\Phi du^2}{(1+H^2uv)^2} + (\frac{1-H^2uv}{1+H^2uv})^2\,\frac{d\Omega_2}{H^2}\,. \tag{4.2}$$

These metrics are obtained from the static patch form of (2.7) written in terms of the null coordinates $u = \frac{1}{H}\sqrt{\frac{1-Hr}{1+Hr}}\exp(Ht)$ and $v = \frac{1}{H}\sqrt{\frac{1-Hr}{1+Hr}}\exp(-Ht)$. To determine the field of a photon in de Sitter geometry, for technical reasons it is simplest to actually consider the case of two photons with the same momentum $p$ which run in the opposite direction in the static patch [70, 71, 72]. So as in [33] we add two antipodal photons on the brane, moving along the geodesics $u = 0$, $\theta = 0$, and $u = 0, \theta = \pi$, by introducing metric discontinuities along the photon worldlines by substituting $dv \to dv - \delta(u)\Phi du$. This yields (4.1), (4.2). To return to a single source we can multiply this solution by the step function $\Theta(\pi/2 - \theta)$ as in [72]. The photon stress-energy tensor is

$$T^{\mu}{}_{\nu} = -\sigma\delta^{\mu}{}_{\nu} + 2\frac{p}{\sqrt{g_5}} g_{4\,uv}\Big(\delta(\theta) + \delta(\theta - \pi)\Big)\delta(\phi)\delta(u)\delta^{v}_{\mu}\delta^{u}_{\nu}\,, \tag{4.3}$$

where we use the notation $g_{4\,uv}$ for the metric on the brane in Eq. (4.2). A straightforward calculation [33] then yields the wave profile equation

$$\frac{M_5^3}{M_4^2 H^2}\Big(\partial_y^2\Phi + 3\epsilon H\partial_{|y|}\Phi + H^2(\Delta_2\Phi + 2\Phi)\Big) + (\Delta_2\Phi + 2\Phi)\delta(y) = \frac{2p}{M_4^2}\Big(\delta(\Omega) + \delta(\Omega')\Big)\delta(y)\,, \tag{4.4}$$

where we use the shorthand $\delta(\Omega) = \delta(\cos\theta - 1)\delta(\phi)$ and $\delta(\Omega') = \delta(\cos\theta + 1)\delta(\phi)$. The operator $\Delta_2$ is the Laplacian on the transverse 2-sphere on the brane. Using the spherical symmetry of the brane geometry transverse to the photon directions, the addition theorem for spherical harmonics and linearity of (4.4), we can decompose the solution as

$$\Phi = \sum_{l=0}^{\infty}\Big(\Phi_l^{(+)}(y)P_l(\cos\theta) + \Phi_l^{(-)}(y)P_l(-\cos\theta)\Big)\,. \tag{4.5}$$

Here $\Phi_l^{(\pm)}(z)$ are the bulk wave functions; $\Phi_l^{(+)}$ is sourced by the photon at $\theta = 0$ and $\Phi_l^{(-)}$ by the photon at $\theta = \pi$. By orthogonality and completeness of Legendre polynomials, the field equation (4.4) yields an identical differential equation for both modes $\Phi_l^{(\pm)}(z)$:

$$\partial_y^2\Phi_l + 3\epsilon H\partial_{|y|}\Phi_l + H^2(2 - l(l+1))\Phi_l = \frac{M_4^2 H^2}{M_5^3}\Big(\frac{(2l+1)p}{2\pi M_4^2} - (2 - l(l+1))\Phi_l\Big)\delta(y)\,. \tag{4.6}$$

Using pillbox integration and recalling $\mathbb{Z}_2$ symmetry which imposes $\Phi_l(-y) = \Phi_l(y)$ we finally determine the boundary value problem for $\Phi_l$:

$$\begin{aligned} &\partial_y^2\Phi_l + 3\epsilon H\partial_y\Phi_l + H^2(2 - l(l+1))\Phi_l = 0\,, \\ &\Phi_l(-y) = \Phi_l(y)\,, \\ &\Phi_l'(0) + \frac{2 - l(l+1)}{\mathsf{g}}H\Phi_l(0) = \frac{H}{\mathsf{g}}\frac{2l+1}{2\pi M_4^2}\,p\,, \end{aligned} \tag{4.7}$$

where $\mathsf{g} = 2M_5^3/(M_4^2 H) = 1/(Hr_c)$. Hence $\Phi_l^{(+)} = \Phi_l^{(-)} = \Phi_l$ since both satisfy the same boundary value problem. Further because $P_l(-x) = (-1)^l P_l(x)$, in the expansion (4.5) only even-indexed terms survive. This yields

$$\Phi = 2\sum_{l=0}^{\infty} \Phi_{2l}(y) P_{2l}(\cos\theta)\,, \tag{4.8}$$

circumventing the unphysical 4D singularities of $l = 1$ terms [70, 72]. Solving the differential equation in (4.7) we see that the modes are of the form $\chi \sim e^{[\pm 2l - (\mp 1 + 3\epsilon)/2]\, H|y|}$. In [33] only normalizable mode, for which $||\Phi||^2 \propto \int dy\, e^{3H|y|}\, |\Phi|^2$ was finite, were kept. Here we want to see what happens when the non-normalizable modes are retained instead, and so we use the general bulk wave function

$$\Phi_{2l} = A_{2l} e^{-(2l + \frac{3\epsilon+1}{2}) H|y|} + B_{2l} e^{(2l - \frac{3\epsilon-1}{2})\, H|y|}\,, \tag{4.9}$$

where $A_{2l}$-mode is normalizable and $B_{2l}$-mode is not. Substituting this into (4.7), (4.8), and introducing the parameter $\alpha_{2l}$ by $B_{2l} = -\frac{p}{4\pi M_4^2} \frac{4l+1}{(2l-1-\frac{1-\epsilon}{2}\mathsf{g})(l+1-\frac{1+\epsilon}{4}\mathsf{g})}\alpha_{2l}$, because it makes the representation (4.10) particularly transparent, after simple algebra we obtain

$$\begin{aligned}\Phi(\Omega, y) = -\frac{p}{2\pi M_4^2} \sum_{l=0}^{\infty} (4l+1) P_{2l}(\cos\theta) \Big( &\frac{1-\alpha_{2l}}{(2l - 1 + \frac{1+\epsilon}{2}\mathsf{g})(l + 1 + \frac{1-\epsilon}{4}\mathsf{g})}\, e^{-(2l+2)\, H|y|} \\ &+ \frac{\alpha_{2l}}{(2l - 1 - \frac{1-\epsilon}{2}\mathsf{g})(l + 1 - \frac{1+\epsilon}{4}\mathsf{g})}\, e^{(2l-1)\, H|y|} \Big)\,.\end{aligned} \tag{4.10}$$

The parameters $\alpha_{2l}$ are selected by the boundary conditions at the bulk infinity, and, in the language of $4D$ theory, they correspond to the choice of the vacuum, since their specification picks out a specific linear combination of the solutions to represent a particle state with a given mass and 4-momentum. Clearly, $\alpha_{2l} = 0$ corresponds to keeping only the normalizable modes in the description, and (4.10) reduces to the shock wave solution of [33], whereas $\alpha_{2l} = 1$ selects only the non-normalizable modes, throwing out the normalizable ones.

Now, how should we read the solution (4.10)? First notice that using the spherical harmonics addition theorem, $\frac{2n+1}{4\pi} P_n(\cos\theta) = \sum_{m=-n}^{n} Y^*_{nm}(0,0)\, Y_{nm}(\theta,\phi)$, and setting $n = 2l$, we can rewrite (4.10) as

$$\begin{aligned}\Phi(\Omega, y) = -\frac{2p}{M_4^2} \sum_{l=0}^{\infty} \sum_{m=-2l}^{2l} & Y^*_{2l\,m}(0,0)\, Y_{2l\,m}(\theta,\phi) \times \\ &\times \Big( \frac{1-\alpha_{2l}}{(2l - 1 + \frac{1+\epsilon}{2}\mathsf{g})(l + 1 + \frac{1-\epsilon}{4}\mathsf{g})}\, e^{-(2l+2)\, H|y|} \\ &+ \frac{\alpha_{2l}}{(2l - 1 - \frac{1-\epsilon}{2}\mathsf{g})(l + 1 - \frac{1+\epsilon}{4}\mathsf{g})}\, e^{(2l-1)\, H|y|} \Big)\,.\end{aligned} \tag{4.11}$$

This is the Green's function of the problem (4.4), describing the gravitational field of a 'particle' of effective mass $p$ in the space transverse to the photon's $u, v$ propagation plane. Now, we recall that in the conventional approach, a tree-level perturbative potential generated by an exchange of a mediating boson is the Fourier transform of the scattering amplitude involving the boson propagator. The formula (4.11) is precisely the same: since the transverse space to the shock is a $S^2_{brane} \times R_{bulk}$, the terms of the expansion (4.11) can be interpreted as the off-shell tree level amplitudes involving the exchange of the discretized Fourier modes on the sphere, described by the spherical harmonics, and weighed by the bulk radial functions which account for the dilution of intermediary's wave function due to the warping of the bulk, when the intermediary slides just outside of the brane. The 'quantum numbers' $l$ measure the Euclidean momentum of the intermediary modes on the $S^2_{brane}$, $q \sim Hl$, and control the momentum transfer in a scattering process between the photon and a distant test particle mediated by a virtual intermediary. The bulk $y$-dependence scales the coupling up or down depending on the location of the probe versus the brane.

Now, to see how the shock wave looks on the self-accelerating brane, we can set $\epsilon = 1$ and $y = 0$ in Eq. (4.10), which yields

$$\Phi = -\frac{2p}{M_4^2} \sum_{l=0}^{\infty} \sum_{m=-2l}^{2l} \Big( \frac{1-\alpha_{2l}}{(2l-1+\mathsf{g})(l+1)} + \frac{\alpha_{2l}}{(2l-1)(l+1-\frac{\mathsf{g}}{2})} \Big) Y^*_{2l\,m}(0,0)\, Y_{2l\,m}(\theta,\phi)\,. \tag{4.12}$$

These formulae are very revealing. First, let us suppose that $0 \le \alpha_{2l} \le 1$. It is clear from the $y$-dependence of (4.10) that general solutions with $\alpha_{2l} \neq 0$ peak far from the brane, indicating that they are very sensitive to the perturbations near the bulk infinity. However, from (4.12) we see that the local physics on the brane is not that sensitive to distant bulk, since for large momentum transfer $l \gg \mathsf{g}$, i.e. at short distances between the source and the probe, $H\mathcal{R} \simeq \sqrt{2(1-\cos\theta)} \ll 1$, the $\alpha_{2l}$ terms which encode bulk boundary conditions completely cancel in the leading order in (4.10), (4.11), (4.12): $\Big( \frac{1-\alpha_{2l}}{(2l-1+\mathsf{g})(l+1)} + \frac{\alpha_{2l}}{(2l-1)(l+1-\frac{\mathsf{g}}{2})} \Big) = \frac{1}{2l} + \mathcal{O}(\frac{\mathsf{g}}{l})$. Thus to the leading order (4.12) behaves the same at short distances as the wave profile with $\alpha_{2l} = 0$, composed only of normalizable modes. Its short distance form will be very well approximated by the Aichelburg-Sexl wave profile, in exactly the same way as the purely normalizable wave profile [33].

However, at very low momentum transfer, or at very large distances, a general solution (4.10) with non-normalizable modes differs very dramatically from the one built purely out of the normalizable modes. The point is that the coefficients of the expansion of the wave profile $\propto \alpha_{2l}$ are *not* positive definite when viewed as a function of $l$. Indeed, when $\mathsf{g} > 2$, all the coefficients in the second term change sign for all $l < \mathsf{g}/2 - 1$ except for $l = 0$, which remains positive because of the $1/(2l-1)$

factor. Now when $\mathtt{g} < 2$, terms with $l \geq 1$ remain positive, but the $l = 0$ term turns negative. Hence these modes come in with *opposite* signs relative to the contributions from their normalizable partners $\propto 1 - \alpha_{2l}$, and because their dependence on the transverse distance from the source is the same as for the normalizable modes, this means that they yield repulsive contributions to the gravitational potential at large distances. Given our interpretation of the contributions of the terms in the expansion of the wave profile (4.11) as the discretized propagator of the exchanged virtual graviton, the repulsive contributions to the potential at very large distances signal a ghost-like behavior among the lightest states in the non-normalizable tensor sector. Note further that when $\mathtt{g}$ is an integer, there are pole-like singularities in the sums (4.10)-(4.12). In the normalizable sector, the only pole in fact occurs when $\mathtt{g} = 1$, that corresponds to the limit of vanishing tension and as discussed in [33] which can be interpreted as the instability that leads to rapid release of energy into the bulk, related to the lightest tensor ghost. The non-normalizable contributions however have poles at all even integer values of $\mathtt{g} > 0$, that indicate that if energy were to be inserted into them at infinity, it would be quickly transferred into the normalizable modes, which could produce large gravitational effects on the brane.

This shows that resurrecting non-normalizable modes, if viewed as $4D$ phenomena, may open the door to new ghost-like modes, over and above the helicity-0/scalar ghost. In this case one further needs to recheck carefully the scalar sector for any pathologies among the non-normalizable scalar modes, which cannot be revealed by the shock wave analysis as they are not sourced by relativistic particles. Note that taking $\alpha_{2l}$ outside of the interval $[0, 1]$ will not remove the repulsive contributions to the shock wave, but would further exacerbate the problem. For $\alpha_{2l} > 1$, all the normalizable mode contributions would switch sign, whereas for $\alpha_{2l} < 0$, all but the lightest non-normalizable modes would be repulsive. Thus it appears that the repulsive terms in the non-normalizable tensor sector will be avoided only if we set $\alpha_{2l} = 0$, but then this retains only the normalizable modes in the helicity-0 sector as well, leaving one with the helicity-0 ghost.

Interestingly, one can check explicitly using (4.10) that the situation on the normal branch is better, in the sense that the relative sign between the normalizable and non-normalizable contributions remains the same for all the terms in the expansion. In fact, on the normal branch, the only term which contributes to the wave profile with a negative sign is the $l = 0$ mode, but this mode is completely constant on the brane and produces no force on brane particles. It will only repel bulk probes, but this is consistent with the picture that domain walls in flat space, when viewed from the wall's rest frame, exert repulsive force on particles in the bulk [55].

## 5. Summary

In this work we have reconsidered the perturbative description of codimension-1 DGP vacua. Our results confirm that in the normalizable sector of modes, there is always a perturbative ghost on the self-accelerating branch. For positive brane tension, it resides in the localized lightest graviton multiplet as the helicity-0 state, whereas for the negative tension it is the scalar 'radion'-like field. In the borderline case of zero tension, describing a background where the brane expansion accelerates solely due to gravity modification, the ghost is an admixture of the helicity-0 and scalar modes, which become completely degenerate. In contrast, the normal branch vacua are free of ghosts, at least when the brane tension is non-negative.

The ghost makes a simple $4D$ perturbative analysis of the self-accelerating dynamics prohibitive. It signals an instability which renders the effective $4D$ description perturbatively meaningless. Off hand, one does not know how to define the ground state of the theory, and has no clear description of the evolutionary end points to which the perturbative ghost may lead. To illustrate the dangers from ghosts, let us review a very simple, intuitive example of the classical ghost instability which, to our surprise, does not seem to be widely discussed. Consider a system of two degenerate harmonic oscillators $x$ and $y$. One could solve it by solving each individual oscillator problem and then define the full set of states as the direct product of the individual oscillators. However, these states do not naturally reflect the $O(2)$ rotational symmetry of the system, which can be made manifest by going to polar coordinates $x = \mathcal{R}\cos\theta$, $y = \mathcal{R}\sin\theta$. Then the angle $\theta$ is a Goldstone mode, whose conserved charge is the angular momentum of the system, while the radial variable is an anharmonic oscillator moving in an effective potential composed of the original parabolic piece far away and the centrifugal barrier erected by the angular momentum close in. This system is classically stable. Now imagine making one of the original harmonic oscillators purely imaginary, e.g. $y \to iy$. This changes the symmetry group to $O(1,1)$, the 'polar' coordinate maps become hyperbolic functions, and the phase is now the ghost, whereas the radial mode remains a normal field. The ghost's centrifugal contribution to the radial motion is now an *infinite well*, instead of a barrier. Thus even a tiniest perturbation with non-zero angular momentum will send the oscillator spiralling into the infinitely deep centrifugal well, spinning it up indefinitely as it goes in. In terms of the original Cartesian variables, the two oscillators gain infinite kinetic energy at the expense of each other, while keeping their difference constant. When the physical oscillator $x$ couples to other normal degrees of freedom, it can transfer its kinetic energy to them, destabilizing the rest of the world. The rate of the instability is controlled by the oscillator period, and in a field theory where one has a tower of ghastly oscillators with arbitrarily high frequencies, the energy transfer rates can therefore be very fast. Interestingly, this situation is reminiscent[9] of what one encounters on the self-accelerating branch of DGP, where among the degenerate lightest gravitons there is a ghost.

We should stress that we do not claim that it is *ultimately* impossible to get rid of the ghost. The question is, what is the price one must pay to exorcize it. We have considered what happens when one restores non-normalizable modes in the bulk, and allows them to couple to brane matter. This may be an interesting arena to explore if by abandoning description in terms of normalizable modes alone, some of the effects of the helicity-0 ghost may be controlled. If one reinstates the bulk scalar modes that could couple to the helicity-0 ghost, however, by bulk general covariance one should also bring in the tensors. All of these modes dwell outside of the realm of $4D$ effective theory at all scales. Even so, our direct calculation of their couplings to relativistic brane matter, modelled by a photon zipping along the brane, shows that at short distances from it, the photon's gravitational field follows closely the $4D$ form, being indistinguishable from the purely normalizable contributions in the leading order of the expansion in $\mathsf{g} = 1/Hr_c$. However, we have noted new repulsive gravitational effects at large distances from the source, arising from the non-normalizable tensor sector, which indicate that low momentum tensors behave as ghosts. This issue clearly deserves more attention, and it would be interesting to develop a description of the processes beyond quadratic perturbation theory to see how the system evolves.

We note in passing that another possibility may be to consider altering the theory at the level of the action itself, for example by adding extra terms in the bulk or on the brane, and compactifying the bulk by adding new branes[10]. In the former case adding the Gauss-Bonnet term in the bulk seems an intriguing possibility since the resulting field equations are of second order and so they maintain the simple distributional brane setup without adding new, possibly dangerous, graviton states of generic higher-derivative models. Cutting the bulk at a finite distance will reintroduce the zero mode graviton, and may change the boundary conditions for the massive modes, affecting the values of masses. If this could lift the localized lightest tensor out of the unitarity-violating window, and/or break degeneracies with scalar modes, the ghost instability may be tamed. One then expects to be left with a strongly coupled massive graviton on the original DGP brane, which could dominate over the zero mode in a range of scales, and so one may consider phenomenological aspects of such a multi-gravity theory. However compactifying the bulk would turn DGP immediately into an effective

[9] The differences are the presence of background de Sitter geometry and a different isospin group ($O(2,1)$ instead of $O(1,1)$) but much of the rest appears the same, at least the linearized perturbation theory level.

[10] We understand that K. Izumi and T. Tanaka are pursuing such approaches.

$4D$ theory at large scales, exposing it to the edge of the Weinberg's venerable no-go theorem [4] for the adjustment of cosmological constant. Thus while removing the ghost by a compactification of the bulk might work, it may automatically restore the usual fine-tunings of the $4D$ vacuum energy, completely obscuring the whole point of self-acceleration.

Thus it is doubtful that self-acceleration in its present guise may serve as a model of the current epoch of cosmic acceleration, since after all it does appear that some perturbative description of our universe at the largest scales should exist. The self-accelerating branch does not seem to fit this bill due to its occult sector. Yet, if one wants to study the implications of modified gravity, one may still find a useful framework among the brane-induced gravity models. The simplest one may be the normal branch solutions. True, they undergo cosmic acceleration at the right rate because one fine-tunes the brane tension to just the right value by hand, $\sigma \simeq (10^{-3}eV)^4$. But there is no ghost, and perturbative description is reliable. If one then also tunes the scale of modification of gravity, $r_c \sim 1/H$, one gets interesting signatures of weakened gravity at the largest scales. Namely although there is a zero mode graviton on the normal branch, the light bulk modes also contribute to gravity at scales smaller than $r_c \sim 1/H$ and the effective graviton that mediates sub-horizon interactions is really a resonance composed of many modes. Thus local gravity is stronger than the horizon scale gravity. At the horizon scale, the force weakens because the effective momentum transfer of the massive admixtures changes from $1/q^2$ to $1/q$ as the extra dimension opens up. This weakening of gravity may simultaneously change the cosmic large scale structure [34] and masquerade the cosmological constant as dark energy with $w < -1$ [73], in a way that could be accessible to observations.

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## 6. Proof of tensor decomposition in Eq. (3.8)

To prove (3.8) we first introduce auxiliary fields $\Sigma_{\mu\nu} = h_{\mu\nu} - \frac{h}{4}\bar{\gamma}_{\mu\nu}$, $B_\mu$ and $B$, where we require that $B_\mu$ and $B$ are solutions of differential equations

$$
\begin{aligned}
&(D^2 + 4H^2)B = \frac{2}{3} D^\mu D^\nu \Sigma_{\mu\nu} \,, \\
&D^2 B_\mu + D^\nu D_\mu B_\nu - \frac{1}{2} D_\mu B = D^\nu \Sigma_{\mu\nu} \,.
\end{aligned}
\tag{6.1}
$$

These differential equations are integrable, despite the cumbersome nature of the vector field equation that appears to mix different components. We can simplify the system by introducing an additional scalar auxiliary field, as follows. First, commute through the derivatives acting on the vector, using the standard rules for commutators of covariant derivatives in de Sitter metric $\bar{\gamma}_{\mu\nu}$ and rewrite the vector equation as

$$
(D^2 + 3H^2)B_\mu = D^\nu \Sigma_{\mu\nu} - D_\mu (D_\nu B^\nu - \frac{1}{2} B) \,. \tag{6.2}
$$

Next multiply this equation by $D_\mu$, commute the derivatives again, and using the defining equation for $B$ to eliminate $D^\mu D^\nu \Sigma_{\mu\nu}$, finally obtain the equation for $\Psi = B - D_\mu B^\mu$ :

$$
(D^2 + 3H^2)\Psi = 0 \,. \tag{6.3}
$$

Thus $\Psi$ is a completely free field on de Sitter background, and this equation can be solved at least in principle.

We can now consider a different system of equations governing the auxiliary fields, where we eliminate $D_\mu B^\mu$ in (6.2) in terms of $\Psi$ and $B$. The full system then becomes

$$\begin{aligned} (D^2+3H^2)\Psi &= 0\,, \\ (D^2+4H^2)B &= \frac{2}{3}D^\mu D^\nu \Sigma_{\mu\nu}\,, \\ (D^2+3H^2)B_\mu &= D^\nu\Sigma_{\mu\nu} - D_\mu(\frac{B}{2}-\Psi)\,. \end{aligned} \tag{6.4}$$

Clearly, once we find a solution $\Psi$ and determine a $B$, which we can do because we know the source in the $B$ equation, given by the original tensor $h_{\mu\nu}$, we can integrate the remaining 4 equations for the vector to get the full solution. It will, clearly, depend on the choice of the auxiliary function $\Psi$. To select *precisely* the required solution of (6.1) we must extract that solution of (6.4) which automatically satisfies $\Psi = B - D_\mu B^\mu$. So indeed multiplying the vector field with $D^\mu$ yields

$$3H^2\Psi = (D^2+6H^2)(B - D_\mu B^\mu)\,, \tag{6.5}$$

and when we exclude the null eigenvalue spurions of $D^2+6H^2$ from $B - D_\mu B^\mu$ by an appropriate choice of boundary conditions[11], the field equation $(D^2+3H^2)\Psi = 0$ implies $(D^2+3H^2)(B-D_\mu B^\mu) = 0$. In this case $\Psi_1 = B - D_\mu B^\mu$ solves the same equation as $\Psi$. But then, since $(D^2+6H^2)\Psi_1 = 3H^2\Psi_1$, comparing with (6.5) gives $\Psi_1 = \Psi$. Thus as required $\Psi = B - D_\mu B^\mu$, and any such solution of (6.4) will be exactly a solution of (6.1). Having a solution, we can now define the tensor

$$\bar{h}_{\mu\nu} = \Sigma_{\mu\nu} - D_\mu B_\nu - D_\nu B_\mu + \frac{B}{2}\bar{\gamma}_{\mu\nu}\,, \tag{6.6}$$

and note, using the differential equations (6.1) that it is transverse, $D_\mu \bar{h}^\mu{}_\nu = 0$. Its trace is

$$\bar{h}^\mu{}_\mu = 2B - 2D_\mu B^\mu\,, \tag{6.7}$$

and so we can eliminate $B$ from the solution writing it as $B = D_\mu B^\mu + \bar{h}/2$. Moreover, we can separate the vector field $B_\mu$ as a Lorentz-gauge vector $A_\mu$, $D_\mu A^\mu = 0$, plus a scalar gradient,

$$B_\mu = A_\mu + \frac{1}{2}D_\mu\phi\,, \tag{6.8}$$

[11] We can always exclude these spurions, because by $(D^2+3H^2)D_\mu\hat{\vartheta} = D_\mu(D^2+6H^2)\hat{\vartheta}$, the gauge shifts $B_\mu \to B_\mu + D_\mu\hat{\vartheta}$ by functions obeying $(D^2+6H^2)\hat{\vartheta} = 0$ drop out from the field equations (6.4) but shift the spurion in $B - D_\mu B^\mu$ by $6H^2\hat{\vartheta}$. So we can simply choose $\hat{\vartheta}$ to completely cancel away the spurion.

and substitute all this back in (6.6). Solving for the original tensor field $h_{\mu\nu}$, we write

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + D_\mu A_\nu + D_\nu A_\mu + D_\mu D_\nu \phi - \frac{1}{4}\bar{\gamma}_{\mu\nu} D^2\phi + \frac{h - \bar{h}}{4}\bar{\gamma}_{\mu\nu} \,, \tag{6.9}$$

Now, we are almost there: the field $\bar{h}_{\mu\nu}$ is transverse, but not yet traceless. However its trace is equal to the auxiliary free field $\Psi$ by Eq. (6.7), which we can pick to be exactly zero by choosing appropriate boundary conditions, amounting to gauge fixing $B_\mu$. Hence the tensor is

$$\bar{h}_{\mu\nu}|_{\Psi=0} = h^{\mathtt{TT}}_{\mu\nu} \,, \tag{6.10}$$

which is also traceless! Thus indeed we recover (3.8), as claimed. Note that nowhere in this decomposition did we need to specify anything about $y$-dependence, which was treated as an extra parameter. Hence, Eq. (3.8) will remain valid for a Fourier transform of the metric perturbation as well. So we see that an arbitrary perturbation in the GN gauge can be separated as a 5-component transverse-traceless tensor, a 3-component Lorentz vector, and 2 scalars, in addition to the brane location $F$, which is a separate $4D$ field.

## 7. Helicity-0 ghosts in de Sitter space

Here we review the proof that a helicity-0 mode of the massive Pauli-Fierz spin-2 theory in de Sitter space is a ghost if the mass obeys $0 < m^2 < 2H^2$. In the early work of Higuchi [44] this was demonstrated by showing that the helicity-0 sector of the Hilbert space contains negative norm states, but since then simpler methods based on Hamiltonian analysis have been developed [46]. Here we largely follow the analysis of [46], although we note that a Lagrangian analysis based on correctly identifying the residual gauge symmetries of the helicity-0 sector would produce equivalent results.

We start with the Pauli-Fierz massive spin-2 theory in a background metric $\bar{\gamma}_{\mu\nu}$, which is given by $\mathcal{L} = \sqrt{-\bar{\gamma}} \;\; - \frac{1}{2} h^{\mu\nu} X^{(m)}_{\mu\nu}$, where $X^{(m)}_{\mu\nu}$ is defined in Eqs. (3.68) and (3.16). We take $\bar{\gamma}_{\mu\nu}$ to be the de Sitter metric (2.8), and $D_\mu$ its covariant derivative. This Lagrangian describes the localized lightest tensor multiplet on the self-accelerating branch, with mass $m = m_d$, which for positive brane tension lies in the unitarity-violating window $0 < m^2 < 2H^2$. The explicit form of the Pauli-Fierz action is

$$S_{PF} = \int d^4x \sqrt{-\bar{\gamma}} \;\; - \frac{1}{4} D^\alpha h^{\mu\nu} D_\alpha h_{\mu\nu} + \tfrac{1}{2} D_\mu h^\mu_\nu D^\alpha h^\nu_\alpha - \tfrac{1}{2} D^\mu h_{\mu\nu} D^\nu h + \tfrac{1}{4} D_\mu h D^\mu h$$
$$- \tfrac{1}{2} H^2 h^{\mu\nu} h_{\mu\nu} - \tfrac{1}{4} H^2 h^2 - \tfrac{m^2}{4} \left(h^{\mu\nu} h_{\mu\nu} - h^2\right) \,, \tag{7.1}$$

where the $h_{\mu\nu}$ are the general metric perturbations. Now, using the after-the-fact wisdom [46], we know that if the mass $m^2$ were zero, this theory would only have two helicity-2 excitations as the propagating modes. Thus the modifications can only arise because some of the scalar and vector perturbations of the general $4D$ metric do not decouple when $m^2 \neq 0$. Further, because the theory remains Lorentz-invariant, the vectors and the scalars decouple from the tensors, and moreover the vectors can only yield the helicity-1 modes. Thus the helicity-0 mode can only arise from the scalar perturbations, which we can parameterize as

$$h_{ij} = 2e^{-\frac{1}{2}Ht} \left(\partial_i\partial_j E + A\delta_{ij}\right) , \qquad h_{it} = e^{-\frac{1}{2}Ht}\, \partial_i B \, , \qquad h_{tt} = 2e^{-\frac{3}{2}Ht}\, \phi \, . \tag{7.2}$$

Here $\partial_i$ are spatial derivatives. We have normalized the perturbations by the appropriate powers of $e^{Ht}$ to simplify the analysis, following [46]. Plugging the perturbations (7.2) into (7.1) yields, with $S_{PF} = \int dt d^3x\ \mathcal{L}$,

$$\begin{aligned}\mathcal{L} = & -6\dot{A}^2 - 4\dot{A}\dot{X} + (4m^2 - 9H^2)AX + A\left(6m^2 - \frac{27}{2}H^2 - 2e^{-2Ht}\Delta\right)A \\ & -\phi\left[4H\dot{X} + 2(m^2 - 3H^2)X + 12H\dot{A} + 6(m^2-3H^2)A - 4e^{-2Ht}\Delta A\right] \\ & +e^{-Ht}\Delta B\left(4H\phi + 4\dot{A} - 6HA\right) - 6H^2\phi^2 - \tfrac{1}{2}m^2 B\Delta B \, ,\end{aligned} \tag{7.3}$$

where we have introduced $X = \Delta E$, with $\Delta = \sum_i \partial_i^2$ the flat 3-space Laplacian, and denoted time differentiation by a dot. To define the Hamiltonian as in [46], we write down the conjugate momenta $\Pi_Q = \frac{\partial \mathcal{L}}{\partial \dot{Q}}$, which are

$$\begin{aligned}& \Pi_X = -4\dot{A} - 4H\phi \, , \qquad \Pi_A = -12\dot{A} - 4\dot{X} - 12H\phi + 4e^{-2Ht}\Delta B \, , \\ & \Pi_B = \Pi_\phi = 0 \, .\end{aligned} \tag{7.4}$$

Thus the Lagrangian and the Hamiltonian are related by $\mathcal{L} = \Pi_X\dot{X} + \Pi_A\dot{A} - \mathcal{H}$, and so using this and (7.4) the Hamiltonian is given by

$$\begin{aligned}\mathcal{H} = & \frac{3}{8}\Pi_X^2 - \frac{1}{4}\Pi_X\Pi_A + A\left(\frac{27}{2}H^2 - 6m^2 + 2e^{-2Ht}\Delta\right)A + (9H^2 - 4m^2)AX \\ & +\phi\left[2(m^2 - 3H^2)X + 6(m^2 - 3H^2)A - 4e^{-2Ht}\Delta A - H\Pi_A\right] \\ & +e^{-Ht}\Delta B\left(\Pi_X + 6HA\right) - \tfrac{1}{2}m^2 B\Delta B\end{aligned} \tag{7.5}$$

From the Hamiltonian we immediately see that the fields $\phi$ and $B$ aren't propagating - which of course comes as no surprise, since they are the scalar remnants of the shift and lapse functions. Varying the Hamiltonian with respect to them yields the Hamiltonian and momentum constraints respectively, which can be written as

$$\begin{aligned}& \Pi_A - \frac{1}{H}\left[2(m^2 - 3H^2)X + 6(m^2-3H^2)A - 4e^{-2Ht}\Delta A\right] = 0 \, , \\ & B + \frac{e^{-Ht}}{m^2}\left(\Pi_X + 6HA\right) = 0 \, ,\end{aligned} \tag{7.6}$$

where we are holding $m^2 \neq 0$ in the last equation. Substituting these equations in the Lagrangian we integrate out the lapse and shift, and integrating by parts find

$$\begin{aligned}\mathcal{L} = &\left(\Pi_X - \frac{2}{H}(m^2 - 3H^2)A\right)\dot{X} - \Pi_X\left(\frac{3}{8} - \frac{e^{-2Ht}\Delta}{2m^2}\right)\Pi_X + \frac{1}{2H}(m^2 - 3H^2)\Pi_X X \\ &+\Pi_X\left[\frac{3}{2H}(m^2 - 3H^2) - \frac{1}{m^2 H}(m^2 - 6H^2)e^{-2Ht}\Delta\right]A + (4m^2 - 9H^2)AX \\ &+A\left[6m^2 - \frac{27}{2}H^2 - \frac{6}{m^2}(m^2 - 3H^2)e^{-2Ht}\Delta\right]A \end{aligned} \tag{7.7}$$

A field redefinition $\Pi_X = p + \frac{2}{H}(m^2 - 3H^2)A$ recasts the Lagrangian as

$$\mathcal{L} = p\dot{X} - p\left(\frac{3}{8} - \frac{e^{-2Ht}\Delta}{2m^2}\right)p + p\frac{e^{-2Ht}\Delta}{H}A + \frac{\nu^2}{2H}pX + \frac{3m^2\nu^2}{2H^2}A^2 + \frac{m^2\nu^2}{H^2}XA\,, \tag{7.8}$$

where $\nu^2 = m^2 - 2H^2$. This shows that with these variables $A$ is not a dynamical field. Its field equation is algebraic, and for $m^2 \neq 2H^2$ it yields $A = -\frac{X}{3} - \frac{He^{-2Ht}\Delta}{3m^2\nu^2}p$. Substituting this into (7.8) gives

$$\mathcal{L} = p\dot{X} - p\left[\frac{e^{-4Ht}\partial^4}{6m^2\nu^2} - \frac{e^{-2Ht}\Delta}{2m^2} + \frac{3}{8}\right]p - p\left[\frac{e^{-2Ht}\Delta}{3H} - \frac{1}{2H}(m^2 - 3H^2)\right]X - \frac{m^2\nu^2}{6H^2}X^2\,. \tag{7.9}$$

At long last, we make the one last field redefinition,

$$X = q + \frac{H}{2m^2\nu^2}\left[3(m^2 - 3H^2) - 2e^{-2Ht}\Delta\right]p\,, \tag{7.10}$$

which casts the Lagrangian in the form

$$\mathcal{L} = p\dot{q} - \frac{m^2\nu^2}{6H^2}q^2 - \frac{3H^2}{2m^2\nu^2}p\left(m^2 - \frac{9H^2}{4} - e^{-2Ht}\Delta\right)p\,. \tag{7.11}$$

This equation looks slightly unusual, since the Lagrangian seems to depend on the spatial gradients of the 'momentum' $p$ rather than the 'field' $q$. However, this is just a mirage, which can be easily removed by a canonical transformation $q = -\pi$, $p = \varphi$, and the integration by parts of $p\dot{q} = -\varphi\dot{\pi} = \pi\dot{\varphi} - \frac{d}{dt}(\varphi\pi)$. Dropping the total derivative, we can extract the final Hamiltonian from the Lagrangian $\mathcal{L} = \pi\dot{\varphi} - \mathcal{H}$, to find

$$\mathcal{H} = \frac{m^2\nu^2}{6H^2}\pi^2 + \frac{3H^2}{2m^2\nu^2}\varphi\left(m^2 - \frac{9H^2}{4} - e^{-2Ht}\Delta\right)\varphi\,. \tag{7.12}$$

We immediately see that when $\nu^2 < 0$ (i.e. $0 < m^2 < 2H^2$) the Hamiltonian is negative definite, and so the field $\varphi$ is a ghost. It can be viewed as literally a massive scalar field covariantly coupled to de Sitter gravity, with the sign of the Lagrangian reversed. When $m^2 = 0$ and $\nu^2 = 0$ the ghost decouples in the pure Pauli-Fierz theory, which can be glimpsed at from the canonically normalized scalar $\varphi = \frac{m|\nu|}{\sqrt{3}H}\varphi_C$, $\pi = \frac{\sqrt{3}H}{m|\nu|}\pi_C$, indicating that all the perturbative Lagrangian couplings of $\varphi$ to matter are proportional to positive powers of $m|\nu|$. This does not occur for the self-accelerating branch DGP because of the additional scalar localized mode, as discussed in the text and in [68].

Also note that the $0 < m^2 < 2H^2$ ghost has a very mild tachyonic instability, induced by de Sitter expansion. One can see it from the field equation for the Fourier components of $\varphi_C$, which from (7.12) is, by setting $\varphi_C = \hat{\varphi}_C(k)e^{i\vec{k}\cdot\vec{x}}$,

$$\ddot{\hat{\varphi}}_C(k) + \Big(m^2 - \frac{9H^2}{4} + \vec{k}^2 e^{-2Ht}\Big)\hat{\varphi}_C(k) = 0\,. \tag{7.13}$$

As time goes on, the 3-momentum term $\propto \vec{k}^2$ becomes insignificant, so that $\hat{\varphi}_C(k) \to \exp(\pm\sqrt{\frac{9H^2}{4} - m^2}\; t)$ for $m^2 < 2H^2$. This behavior changes for all gravitons above de Sitter gap $m^2 \geq \frac{9H^2}{4}$ (as discussed in [46]), and when $m^2 = 0$ the ghost is absent in the first place. Yet it is clear that these instabilities simply correspond to the freezing out of long wavelength ghost modes at super-horizon scales, as is common in inflation, and it would be interesting to explore the implications of this mechanism for DGP.

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