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A UNIFYING APPROACH TO INTEGRATION FOR BOUNDED POSITIVE CHARGES

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Abstract. This paper deals with n-monotone functionals, which constitute a generalisation of n-monotone set functions. Using the notion of exactness of a functional, we introduce a new notion of lower and upper integral which subsumes as particular cases most of the approaches to integration in the literature. As a consequence, we can characterise which types of integrals can be used to calculate the natural extension (the lower envelope of all linear extensions) of a positive bounded charge.

1. Introduction

Coherent lower previsions ([19]), and exact functionals ([16]) are among the most interesting uncertainty models in what has been called the theory of imprecise probabilities; this is the theory which extends the Bayesian theory of probability by allowing for indecision.

This paper deals with a special subclass of these functionals, namely those that are n-monotone, for $n \geq 1$. In a companion paper ([6]), we have introduced and studied in the notion of n-monotone exact functionals, building on Choquet's [4] original and very general definition of n-monotonicity for functions defined on arbitrary lattices. We have summarised the main results of that paper in the introductory Section 2. We use those results to introduce, in Section 3, a new and quite flexible type of (lower) integral that subsumes many of the well-known lower integrals in the literature, such as the lower Riemann–(Stieltjes), S-, Dunford, Lebesgue, and Young–Stieltjes integrals. We show that our lower integral (and therefore all the others) is an exact functional that is completely monotone, i.e., n-monotone for all $n \geq 1$.

Taking the generalisation yet a step further, we are led in Section 4 to define a notion of integrability with respect to exact functionals that unifies and generalises many types of integrability known from the literature. In this way, we are able to characterise which types of (lower) integration coincide with the natural extension of a bounded positive charge, i.e., the lower envelope of all the positive linear functionals that extend the charge from events to bounded mappings. By conjugacy, the associated upper integral is then the upper envelope of this set of positive linear functionals.

We conclude in Section 5 with some additional comments and remarks.

2. Coherence, exactness, and n-monotonicity

Let us first mention a few basic notions about coherent lower previsions and exact functionals.

2.1. **Coherent lower previsions and linear previsions.**

We begin with coherent lower previsions, which have been studied in depth by Walley in [19].

Consider a non-empty set Ω. A *gamble* f on Ω is a bounded real-valued mapping on Ω. The set of all gambles on Ω is denoted by $\mathcal{L}$. A special class of gambles only take values in $\{0, 1\}$: let A be any subset of Ω, also called an *event*, then the gamble I_A, defined by $I_A(\omega) := 1$ if

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$\omega \in A$ and $I_A(\omega) := 0$ otherwise, is called the *indicator* of A. This establishes a correspondence between events and $\{0,1\}$-valued gambles. Often, for an event A, we shall also denote I_A by A.

We shall call *functional* any real-valued map $\underline{\Gamma}$ defined on a subset of $\mathcal{L}$, called the *domain* $\operatorname{dom}\underline{\Gamma}$ of $\underline{\Gamma}$. When $\operatorname{dom}\underline{\Gamma}$ contains only indicators of events, we shall call $\underline{\Gamma}$ a *set function.* The *conjugate* $\overline{\Gamma}$ of $\underline{\Gamma}$ is a functional defined on the set $\operatorname{dom}\overline{\Gamma} = -\operatorname{dom}\underline{\Gamma} := \{-f : f \in \operatorname{dom}\underline{\Gamma}\}$ by $\overline{\Gamma}(f) = -\underline{\Gamma}(-f)$.

A functional $\underline{P}$ defined on $\mathcal{L}$ is called a *coherent lower prevision* if the following three properties are satisfied for all f, g in $\operatorname{dom}\underline{P}$ and all non-negative real λ:

(C1) $\underline{P}(f) \geq \inf f$ (accepting sure gains);
(C2) $\underline{P}(\lambda f) = \lambda\underline{P}(f)$ (positive homogeneity);
(C3) $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$ (super-additivity).

A functional $\underline{P}$ with a general domain (not necessarily a linear space) is called a *coherent lower prevision* if it can be extended to a coherent lower prevision on all gambles. This is the case if and only if $\sup\left[\sum_{i=1}^n f_i - mf_0\right] \geq \sum_{i=1}^n \underline{P}(f_i) - m\underline{P}(f_0)$ for any natural numbers $n \geq 0$ and $m \geq 0$, and f_0, f_1, …, f_n in the domain of $\underline{P}$. For any gamble f in $\operatorname{dom}\underline{P}$, $\underline{P}(f)$ is then called the lower prevision of f. If the domain of $\underline{P}$ contains only (indicators of) events A, then we also call $\underline{P}$ a *coherent lower probability*, and we write $\underline{P}(I_A)$ also as $\underline{P}(A)$, the lower probability of A.

The conjugate *coherent upper prevision* $\overline{P}$ of $\underline{P}$ is defined on $\operatorname{dom}\overline{P} = -\operatorname{dom}\underline{P} := \{-f : f \in \operatorname{dom}\underline{P}\}$ by $\overline{P}(f) := -\underline{P}(-f)$ for every $-f$ in the domain of $\underline{P}$. This conjugacy relationship shows that we can restrict our attention to the study of coherent lower previsions only. If the domain of $\overline{P}$ contains indicators only, then we also call $\overline{P}$ an *upper probability.* It generally holds for coherent lower previsions $\underline{P}$ that $\underline{P}(f) \leq \overline{P}(f)$ for all $f \in \operatorname{dom}\underline{P} \cap \operatorname{dom}\overline{P}$.

Given a coherent lower prevision $\underline{P}$ on some domain $\operatorname{dom}\underline{P}$, there is a point-wise smallest coherent lower prevision $\underline{E}_{\underline{P}}$ on $\mathcal{L}$ that coincides with $\underline{P}$ on $\operatorname{dom}\underline{P}$. It is called the *natural extension* of $\underline{P}$, and is given by (Walley [19, Lemma 3.1.3(b)])

$$\underline{E}_{\underline{P}}(f) = \sup\left\{\sum_{k=1}^n \lambda_k\underline{P}(f_k) + \lambda : n \geq 1, \lambda_k \in \mathbb{R}_+, \lambda \in \mathbb{R}, f_k \in \operatorname{dom}\underline{P}, \sum_{k=1}^n \lambda_k f_k + \lambda \leq f\right\}$$

for all gambles f in $\mathcal{L}$, where $\mathbb{R}_+$ is the set of non-negative reals and $\mathbb{R}$ is the set of reals.

A *linear prevision* P on $\mathcal{L}$ is a non-negative, normed [$P(1) = 1$], real-valued, linear functional on $\mathcal{L}$. The restriction of such a linear prevision on $\mathcal{L}$ to (indicators of) events is a *probability charge* (or finitely additive probability measure) on $\wp(\Omega)$, the class of all subsets of Ω. It can be checked that $\underline{E}_{\underline{P}}$ is equal to the lower envelope of of those linear previsions on $\mathcal{L}$ that dominate $\underline{P}$ on its domain.

2.2. **Exact functionals.** Maaß [16] has extended the notion of coherent lower previsions to that of *exact functionals.* Let $\underline{\Gamma}$ be a functional with domain $\operatorname{dom}\underline{\Gamma}$. If $\operatorname{dom}\underline{\Gamma} = \mathcal{L}$, then $\underline{\Gamma}$ is called *exact* whenever for any gambles f and g on Ω, any non-negative real number λ, and any real number μ, it holds that

(E1) if $f \geq g$ then $\underline{\Gamma}(f) \geq \underline{\Gamma}(g)$ (monotonicity);
(E2) $\underline{\Gamma}(\lambda f) = \lambda\underline{\Gamma}(f)$ (positive homogeneity);
(E3) $\underline{\Gamma}(f+g) \geq \underline{\Gamma}(f) + \underline{\Gamma}(g)$ (super-additivity);
(E4) $\underline{\Gamma}(f+\mu) = \underline{\Gamma}(f) + \underline{\Gamma}(\mu)$ (constant additivity).

A functional defined on an arbitrary subset of $\mathcal{L}$ is called *exact* if it can be extended to an exact functional on all of $\mathcal{L}$. If $\underline{\Gamma}$ is exact, then $\underline{\Gamma}(f) \leq \overline{\Gamma}(f)$ for all $f \in \operatorname{dom}\underline{\Gamma} \cap \operatorname{dom}\overline{\Gamma}$.

Exact functionals are important not only because they generalise coherent lower previsions; they can also be seen as the negatives of coherent risk measures ([1, 7]) and they are very closely related to exact cooperative games ([18]).

Maaß [16, Eq. (1.2), p. 4] defines also the following norm on the space of all functionals:

$$\|\underline{\Gamma}\| := \inf\left\{c \in \mathbb{R}_+ : f \geq \sum_{k=1}^{n} \lambda_k f_k + \lambda \Rightarrow \underline{\Gamma}(f) \geq \sum_{k=1}^{n} \lambda_k \underline{\Gamma}(f_k) + \lambda c\right\}$$

where the condition must hold for all natural n, non-negative reals $\lambda_1, \ldots, \lambda_n$, real λ, and gambles f, $f_1, \ldots, f_n$ in $\operatorname{dom}\underline{\Gamma}$. Maaß also shows that a functional $\underline{\Gamma}$ is exact if and only if its norm is finite: $\|\underline{\Gamma}\| < +\infty$. He also shows that if $1 \in \operatorname{dom}\underline{\Gamma}$ then $\|\underline{\Gamma}\| = \underline{\Gamma}(1)$. The norm serves as a Lipschitz constant because for any two gambles f and g in $\operatorname{dom}\underline{\Gamma}$, $|\underline{\Gamma}(f) - \underline{\Gamma}(g)| \leq \|\underline{\Gamma}\| \sup|f - g|$. Hence, exactness implies (uniform) continuity with respect to the supremum norm.

We shall call a functional Γ *linear exact* if it can be extended to an exact functional Ψ on $\mathcal{L}$ which is at the same time also a linear functional, *i.e.*, which also satisfies $\Psi(f) + \Psi(g) = \Psi(f + g)$ for any f and g in $\mathcal{L}$. For a linear exact functional $\underline{\Gamma}$, it holds that $\underline{\Gamma}(f) = \overline{\Gamma}(f)$ for all $f \in \operatorname{dom}\underline{\Gamma} \cap \operatorname{dom}\overline{\Gamma}$.

An exact functional $\underline{\Gamma}$ has by its very definition exact extensions to all of $\mathcal{L}$. Among those, there are exact extension whose norm is equal to $\|\underline{\Gamma}\|$, and the point-wise smallest such exact extension $\underline{E}_{\underline{\Gamma}}$ is called the *natural extension* of the exact functional $\underline{\Gamma}$. It is also the lower envelope of the set of dominating positive linear functionals that dominate $\underline{\Gamma}$ and have the same norm; see Maaß [16, Theorem 1.2.5 and Corollary 1.5.8] for more details. This gives very special importance to the notion of natural extension. In this paper, we will be especially concerned with the natural extension of positive bounded charges.

The relationship between coherent lower previsions, exact functionals, and their respective natural extensions is given by the following theorem.

Theorem 1. [6, Thm. 2] *Let $\underline{\Gamma}$ be a functional defined on a subset of $\mathcal{L}$.*

(i) If $\underline{\Gamma}$ is exact, then there is a coherent lower prevision $\underline{P}$ defined on $\operatorname{dom}\underline{\Gamma}$ *such that $\underline{\Gamma} = \|\underline{\Gamma}\|\underline{P}$, and moreover $\underline{E}_{\underline{\Gamma}} = \|\underline{\Gamma}\|\underline{E}_{\underline{P}}$.*

(ii) $\underline{\Gamma}$ is exact if and only if there is a coherent lower prevision $\underline{P}$ defined on $\operatorname{dom}\underline{\Gamma}$*, and a non-negative real number λ, such that $\underline{\Gamma} = \lambda\underline{P}$. In that case, $\lambda\underline{E}_{\underline{P}}$ is an exact extension of $\underline{\Gamma}$ with norm λ.*

2.3. ***n*-monotonicity.** We now turn to the notion of n-monotonicity for functionals, which is a special case of the general definition of n-monotonicity that Choquet [4] has given for functions from an Abelian semi-group to an Abelian group. Recall that a subset $\mathcal{S}$ of $\mathcal{L}$ is called a *lattice* if it is closed under point-wise maximum $\vee$ and point-wise minimum $\wedge$, i.e., if for all f and g in $\mathcal{S}$, both $f \vee g$ and $f \wedge g$ also belong to $\mathcal{S}$. We denote by $\mathbb{N}$ the set of all (strictly positive) natural numbers, by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of all non-negative integers, and by $\mathbb{N}^* = \mathbb{N} \cup \{+\infty\}$ the set of extended natural numbers.

Definition 1. Let $n \in \mathbb{N}^*$, and let $\underline{\Gamma}$ be a functional whose domain $\operatorname{dom}\underline{\Gamma}$ is a lattice of gambles on Ω. Then we call $\underline{\Gamma}$ *n-monotone* if for all $p \in \mathbb{N}$, $p \leq n$, and all f, $f_1, \ldots, f_p$ in $\operatorname{dom}\underline{\Gamma}$:

$$\sum_{I \subseteq \{1,\ldots,p\}} (-1)^{|I|} \underline{\Gamma}\left(f \wedge \bigwedge_{i \in I} f_i\right) \geq 0.$$

The conjugate of an n-monotone functional is called *n-alternating*. An ∞-monotone functional (i.e, a functional which is n-monotone for all $n \in \mathbb{N}$) is also called *completely monotone*, and its conjugate *completely alternating*.

In what follows, we shall use the following consequences of n-monotonicity. Recall that two gambles f and g on Ω are called *comonotone* if $f(\omega) > f(\varpi)$ implies that $g(\omega) \geq g(\varpi)$ for all ω and ϖ in Ω. A functional $\underline{\Gamma}$ is called *comonotone additive* if $\underline{\Gamma}(f + g) = \underline{\Gamma}(f) + \underline{\Gamma}(g)$ for all comonotone f and g in $\operatorname{dom}\underline{\Gamma}$ such that $f + g \in \operatorname{dom}\underline{\Gamma}$.

Theorem 2. [6, Thms. 10,11,13 and 17] *Consider a functional* $\underline{\Gamma}$ *defined on a lattice* $\operatorname{dom}\underline{\Gamma}$ *of* $\mathcal{L}$.

(i) *If* $\underline{\Gamma}$ *is minimum preserving, then it is completely monotone.*
(ii) *If* $\underline{\Gamma}$ *is an exact linear functional, then it is both completely monotone and completely alternating.*
(iii) *Let* $\underline{\Gamma}$ *be exact and let* $\operatorname{dom}\underline{\Gamma}$ *be a linear lattice of gambles that contains all constant gambles. Then* $\underline{\Gamma}$ *is comonotone additive if and only if it is* 2*-monotone, and in both cases we have for all* f *in* $\operatorname{dom}\underline{\Gamma}$ *that*

$$\underline{\Gamma}(f) = (C)\int f \,\mathrm{d}\underline{\Gamma}_* = \|\underline{\Gamma}\| \inf f + (R)\int_{\inf f}^{\sup f} \underline{\Gamma}_*(\{f \geq x\})\,\mathrm{d}x, \tag{1}$$

where the first integral is a Choquet integral, the second a Riemann integral, and $\underline{\Gamma}_*$ *is the* inner extension *of* $\underline{\Gamma}$*, given by* $\underline{\Gamma}_*(f) = \sup\{\underline{\Gamma}(g)\colon g \in \operatorname{dom}\underline{\Gamma}, g \leq f\}$.
(iv) *If* $\operatorname{dom}\underline{\Gamma}$ *is a lattice of events that contains both* $\emptyset$ *and* Ω*, and if* $\underline{\Gamma}$ *is an* n*-monotone set function on* $\operatorname{dom}\underline{\Gamma}$*, then it is exact and its natural extension* $\underline{E}_{\underline{\Gamma}}$ *to the set of all gambles is* n*-monotone as well, and it is given by Eq.* (1)*, where now* $\underline{\Gamma}_*$ *is the inner set function of* $\underline{\Gamma}$.

The third point of this theorem essentially states that for exact functionals defined on all gambles, 2-monotonicity and comonotone additivity are equivalent. This was shown in [6] to follow from two more basic results: (i) Greco's representation theorem [8], which states that under some additional technical conditions a real functional is monotone and comonotone additive if and only if it can be represented as a Choquet functional associated with a monotone set function; and (ii) a representation result we proved in [6, Theorem 17 and Corollary 19] which states that a 2-monotone exact functional defined on all gambles is always the Choquet functional associated with its restriction to events. It follows immediately that under exactness, 2-monotonicity implies comotonone additivity. The converse result is proven by showing that the smallest monotone set function that represents a comonotone additive exact functional is 2-monotone.

One consequence of this theorem to be remembered, then, is that an n-monotone exact functional on all gambles is always the natural extension (Choquet integral) of its restriction to all events. It also deserves to be mentioned here that both exactness and n-monotonicity are preserved under taking point-wise limits and non-negative linear combinations.

3. A general notion of lower integral as an instance of a completely monotone exact functional

Let us now show that many of the lower integrals in the literature are actually instances of completely monotone exact functionals. The way such lower integrals are obtained can be formalised nicely through the introduction of a new and very flexible type of integration. It actually goes back to an idea suggested by Moore and Smith [17, Sect. 5, p. 114, ll. 10–13], who provided an alternative definition of the Lebesgue integral for bounded real-valued functions (i.e., gambles).

3.1. **The definition of (lower and upper) $\mathbb{V}$-integrals.** Consider a bounded positive charge μ on a field $\mathcal{F}$ of subsets of Ω, i.e., a real-valued set function on $\mathcal{F}$ such that $\mu(\emptyset) = 0$, $\mu(A) \geq 0$, and $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$ for any A and B in $\mathcal{F}$. Such a bounded positive charge can be identified with a functional Γ_μ on the set of gambles $I_{\mathcal{F}} := \{I_A\colon A \in \mathcal{F}\}$ by letting $\Gamma_\mu(I_A) := \mu(A)$ for all A in $\mathcal{F}$. Note that this functional is completely monotone.

Next, we define a *refinement relation* $\preceq$ on the collection of subsets of $\mathcal{F}$, as follows: for any two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{F}$, we write $\mathcal{V}_1 \preceq \mathcal{V}_2$ if $\mathcal{V}_2$ *refines* $\mathcal{V}_1$ in the sense that every element of $\mathcal{V}_2$ is included in some element of $\mathcal{V}_1$.

Now consider a collection $\mathbb{V}$ of *finite* collections of *non-empty* subsets of $\mathcal{F}$ (in other words, every element $\mathcal{V}$ of $\mathbb{V}$ is a finite collection of non-empty subsets of $\mathcal{F}$) that satisfies the following properties:

($\mathbb{V}$1) $\mathbb{V}$ *is directed*: for any $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\mathbb{V}$, there is some $\mathcal{V}$ in $\mathbb{V}$ such that $\mathcal{V}_1 \preceq \mathcal{V}$ and $\mathcal{V}_2 \preceq \mathcal{V}$;

($\mathbb{V}$2) *covering:* for any $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\mathbb{V}$ such that $\mathcal{V}_1 \preceq \mathcal{V}_2$, it holds that

$$\bigcup\{V_2 \in \mathcal{V}_2 \colon V_2 \subseteq V_1\} = V_1$$

for all V_1 in $\mathcal{V}_1$;

($\mathbb{V}$3) *additivity*: for any $\mathcal{V}$ in $\mathbb{V}$ and any $\mathcal{D} \subseteq \mathcal{V}$, it holds that

$$\mu(\bigcup\mathcal{D}) = \sum_{V\in\mathcal{D}} \mu(V);$$

($\mathbb{V}$4) *smallest element*: $\{\Omega\}$ belongs to $\mathbb{V}$.

As an immediate consequence of ($\mathbb{V}$2)–($\mathbb{V}$4) we find that for any $\mathcal{V}$ in $\mathbb{V}$:

$$\bigcup_{V\in\mathcal{V}} V = \Omega, \text{ and } \sum_{V\in\mathcal{V}} \mu(V) = \mu(\Omega). \tag{2}$$

Condition ($\mathbb{V}$1) states that $\preceq$ satisfies the *composition property* or the *Moore–Smith property.* Since $\preceq$ is also a reflexive and transitive relation, it follows that $\mathbb{V}$ is a directed set with respect to $\preceq$. Therefore, given a *net* α on $\mathbb{V}$, i.e., a mapping $\alpha\colon \mathbb{V} \to \mathbb{R}$, we can take the *Moore–Smith limit* of α with respect to the directed set $(\mathbb{V}, \preceq)$ (see Moore and Smith [17, Sect. I, p. 103]), which, if it exists, is defined as the unique real number a such that, for every $\epsilon > 0$, there is a $\mathcal{V}_\epsilon$ in $\mathbb{V}$, such that $|\alpha(\mathcal{V}) - a| < \epsilon$ for all $\mathcal{V} \succeq \mathcal{V}_\epsilon$. The Moore–Smith limit a of α is denoted by $\lim_{\mathcal{V}\in\mathbb{V}} \alpha(\mathcal{V})$. Note that this limit always exists if α is non-decreasing and bounded from above, or if α is non-increasing and bounded from below; we shall use this result further on.

Let $\underline{P}_V(f)$ denote the *vacuous lower prevision* of f relative to the non-empty subset V of Ω, and $\overline{P}_V(f)$ the *vacuous upper prevision* of f relative to V, which are defined for any f in $\mathcal{L}$ as

$$\underline{P}_V(f) := \inf_{\omega\in V} f(\omega), \text{ and } \overline{P}_V(f) := \sup_{\omega\in V} f(\omega).$$

Proposition 3. *Let V be a non-empty subset of Ω, then the vacuous lower prevision $\underline{P}_V$ relative to V is a completely monotone coherent lower prevision on $\mathcal{L}$. $\overline{P}_V$ is its conjugate upper prevision.*

Proof. Immediately from the first statement of Theorem 2, and the fact that $\underline{P}_V$ is easily seen to verify the coherence conditions (C1)–(C3), and is therefore a coherent lower prevision. Also, it is obvious that $\overline{P}_V(-f) = \sup_{\omega\in V} -f(\omega) = -\inf_{\omega\in V} f(\omega) = -\underline{P}_V(f)$, so $\overline{P}_V$ is the conjugate upper prevision. □

Lemma 4. *Consider $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\mathbb{V}$ such that $\mathcal{V}_1 \preceq \mathcal{V}_2$. Consider any W in $\mathcal{V}_2$ and suppose that there are different V_1 and U_1 in $\mathcal{V}_1$ such that $W \subseteq V_1$ and $W \subseteq U_1$. Then $\mu(W) = 0$.*

Proof. Consider different V_1 and U_1 in $\mathcal{V}_1$. It follows from ($\mathbb{V}$2) and ($\mathbb{V}$3) that

$$\mu(V_1 \cup U_1) = \mu(V_1) + \mu(U_1) = \sum_{V_2\in\mathcal{V}_2, V_2\subseteq V_1} \mu(V_2) + \sum_{U_2\in\mathcal{V}_2, U_2\subseteq U_1} \mu(U_2). \tag{3}$$

Consider on the other hand the subset

$$\mathcal{D} = \{W_2 \in \mathcal{V}_2 \colon W_2 \subseteq V_1 \text{ or } W_2 \subseteq U_1\}$$

of $\mathcal{V}_2$, then it follows from ($\mathbb{V}$2) that $\bigcup\mathcal{D} = V_1 \cup U_1$ and from ($\mathbb{V}$3) that

$$\mu(V_1 \cup U_1) = \sum_{W_2\in\mathcal{V}_2, W_2\subseteq V_1 \text{ or } W_2\subseteq U_1} \mu(W_2) \tag{4}$$

Now if W in $\mathcal{V}_2$ is such that $W \subseteq V_1$ *and* $W \subseteq U_1$, then the term $\mu(W)$ will appear twice in the summation on the right hand side of Eq. (3), but only once the summation on the right hand side of Eq. (4). This implies that $\mu(W)$ must be zero. □

Lemma 5. *For any $\mathcal{V}_1$ and $\mathcal{V}_2$ in $\mathbb{V}$ such that $\mathcal{V}_1 \preceq \mathcal{V}_2$ we have that for any $f \in \mathcal{L}$:*

$$\sum_{V_1 \in \mathcal{V}_1} \underline{P}_{V_1}(f)\mu(V_1) \leq \sum_{V_2 \in \mathcal{V}_2} \underline{P}_{V_2}(f)\mu(V_2)$$
$$\sum_{V_1 \in \mathcal{V}_1} \overline{P}_{V_1}(f)\mu(V_1) \geq \sum_{V_2 \in \mathcal{V}_2} \overline{P}_{V_2}(f)\mu(V_2).$$

Moreover, for any $\mathcal{V}$ in $\mathbb{V}$ we have

$$\mu(\Omega)\inf f \leq \sum_{V \in \mathcal{V}} \underline{P}_V(f)\mu(V) \leq \sum_{V \in \mathcal{V}} \overline{P}_V(f)\mu(V) \leq \mu(\Omega)\sup f.$$

Proof. Since the inequalities trivially hold if $\mu(\Omega) = 0$, we shall assume that $\mu(\Omega) > 0$. It suffices to prove the first of these inequalities, as the second then follows by conjugacy. Consider any V_1 in $\mathcal{V}_1$. Since $V_1 \neq \emptyset$, we know from ($\mathbb{V}$2) that there is at least one V_2 in $\mathcal{V}_2$ such that $V_2 \subseteq V_1$. Consider any such V_2, then immediately $\underline{P}_{V_2}(f) \geq \underline{P}_{V_1}(f)$, and therefore, also using ($\mathbb{V}$2) and ($\mathbb{V}$3),

$$\sum_{V_2 \subseteq V_1} \underline{P}_{V_2}(f)\mu(V_2) \geq \underline{P}_{V_1}(f) \sum_{V_2 \subseteq V_1} \mu(V_2) = \underline{P}_{V_1}(f)\mu(V_1).$$

By summing over all V_1 in $\mathcal{V}_1$, we get

$$\sum_{V_1 \in \mathcal{V}_1} \sum_{V_2 \subseteq V_1} \underline{P}_{V_2}(f)\mu(V_2) \geq \sum_{V_1 \in \mathcal{V}_1} \underline{P}_{V_1}(f)\mu(V_1),$$

and taking into account Lemma 4 and the fact that $\mathcal{V}_1 \preceq \mathcal{V}_2$ we infer that the left-hand side is equal to $\sum_{V_2 \in \mathcal{V}_2} \underline{P}_{V_2}(f)\mu(V_2)$. The rest of the proof is obvious, since by Eq. (2), the sums in these inequalities are convex mixtures (after division by $\mu(\Omega)$). □

Using the Moore–Smith limit, we can define the following functionals on $\mathcal{L}$:

$$(\mathbb{V})\underline{\int} f \,\mathrm{d}\mu := \lim_{\mathcal{V} \in \mathbb{V}} \sum_{V \in \mathcal{V}} \underline{P}_V(f)\mu(V) \tag{5}$$

is called the *lower $\mathbb{V}$-integral* of f with respect to μ, and

$$(\mathbb{V})\overline{\int} f \,\mathrm{d}\mu := \lim_{\mathcal{V} \in \mathbb{V}} \sum_{V \in \mathcal{V}} \overline{P}_V(f)\mu(V)$$

the *upper $\mathbb{V}$-integral* of f with respect to μ. Indeed, Lemma 5 tells us that both Moore–Smith limits converge to real numbers, and that moreover

$$\mu(\Omega)\inf f \leq (\mathbb{V})\underline{\int} f \,\mathrm{d}\mu \leq (\mathbb{V})\overline{\int} f \,\mathrm{d}\mu \leq \mu(\Omega)\sup f.$$

It makes sense to say that a gamble f is *$\mathbb{V}$-integrable with respect to μ* whenever its lower and upper $\mathbb{V}$-integral with respect to μ coincide. For such gambles, we denote the common value of $(\mathbb{V})\underline{\int} f \,\mathrm{d}\mu$ and $(\mathbb{V})\overline{\int} f \,\mathrm{d}\mu$ by

$$(\mathbb{V}) \int f \,\mathrm{d}\mu$$

and we call it the *$\mathbb{V}$-integral of f with respect to μ.*

3.2. **Examples of lower $\mathbb{V}$-integrals.** We now discuss a number of integrals in the literature that can be considered as special instances of the general notion of a $\mathbb{V}$-integral. In all examples, a and b are (finite) real numbers such that $a < b$.

3.2.1. *The Riemann integral.* Let $\mathcal{F}$ be any field that contains all closed intervals in $[a, b]$; the Borel σ-field $\mathcal{B}([a, b])$ on $[a, b]$ is an instance of such field. Let μ be any bounded positive charge on $[a, b]$ that satisfies $\mu([c, d]) = d - c$ for any $a \leq c \leq d \leq b$; in case $\mathcal{F}$ is the Borel σ-field $\mathcal{B}([a, b])$, the Borel-Lebesgue measure on $[a, b]$ is an instance of such positive bounded charge. Let $\mathbb{V}$ be the collection of all finite collections of *closed* intervals that overlap only on their borders, and whose union is $[a, b]$.

It is easily verified that Conditions ($\mathbb{V}$1)–($\mathbb{V}$4) are satisfied. The lower $\mathbb{V}$-integral with respect to μ is precisely the lower Riemann integral $(R)\underline{\int}_a^b \cdot \,\mathrm{d}x$ (see Darboux [5, Sect. II, p. 64]), and a gamble is $\mathbb{V}$-integrable with respect to μ if and only if it is Riemann integrable.

3.2.2. *The Riemann–Stieltjes integral.* Let $F \colon [a, b] \to \mathbb{R}$ be any non-decreasing mapping. Consider the smallest field $\mathcal{F}$ that contains all closed intervals in $[a, b]$; $\mathcal{F}$ contains exactly all finite unions of (not only closed, but all) intervals in $[a, b]$. Now consider the bounded positive charge μ_F on $\mathcal{F}$ that is uniquely characterised by $\mu_F([c, d]) = F(d) - F(c)$ for any $a \leq c \leq d \leq b$. Let $\mathbb{V}$ again be the collection of all finite collections of closed intervals that overlap only on their borders and whose union is $[a, b]$.

Here too, Conditions ($\mathbb{V}$1)–($\mathbb{V}$4) are satisfied. The lower $\mathbb{V}$-integral with respect to μ_F is now the Darboux version of the lower Riemann–Stieltjes integral $(RS)\underline{\int}_a^b \cdot \,\mathrm{d}F(x)$ with respect to F (see Hildebrandt [11, Chap. II, pp. 27–32]), and a gamble is $\mathbb{V}$-integrable with respect to μ_F if and only if it is Riemann–Stieltjes integrable with respect to F.

3.2.3. *The S-integral, the Dunford integral, and another Lebesgue-like way of defining an integral.* Let μ be any positive bounded charge on a field $\mathcal{F}$, and let $\mathbb{V}$ be the collection of all finite partitions in $\mathcal{F}$. Then the lower S-integral $(S)\underline{\int} \cdot \,\mathrm{d}\mu$ with respect to μ, as defined by Bhaskara Rao and Bhaskara Rao [2, Sect. 4.5], is equal to the lower $\mathbb{V}$-integral with respect to μ, and a gamble is $\mathbb{V}$-integrable with respect to μ if and only if it is S-integrable with respect to μ. Note that the S-integral coincides with the Dunford integral on gambles (again see Bhaskara Rao and Bhaskara Rao [2, Sect. 4.5, Thm. 4.5.7 and Prop. 4.5.8]), so a similar result holds for the Dunford integral.

Yet another way to obtain the lower S-integral, is fashioned after a method due to Lebesgue [14, p. 47, l. 5], originally aimed at reconciling Cauchy's (geometric) way of defining an integral with Riemann's (analytic) way: let f be any gamble on Ω, and consider the sets

$$A_f^+ := \{(\omega, x) \colon \omega \in \Omega,\, 0 \leq x \leq f(\omega)\}, \text{ and } A_f^- := \{(\omega, x) \colon \omega \in \Omega,\, f(\omega) \leq x \leq 0\};$$

these two sets constitute the surface between the gamble f and the Ω-axis. A lower approximation of the (signed) area of this surface is:

$$\kappa_*(A_f^+) - \kappa^*(A_f^-),$$

where κ is the product of the charge μ and the Lebesgue–Borel measure λ on $\mathbb{R}$, and κ_* and κ^* are respectively the inner and outer set functions induced by κ; so,

$$\kappa_*(A) := \sup\left\{ \sum_{i=1}^{n} \mu(B_i)\lambda(C_i) \colon n \in \mathbb{N},\, B_1, \dots, B_n \in \operatorname{dom}\mu,\, C_1, \dots, C_n \in \operatorname{dom}\lambda, \right.$$
$$\left. B_1 \times C_1, \dots, B_n \times C_n \text{ disjoint}, \bigcup_{i=1}^{n} B_i \times C_i \subseteq A \right\},$$

and

$$\kappa^*(A) := \inf\left\{ \sum_{i=1}^n \mu(B_i)\lambda(C_i)\colon n \in \mathbb{N},\, B_1,\dots,B_n \in \operatorname{dom}\mu,\, C_1,\dots,C_n \in \operatorname{dom}\lambda,\right.$$
$$\left. B_1\times C_1,\dots,B_n\times C_n \text{ disjoint},\, \bigcup_{i=1}^n B_i\times C_i \supseteq A\right\}.$$

Proposition 6. *For any gamble f on Ω, it holds that*

$$(S)\underline{\int} f\,\mathrm{d}\mu = \kappa_*(A_f^+) - \kappa^*(A_f^-).$$

Proof. Let f^+ denote $f\vee 0$, and let f^- denote $f\wedge 0$. Let's first prove that $(S)\underline{\int} f^+\,\mathrm{d}\mu = \kappa_*(A_f^+)$. Recall that the lower S-integral with respect to μ is the lower $\mathbb{V}$-integral with respect to μ where we take for $\mathbb{V}$ the collection of all finite partitions in $\mathcal{F}$.

Consider again the definition of κ_*:

$$\kappa_*(A_f^+) := \sup\left\{ \sum_{i=1}^n \mu(B_i)\lambda(C_i)\colon n \in \mathbb{N},\, B_1,\dots,B_n \in \operatorname{dom}\mu,\, C_1,\dots,C_n \in \operatorname{dom}\lambda,\right. \tag{6}$$
$$\left. B_1\times C_1,\dots,B_n\times C_n \text{ disjoint},\, \bigcup_{i=1}^n B_i\times C_i \subseteq A_f^+\right\}.$$

Let $n\in\mathbb{N}$, $B_1,\dots,B_n\in\operatorname{dom}\mu$, and $C_1,\dots,C_n\in\operatorname{dom}\lambda$. Since each B_i belongs to $\operatorname{dom}\mu$, there is a finite partition $\mathcal{V}\in\mathbb{V}$, such that each B_i is a union of elements of $\mathcal{V}$; denote by $\mathcal{V}_{B_i}$ the set of elements of $\mathcal{V}$ that make up B_i. Consider any $i\in\{1,\dots,n\}$: since $B_i\times C_i\subseteq A_f^+$, we find that for each element V of $\mathcal{V}_{B_i}$, it holds that

$$C_i \subseteq \{x\colon (\forall\omega\in B_i)(0\leq x\leq f(\omega))\} = [0,\underline{P}_{B_i}(f^+)] \subseteq [0,\underline{P}_V(f^+)].$$

As this holds for any $i\in\{1,\dots,n\}$, we deduce that

$$\bigcup_{i=1}^n B_i\times C_i = \bigcup_{i=1}^n \bigcup_{V\in\mathcal{V}_{B_i}} V\times C_i \subseteq \bigcup_{i=1}^n \bigcup_{V\in\mathcal{V}_{B_i}} V\times[0,\underline{P}_V(f^+)] \subseteq \bigcup_{V\in\mathcal{V}} V\times[0,\underline{P}_V(f^+)] \subseteq A_f^+.$$

This shows that, without loss of generality, we can restrict the supremum in Eq. (6) to those sets B_i that make up a finite partition of elements of $\operatorname{dom}\mu$, and sets $C_i=[0,\underline{P}_{B_i}(f^+)]$:

$$\kappa_*(A_f^+) = \sup_{\mathcal{V}\in\mathbb{V}} \sum_{V\in\mathcal{V}} \underline{P}_V(f^+)\mu(V) = \lim_{\mathcal{V}\in\mathbb{V}} \sum_{V\in\mathcal{V}} \underline{P}_V(f^+)\mu(V) = (S)\underline{\int} f^+\,\mathrm{d}\mu.$$

In a similar manner, we can show that $-\kappa^*(A_f^-) = (S)\underline{\int} f^-\,\mathrm{d}\mu$. Briefly, now with $\bigcup_{i=1}^n B_i\times C_i\supseteq A_f^-$:

$$\bigcup_{i\colon V\in\mathcal{V}_{B_i}} C_i \supseteq \{x\colon (\forall\omega\in V)(f(\omega)\leq x\leq 0)\} = [\underline{P}_V(f^-),0],$$

so

$$\bigcup_{i=1}^n B_i\times C_i = \bigcup_{V\in\mathcal{V}} \bigcup_{i\colon V\in\mathcal{V}_{B_i}} B_i\times C_i \supseteq \bigcup_{V\in\mathcal{V}} V\times[\underline{P}_V(f^-),0],$$

and therefore,

$$-\kappa^*(A_f^-) = -\inf_{\mathcal{V}\in\mathbb{V}} \sum_{V\in\mathcal{V}} (-\underline{P}_V(f^-))\mu(V) = \lim_{\mathcal{V}\in\mathbb{V}} \sum_{V\in\mathcal{V}} \underline{P}_V(f^-)\mu(V) = (S)\underline{\int} f^-\,\mathrm{d}\mu.$$

Finally, it follows by the comonotone additivity of the lower S-integral (apply Theorem 8 further on and the third statement of Theorem 2), that

$$(S)\underline{\int} f\,\mathrm{d}\mu = (S)\underline{\int} f^{+}\,\mathrm{d}\mu + (S)\underline{\int} f^{-}\,\mathrm{d}\mu = \kappa^{*}(A_f^{+}) - \kappa^{*}(A_f^{-}).$$

□

Proposition 6 provides a geometric interpretation for the lower S-integral. It is worth mentioning already that, as a consequence of Theorem 16 below, the proposition also provides a geometric interpretation for many other $\mathbb{V}$-integrals, and of the natural extension of a probability charge.

It also shows that the lower Lebesgue integral on $\mathbb{R}$, as introduced by Lebesgue [14], coincides with the lower S-integral, if we take for μ the Borel-Lebesgue measure as well (so that κ is the 2-dimensional Borel-Lebesgue measure).

3.2.4. *The Young–Stieltjes integral.* Let a and b be two finite real numbers such that $a < b$ and a non-decreasing *continuous* mapping $F\colon [a,b] \to \mathbb{R}$. Take for $\mathcal{F}$ the Borel σ-field on $[a,b]$, for μ_F the Lebesgue-Stieltjes measure on $[a,b]$ and for $\mathbb{V}$ the collection of all finite partitions in $\mathcal{F}$. Again, Conditions ($\mathbb{V}$1)–($\mathbb{V}$4) are satisfied, and now the lower $\mathbb{V}$-integral with respect to μ_F coincides with the lower Young–Stieltjes integral on gambles on $[a,b]$ as defined by Hildebrandt [11, Chap. VII, Def. 3.3]; such a gamble is $\mathbb{V}$-integrable with respect to μ_F if and only if it is Young–Stieltjes integrable with respect to F.

3.2.5. *The textbook integral.* In many textbooks (see for instance [3, 9, 10, 13, 15]), "the integral" $\int \cdot\,\mathrm{d}\mu$ of an $\mathcal{F}$-measurable gamble f with respect to a positive bounded measure μ on a σ-field $\mathcal{F}$ on Ω is defined as follows:

$$\int f\,\mathrm{d}\mu := (S)\underline{\int} \big(f \vee 0\big)\,\mathrm{d}\mu - (S)\underline{\int} \big((-f) \vee 0\big)\,\mathrm{d}\mu;$$

the difference on the right hand side is always well-defined since a gamble f is a bounded random variable. But, any $\mathcal{F}$-measurable gamble is S-integrable, and hence, so must be $f \vee 0$ and $(-f) \vee 0$ (this follows from Theorem 11 below). Therefore, for $\mathcal{F}$-measurable gambles, this textbook integral coincides with the S-integral, and is thus a particular instance of a $\mathbb{V}$-integral as well.

3.2.6. *Kadane and O'Hagan's uniform distribution on* $\mathbb{N}_0$. For a completely different example of a $\mathbb{V}$-integral, define the *residue sets* $R_m^r := \{km + r\colon k \in \mathbb{N}_0\} \subseteq \mathbb{N}_0$, for any m in $\mathbb{N}$ and $r = 0, \dots, m-1$. This means that $\ell \in R_m^r$ if and only if $\ell = r \mod m$, or in other words, if dividing ℓ by m leaves a remainder r.

Kadane and O'Hagan [12, p. 628–629, Sect. 4] suggest defining a uniform distribution on the set of natural numbers (with zero) $\mathbb{N}_0$ as any probability charge that takes the value $\frac{1}{m}$ in all the sets R_m^r, $m \in \mathbb{N}$ and $r = 0, \dots, m-1$. To see that such a probability charge indeed exists, consider for any $m \in \mathbb{N}$ the finite partition of $\mathbb{N}_0$ given by

$$\mathcal{V}_m := \{R_m^r\colon r = 0, \dots, m-1\},$$

and let $\mathbb{V} := \{\mathcal{V}_m\colon m \in \mathbb{N}\}$ be the collection of all such partitions. Note that the partition $\mathcal{V}_m$ has m different elements. The following lemma tells us that the set $\mathbb{V}$ is directed under the refinement relation $\preceq$: any $\mathcal{V}_m$ and $\mathcal{V}_n$ have for instance $\mathcal{V}_{mn}$ as a common refinement.

Lemma 7. *Let m and $n \in \mathbb{N}$. Then $\mathcal{V}_m \preceq \mathcal{V}_n$ if and only if there is some $k \in \mathbb{N}$ such that $n = km$.*

Proof. "if". We must show that $\mathcal{V}_m \preceq \mathcal{V}_{km}$, or equivalently, that for any $r = 0, \dots, m-1$, there is an $s \in \{0, \dots, km-1\}$, such that $R_{km}^s \subseteq R_m^r$. Simply take $s = r$.

"only if". If $\mathcal{V}_m \preceq \mathcal{V}_n$, then there is an $s \in \{0, \dots, m-1\}$ such that $R_n^0 \subseteq R_m^s$. Since $0 \in R_n^0$, also $0 \in R_m^s$, whence $s = 0$. Consequently, $\{\ell n \colon \ell \in \mathbb{N}_0\} \subseteq \{jm \colon j \in \mathbb{N}_0\}$, and therefore, considering $\ell = 1$, there is a $k \in \mathbb{N}$ such that $n = km$. $\square$

Now let $\mathcal{F}_m$ be the field generated by the partition $\mathcal{V}_m$, and let μ_m be the unique probability charge on $\mathcal{F}_m$ that satisfies $\mu_m(R_m^r) = \frac{1}{m}$ for all $r = 0, \dots, m-1$. If we define $\mathcal{F} := \bigcup_{m\in\mathbb{N}} \mathcal{F}_m$ then it is easy to see that $\mathcal{F}$ is a field as well, and that we can consistently define a unique probability charge μ on $\mathcal{F}$ that coincides with the μ_m on $\mathcal{F}_m$ and therefore on $\mathcal{V}_m$, as follows: if $A \in \mathcal{F}$ then there is some $m \in \mathbb{N}$ such that $A \in \mathcal{F}_m$ and then we let $\mu(A) := \mu_m(A)$. This probability charge coincides with the *natural density* on the natural numbers that is used in number theory.

Again, it is easy to see that $\mathbb{V}$ and μ satisfy the properties ($\mathbb{V}$1)–($\mathbb{V}$4): we have already argued above that ($\mathbb{V}$1) is satisfied; ($\mathbb{V}$2) and ($\mathbb{V}$3) follow from the fact that all elements of $\mathbb{V}$ are finite partitions; and for ($\mathbb{V}$4), simply note that $\mathcal{V}_1 = \{\mathbb{N}_0\}$.

So, we can define the $\mathbb{V}$-integral with respect to μ, and instead of using a Moore–Smith (net) limit, it is immediate that for this integral we can simply write a (sequence) limit over $\mathbb{N}$:

$$(\mathbb{V})\underline{\int} f \,\mathrm{d}\mu = \lim_{m\to+\infty} \frac{1}{m} \sum_{r=0}^{m-1} \inf_{k\in\mathbb{N}} f(km+r) = \sup_{m\in\mathbb{N}} \frac{1}{m} \sum_{r=0}^{m-1} \inf_{k\in\mathbb{N}} f(km+r). \tag{7}$$

Note that for events, we find that

$$(\mathbb{V})\underline{\int} I_A \,\mathrm{d}\mu = \lim_{m\to+\infty} \frac{1}{m} |\{r \colon R_m^r \subseteq A\}| = \sup_{m\in\mathbb{N}} \frac{1}{m} |\{r \colon R_m^r \subseteq A\}|. \tag{8}$$

We shall see further on in Section 4.2.3 that this completely agrees with a result proven in an entirely different manner by Kadane and O'Hagan [12, Thm. 6].

3.3. **Complete monotonicity theorem.** We now prove a simple theorem, which has many interesting consequences.

Theorem 8. $(\mathbb{V})\underline{\int} \cdot \,\mathrm{d}\mu$ *is a completely monotone exact functional on* $\mathcal{L}$, *and* $(\mathbb{V})\overline{\int} \cdot \,\mathrm{d}\mu$ *is its conjugate.*

Proof. First, for any non-empty subset V of $\mathcal{F}$, we have from Proposition 3 that $\underline{P}_V$ is a completely monotone coherent lower prevision on $\mathcal{L}$. This means that $(\mathbb{V})\underline{\int} \cdot \,\mathrm{d}\mu$ is the point-wise Moore–Smith limit of non-negative linear combinations (by Eqs. (2) and (5)) of completely monotone coherent lower previsions. Since exactness and complete monotonicity are preserved under non-negative linear combination and point-wise limits, this implies that $(\mathbb{V})\underline{\int} \cdot \,\mathrm{d}\mu$ must be exact and completely monotone as well.

Finally, for any gamble f on Ω and any $\mathcal{V}$ in $\mathbb{V}$ it holds that

$$\sum_{V\in\mathcal{V}} \overline{P}_V(-f)\mu(V) = -\sum_{V\in\mathcal{V}} \underline{P}_V(f)\mu(V);$$

by taking the limit over $\mathcal{V} \in \mathbb{V}$ on both sides of this equality, we find that $(\mathbb{V})\overline{\int} -f \,\mathrm{d}\mu = -(\mathbb{V})\underline{\int} f \,\mathrm{d}\mu$, which completes the proof. $\square$

By Theorem 2, it follows that $(\mathbb{V})\underline{\int} \cdot \,\mathrm{d}\mu$ is comonotone additive on $\mathcal{L}$, and representable by a Choquet integral with respect to the restriction of $(\mathbb{V})\underline{\int} \cdot \,\mathrm{d}\mu$ to events. If we denote this restriction by $\underline{\mu}_{\mathbb{V}}$, then $\underline{\mu}_{\mathbb{V}}$ is the completely monotone exact set function on $\wp(\Omega)$, given by

$$\underline{\mu}_{\mathbb{V}}(A) = \lim_{\mathcal{V}\in\mathbb{V}} \sum_{V\in\mathcal{V}, V\subseteq A} \mu(V) \tag{9}$$

for any subset A of Ω, and we find that for any gamble f on Ω

$$(\mathbb{V})\underline{\int} f\,\mathrm{d}\mu = (C)\int f\,\mathrm{d}\underline{\mu}_{\mathbb{V}} = \mu(\Omega)\inf f + (R)\int_{\inf f}^{\sup f} \underline{\mu}_{\mathbb{V}}(\{f \geq x\})\,\mathrm{d}x. \tag{10}$$

Similarly, if we denote the restriction of $(\mathbb{V})\overline{\int}\cdot\,\mathrm{d}\mu$ by $\overline{\mu}_{\mathbb{V}}$, then $\overline{\mu}_{\mathbb{V}}$ is the completely alternating conjugate exact set function on $\wp(\Omega)$, given by

$$\overline{\mu}_{\mathbb{V}}(A) = 1 - \underline{\mu}_{\mathbb{V}}(A^c) = \lim_{\mathcal{V}\in\mathbb{V}} \sum_{V\in\mathcal{V}, V\cap A\neq\emptyset} \mu(V)$$

for any subset A of Ω, and we find that for any gamble f on Ω

$$(\mathbb{V})\overline{\int} f\,\mathrm{d}\mu = (C)\int f\,\mathrm{d}\overline{\mu}_{\mathbb{V}} = \mu(\Omega)\inf f + (R)\int_{\inf f}^{\sup f} \overline{\mu}_{\mathbb{V}}(\{f \geq x\})\,\mathrm{d}x. \tag{11}$$

Call a subset A of Ω $\mathbb{V}$-integrable if its indicator I_A is, or in other words if $\overline{\mu}_{\mathbb{V}}(A) = \underline{\mu}_{\mathbb{V}}(A)$. The following result gives an interesting characterisation of the $\mathbb{V}$-integrability of events and gambles.

Proposition 9. *A subset A of Ω is $\mathbb{V}$-integrable if and only if for all $\epsilon > 0$ there is some $\mathcal{V}_\epsilon$ in $\mathbb{V}$ such that:*

$$\sum_{\substack{V\in\mathcal{V}_\epsilon\\ V\cap A\neq\emptyset, V\cap A^c\neq\emptyset}} \mu(V) < \epsilon.$$

Moreover, a gamble f on Ω is $\mathbb{V}$-integrable if and only if $\{f \geq x\}$ is $\mathbb{V}$-integrable for all but a countable number of elements x of $[\inf f, \sup f]$.

Proof. To prove the first statement, observe that

$$(\mathbb{V})\overline{\int} I_A\,\mathrm{d}\mu - (\mathbb{V})\underline{\int} I_A\,\mathrm{d}\mu = \overline{\mu}_{\mathbb{V}}(A) - \underline{\mu}_{\mathbb{V}}(A) = \lim_{\mathcal{V}\in\mathbb{V}} \left[\sum_{V\in\mathcal{V}} \overline{P}_V(A)\mu(V) - \sum_{V\in\mathcal{V}} \underline{P}_V(A)\mu(V)\right],$$

and since from Lemma 5 the expression between brackets is non-increasing in $\mathcal{V}$ with respect to the relation $\preceq$ on $\mathbb{V}$, we find that A is $\mathbb{V}$-integrable if and only if for all $\epsilon > 0$ there is some $\mathcal{V}_\epsilon$ in $\mathbb{V}$ such that

$$\sum_{V\in\mathcal{V}_\epsilon} \left[\overline{P}_V(A) - \underline{P}_V(A)\right]\mu(V) < \epsilon.$$

Now observe that $\overline{P}_V(A) - \underline{P}_V(A)$ equals one if both $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$, and is zero otherwise.

Let's now prove the second statement. First assume that $\{f \geq x\}$ is $\mathbb{V}$-integrable, so that

$$\overline{\mu}_{\mathbb{V}}(\{f \geq x\}) = \underline{\mu}_{\mathbb{V}}(\{f \geq x\})$$

for all but a countable number of elements x of $[\inf f, \sup f]$. Since two Riemann-integrable bounded functions that differ only in a countable set of points have the same Riemann integral, we see, using Eqs. (10) and (11), that $(\mathbb{V})\overline{\int} f\,\mathrm{d}\mu = (\mathbb{V})\underline{\int} f\,\mathrm{d}\mu$, so f is indeed $\mathbb{V}$-integrable. Conversely, assume that f is $\mathbb{V}$-integrable, then it follows from Eqs. (10) and (11) that

$$(R)\int_{\inf f}^{\sup f} \left[\overline{\mu}_{\mathbb{V}}(\{f \geq x\}) - \underline{\mu}_{\mathbb{V}}(\{f \geq x\})\right]\,\mathrm{d}x = 0.$$

Now the Riemann integral of a non-negative Riemann integrable function g is zero if and only if the function g is non-zero only in its points of discontinuity (see for instance [11, p. 76]). Since here $g(x) = \overline{\mu}_{\mathbb{V}}(\{f \geq x\}) - \underline{\mu}_{\mathbb{V}}(\{f \geq x\})$ is a difference of two non-increasing functions, it is Riemann integrable and has at most a countable number of discontinuities. □

4. A general notion of integrability

4.1. **Integrability for exact functionals.** The results of the previous sections naturally lead us to define a notion of integrability for arbitrary exact functionals, which generalises all existing notions of integrability mentioned before.

Let $\underline{\Gamma}$ be an exact functional defined on some set of gambles $\operatorname{dom}\underline{\Gamma} \subseteq \mathcal{L}$. Let $\underline{E}_{\underline{\Gamma}}$ be its natural extension. Let's then call a gamble f on Ω $\underline{\Gamma}$-*integrable* if $\underline{E}_{\underline{\Gamma}}(f) = \overline{E}_{\underline{\Gamma}}(f)$. This means that all the positive linear functionals on $\mathcal{L}$ that dominate $\underline{\Gamma}$ on its domain and have the same norm as $\underline{\Gamma}$, assign the same value $\underline{E}_{\underline{\Gamma}}(f) = \overline{E}_{\underline{\Gamma}}(f)$ to the gamble f. We denote the set of all $\underline{\Gamma}$-integrable gambles by

$$\mathcal{L}_{\underline{\Gamma}} = \left\{ f \in \mathcal{L} \colon \underline{E}_{\underline{\Gamma}}(f) = \overline{E}_{\underline{\Gamma}}(f) \right\}.$$

We denote the restriction of $\underline{E}_{\underline{\Gamma}}$ to $\mathcal{L}_{\underline{\Gamma}}$ by $E_{\underline{\Gamma}}$. When $\underline{\Gamma}$ is (essentially) a bounded positive charge μ, then this set of integrable gambles $\mathcal{L}_{\underline{\Gamma}}$ is closely related to the Jordan field of μ, as we shall explain in the following subsection. For now, we concentrate on the properties of the set of integrable gambles $\mathcal{L}_{\underline{\Gamma}}$ for *any* exact functional $\underline{\Gamma}$.

Proposition 10. *$\mathcal{L}_{\underline{\Gamma}}$ is a uniformly closed linear space that contains all constant gambles, and $E_{\underline{\Gamma}}$ is a linear exact functional on $\mathcal{L}_{\underline{\Gamma}}$ whose norm is $\|\underline{\Gamma}\|$. Moreover,*

$$\mathcal{L}_{\underline{\Gamma}} = \left\{ f \in \mathcal{L} \colon (\forall g \in \mathcal{L})(\underline{E}_{\underline{\Gamma}}(f+g) = \underline{E}_{\underline{\Gamma}}(f) + \underline{E}_{\underline{\Gamma}}(g)) \right\}. \tag{12}$$

Proof. Consider f and g in $\mathcal{L}_{\underline{\Gamma}}$, and non-negative real a and b. Then since $\underline{E}_{\underline{\Gamma}}$ is an exact functional, we get

$$\underline{E}_{\underline{\Gamma}}(af+bg) \leq \overline{E}_{\underline{\Gamma}}(af+bg) \leq a\overline{E}_{\underline{\Gamma}}(f) + b\overline{E}_{\underline{\Gamma}}(g) = a\underline{E}_{\underline{\Gamma}}(f) + b\underline{E}_{\underline{\Gamma}}(g) \leq \underline{E}_{\underline{\Gamma}}(af+bg),$$

so $af+bg \in \mathcal{L}_{\underline{\Gamma}}$ as well. Similarly $\overline{E}_{\underline{\Gamma}}(-f) = -\underline{E}_{\underline{\Gamma}}(f) = -\overline{E}_{\underline{\Gamma}}(f) = \underline{E}_{\underline{\Gamma}}(-f)$, so $-f \in \mathcal{L}_{\underline{\Gamma}}$ too. Also for any constant gamble μ, $\underline{E}_{\underline{\Gamma}}(\mu) = \overline{E}_{\underline{\Gamma}}(\mu) = \mu\|\underline{\Gamma}\|$, so $\mu \in \mathcal{L}_{\underline{\Gamma}}$. This means that $\mathcal{L}_{\underline{\Gamma}}$ is a linear space that contains all constant gambles, and that $E_{\underline{\Gamma}}$ is a linear functional on that space. Since $\underline{E}_{\underline{\Gamma}}$ is the lower envelope of all positive linear functionals on $\mathcal{L}$ that dominate $\underline{\Gamma}$, and all such positive linear functionals coincide with $\underline{E}_{\underline{\Gamma}}$, and therefore with $E_{\underline{\Gamma}}$, on $\mathcal{L}_{\underline{\Gamma}}$, $E_{\underline{\Gamma}}$ has exact and linear extensions to all of $\mathcal{L}$, and is therefore a linear exact functional. Its norm is given by $\|E_{\underline{\Gamma}}\| = E_{\underline{\Gamma}}(1) = \underline{E}_{\underline{\Gamma}}(1) = \|\underline{\Gamma}\|$.

Next, suppose that a sequence f_n, $n \in \mathbb{N}$ of gambles in $\mathcal{L}_{\underline{\Gamma}}$ converges uniformly to a gamble f. Then, because both $\underline{E}_{\underline{\Gamma}}$ and $\overline{E}_{\underline{\Gamma}}$ are continuous with respect to the supremum norm, by exactness, we see that

$$\underline{E}_{\underline{\Gamma}}(f) = \lim_{n\to\infty} \underline{E}_{\underline{\Gamma}}(f_n) = \lim_{n\to\infty} \overline{E}_{\underline{\Gamma}}(f_n) = \overline{E}_{\underline{\Gamma}}(f),$$

so $f \in \mathcal{L}_{\underline{\Gamma}}$ as well, and $\mathcal{L}_{\underline{\Gamma}}$ is uniformly closed.

Finally, to prove that Eq. (12) holds, consider any gamble f. First assume that it belongs to the set on the right hand side. Then for $g = -f$, we see that

$$0 = \underline{E}_{\underline{\Gamma}}(f-f) = \underline{E}_{\underline{\Gamma}}(f) + \underline{E}_{\underline{\Gamma}}(-f) = \underline{E}_{\underline{\Gamma}}(f) - \overline{E}_{\underline{\Gamma}}(f),$$

so indeed $f \in \mathcal{L}_{\underline{\Gamma}}$. Conversely, assume that $f \in \mathcal{L}_{\underline{\Gamma}}$, and consider a arbitrary gamble g. Then using the super-additivity of the exact functional $\underline{E}_{\underline{\Gamma}}$, we get

$$\underline{E}_{\underline{\Gamma}}(f+g) \geq \underline{E}_{\underline{\Gamma}}(f)+\underline{E}_{\underline{\Gamma}}(g) = \overline{E}_{\underline{\Gamma}}(f)+\underline{E}_{\underline{\Gamma}}(f+g-f) \geq \overline{E}_{\underline{\Gamma}}(f)+\underline{E}_{\underline{\Gamma}}(f+g)+\underline{E}_{\underline{\Gamma}}(-f) = \underline{E}_{\underline{\Gamma}}(f+g),$$

whence indeed $\underline{E}_{\underline{\Gamma}}(f+g) = \underline{E}_{\underline{\Gamma}}(f) + \underline{E}_{\underline{\Gamma}}(g)$. □

We have the following stronger result when $\underline{E}_{\underline{\Gamma}}$ is 2-monotone on a sufficiently rich domain.

Theorem 11. *If $\underline{E}_{\underline{\Gamma}}$ is 2-monotone on some linear lattice that includes $\mathcal{L}_{\underline{\Gamma}}$, then $\mathcal{L}_{\underline{\Gamma}}$ is a uniformly closed linear lattice that contains all constant gambles.*

Proof. To prove that $\mathcal{L}_{\underline{\Gamma}}$ is a uniformly closed linear lattice, it suffices to show that $|f|$ is $\underline{\Gamma}$-integrable if f is. Let $\mathcal{K}$ be a linear lattice that includes $\mathcal{L}_{\underline{\Gamma}}$. Consider f in $\mathcal{L}_{\underline{\Gamma}}$, then both f and $|f|$ belong to $\mathcal{K}$. Since $\underline{E}_{\underline{\Gamma}}$ is 2-monotone on $\mathcal{K}$, we find by Lemma 12 that

$$0 = \overline{E}_{\underline{\Gamma}}(f) - \underline{E}_{\underline{\Gamma}}(f) \geq \overline{E}_{\underline{\Gamma}}(|f|) - \underline{E}_{\underline{\Gamma}}(|f|) \geq 0,$$

whence $\overline{E}_{\underline{\Gamma}}(|f|) = \underline{E}_{\underline{\Gamma}}(|f|)$, so $|f|$ is indeed $\underline{\Gamma}$-integrable. □

Lemma 12. *Let $\underline{\Gamma}$ be a 2-monotone exact functional defined on a linear lattice of gambles. Then $\overline{\Gamma}(f) - \underline{\Gamma}(f) \geq \overline{\Gamma}(|f|) - \underline{\Gamma}(|f|)$ for all f in $\operatorname{dom}\underline{\Gamma}$.*

Proof. Assume that $\underline{\Gamma}$ is 2-monotone on $\operatorname{dom}\underline{\Gamma}$, and let $f \in \operatorname{dom}\underline{\Gamma}$. Since $|f| = f \vee -f$ and $-|f| = f \wedge -f$ both belong to $\operatorname{dom}\underline{\Gamma}$, we find from the 2-monotonicity of $\underline{\Gamma}$ that

$$\underline{\Gamma}(|f|) + \underline{\Gamma}(-|f|) \geq \underline{\Gamma}(f) + \underline{\Gamma}(-f),$$

and the desired inequality follows immediately. □

Corollary 13. *Suppose that $\underline{E}_{\underline{\Gamma}}$ is 2-monotone on some linear lattice of gambles that includes $\mathcal{L}_{\underline{\Gamma}}$, and let f, g be two $\underline{\Gamma}$-integrable gambles. Let $N = \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\}$. Then*

$$\underline{E}_{\underline{\Gamma}}(N) = 0 \Rightarrow \overline{E}_{\underline{\Gamma}}(|f - g|) = 0 \Rightarrow E_{\underline{\Gamma}}(f) = E_{\underline{\Gamma}}(g).$$

Proof. Consider two $\underline{\Gamma}$-integrable gambles f and g, such that $\underline{E}_{\underline{\Gamma}}(N) = 0$. Let $\lambda = \sup|f - g|$, then $|f - g| \leq I_N\lambda$. Since $\mathcal{L}_{\underline{\Gamma}}$ is a linear lattice, we know that $|f - g|$ is $\underline{\Gamma}$-integrable, and consequently

$$0 \leq \overline{E}_{\underline{\Gamma}}(|f - g|) = \underline{E}_{\underline{\Gamma}}(|f - g|) \leq \lambda\underline{E}_{\underline{\Gamma}}(N) = 0.$$

Since it follows from the exactness of $\underline{E}_{\underline{\Gamma}}$ that

$$0 \leq |\underline{E}_{\underline{\Gamma}}(f) - \underline{E}_{\underline{\Gamma}}(g)| \leq \overline{E}_{\underline{\Gamma}}(|f - g|),$$

we find that indeed $E_{\underline{\Gamma}}(f) = \underline{E}_{\underline{\Gamma}}(f) = \underline{E}_{\underline{\Gamma}}(g) = E_{\underline{\Gamma}}(g)$. □

The converses of the two implications in this corollary do not hold in general: for the first, let Γ be a linear functional on the set of all gambles on $\mathbb{N}$ such that $\Gamma(A) = 0$ for any finite set A, and let f and g be gambles such that $f(n) = g(n) + \frac{1}{n}$ for all $n \in \mathbb{N}$; for the second consider for $\underline{\Gamma}$ the infimum operator on $\mathcal{L}$ and let f and g be indicators of (proper and different) subsets of Ω.

4.2. Representation of natural extension by lower $\mathbb{V}$-integrals.

4.2.1. *$\mathbb{V}$-integrability and the Jordan Extension.* To complete the paper, let us consider the special case that $\underline{\Gamma}$ is defined on the set $\mathcal{L}$ of all gambles on Ω as the completely monotone exact functional $(\mathbb{V})\underline{\int}\cdot\,\mathrm{d}\mu$, i.e., the lower $\mathbb{V}$-integral with respect to a positive bounded charge μ, defined on a field $\mathcal{F}$ of subsets of Ω.

In this case, since $\underline{\Gamma}$ is exact on $\mathcal{L}$, it coincides with its natural extension on $\mathcal{L}$, and a gamble f is $\underline{\Gamma}$-integrable if and only if it is $\mathbb{V}$-integrable with respect to μ:

$$\mathcal{L}_{\underline{\Gamma}} = \mathcal{L}_{(\mathbb{V},\mu)} := \left\{f \in \mathcal{L} \colon (\mathbb{V})\underline{\int} f\,\mathrm{d}\mu = (\mathbb{V})\overline{\int} f\,\mathrm{d}\mu\right\}.$$

By Theorem 11, $\mathcal{L}_{(\mathbb{V},\mu)}$ is a uniformly closed linear lattice that contains all constant gambles. Its restriction to events is the field $\mathcal{J}_{(\mathbb{V},\mu)}$ satisfying

$$\mathcal{J}_{(\mathbb{V},\mu)} := \left\{A \subseteq \Omega \colon \underline{\mu}_{\mathbb{V}}(A) = \overline{\mu}_{\mathbb{V}}(A)\right\},$$

and we deduce from Proposition 9 that $f \in \mathcal{L}_{(\mathbb{V},\mu)}$ if and only if $\{f \geq x\} \in \mathcal{J}_{(\mathbb{V},\mu)}$ for all but a countable number of x in $[\inf f, \sup f]$. We shall sometimes use the notation $\mu_{\mathbb{V}}$ for the restriction of $\underline{\mu}_{\mathbb{V}}$ (or $\overline{\mu}_{\mathbb{V}}$, it doesn't matter which) to $\mathcal{J}_{(\mathbb{V},\mu)}$.

On the other hand, we can identify the bounded positive charge μ with a functional Γ_μ on $I_{\mathcal{F}}$. Since μ is also completely monotone on $\mathcal{F}$, the fourth statement of Theorem 2 tells us that its natural extension $\underline{E}_\mu$ is given by $\underline{E}_\mu(f) = (C)\int f \,\mathrm{d}\mu_*$, where μ_* is the inner set function of μ, given by $\mu_*(A) = \sup\{\mu(B) \colon B \in \mathcal{F} \text{ and } B \subseteq A\}$ for any $A \subseteq \Omega$. Moreover, the conjugate is given by $\overline{E}_\mu(f) = (C)\int f \,\mathrm{d}\mu^*$, where μ^* is the conjugate of the inner set function μ_*, that is, $\mu^*(A) = \mu(\Omega) - \mu_*(A^c) = \inf\{\mu(B) \colon B \in \mathcal{F} \text{ and } A \subseteq B\}$ for any $A \subseteq \Omega$. We also deduce from the fourth statement of Theorem 2 that $\underline{E}_\mu$ is completely monotone on all of $\mathcal{L}$. The set $\mathcal{L}_\mu := \mathcal{L}_{\Gamma_\mu}$ of μ-integrable (or more precisely Γ_μ-integrable) gambles is given by

$$\mathcal{L}_\mu := \left\{ f \in \mathcal{L} \colon (C)\int f \,\mathrm{d}\mu_* = (C)\int f \,\mathrm{d}\mu^* \right\}.$$

By Theorem 11, this is a uniformly closed linear lattice that contains all constant gambles, and it is the largest set of gambles to which the set function μ can be uniquely extended as a linear exact functional. Its restriction $\mathcal{J}_\mu$ to events is the *Jordan*, or *Carathéodory*, *field* of μ:

$$\mathcal{J}_\mu = \{A \subseteq \Omega \colon \mu_*(A) = \mu^*(A)\},$$

the largest field to which the positive bounded charge μ can be uniquely extended as a positive bounded charge—which is usually called the *Jordan extension* of μ. Observe that $\mathcal{F} \subseteq \mathcal{J}_\mu$. An immediate counterpart of Proposition 9 tells us that a gamble f is μ-integrable, i.e., $f \in \mathcal{L}_\mu$, if and only if $\{f \geq x\}$ belongs to the Jordan field $\mathcal{J}_\mu$ for all but a countable number of x in $[\inf f, \sup f]$.

Together with the first part of Proposition 9, we immediately infer the following interesting characterisation of the Jordan field $\mathcal{J}_\mu$ of a positive bounded charge μ. A similar result was proven in a completely different way by Walley [19, Cor. 3.1.9].

Proposition 14. *A subset A of Ω belongs to the Jordan field $\mathcal{J}_\mu$ of a positive bounded charge μ defined on a field $\mathcal{F}$ (or in other words, μ can be uniquely extended as a positive bounded charge to A) if and only if for all $\epsilon > 0$ there are disjoint F_1 and F_2 in $\mathcal{F}$ such that $F_1 \subseteq A$, $F_2 \subseteq A^c$ and $\mu(\Omega \setminus (F_1 \cup F_2)) < \epsilon$.*

4.2.2. *$\mathbb{V}$-Integral Representation Theorem.* We now ask ourselves what is the relationship between the sets $\mathcal{L}_{(\mathbb{V},\mu)}$ and $\mathcal{L}_\mu$, and between $\mathcal{J}_{(\mathbb{V},\mu)}$ and $\mathcal{J}_\mu$, respectively.

Proposition 15. *For any $A \subseteq \Omega$, it holds that*

$$\underline{\mu}_{\mathbb{V}}(A) \leq \mu_*(A) \leq \mu^*(A) \leq \overline{\mu}_{\mathbb{V}}(A),$$

and therefore also that $\mathcal{J}_{(\mathbb{V},\mu)} \subseteq \mathcal{J}_\mu$ and $\mathcal{L}_{(\mathbb{V},\mu)} \subseteq \mathcal{L}_\mu$.

Proof. Conjugacy arguments tell us it suffices to prove that $\underline{\mu}_{\mathbb{V}}(A) \leq \mu_*(A)$. Consider any $\mathcal{V}$ in $\mathbb{V}$ and let $A_{\mathcal{V}} := \bigcup\{V \in \mathcal{V} \colon V \subseteq A\}$. Then clearly $A_{\mathcal{V}} \subseteq A$, and since $\mathcal{V}$ is a finite collection of elements of $\mathcal{F}$, we have that $A_{\mathcal{V}} \in \mathcal{F}$. Consequently, using ($\mathbb{V}$3) and the definition of μ_*, we find that

$$\mu_*(A) \geq \mu(A_{\mathcal{V}}) = \sum_{V \in \mathcal{V}, V \subseteq A} \mu(V),$$

and then it suffices to take the Moore–Smith limit over $\mathcal{V} \in \mathbb{V}$ on both sides of the inequality. □

Theorem 16. *The following statements are equivalent:*

(i) $\underline{\mu}_{\mathbb{V}}$ coincides with μ on $\mathcal{F}$;
(ii) $\overline{\mu}_{\mathbb{V}}$ coincides with μ on $\mathcal{F}$;

(iii) *for all A in $\mathcal{F}$ and for any $\epsilon > 0$ there is a $\mathcal{V}_{\epsilon,A}$ in $\mathbb{V}$ such that* $\sum_{V\in\mathcal{V}_{\epsilon,A},V\subseteq A}\mu(V) > \mu(A)-\epsilon$;

(iv) *for all A in $\mathcal{F}$ and for any $\epsilon > 0$ there is a $\mathcal{V}_{\epsilon,A}$ in $\mathbb{V}$ such that* $\sum_{V\in\mathcal{V}_{\epsilon,A},V\cap A\neq\emptyset}\mu(V) < \mu(A)+\epsilon$;

(v) $\underline{\mu}_{\mathbb{V}}$ *coincides with* μ_* *on all events;*

(vi) $\overline{\mu}_{\mathbb{V}}$ *coincides with* μ^* *on all events;*

(vii) $(\mathbb{V})\underline{\int}\cdot\,\mathrm{d}\mu$ *coincides with* $\underline{E}_\mu$ *on all gambles;*

(viii) $(\mathbb{V})\overline{\int}\cdot\,\mathrm{d}\mu$ *coincides with* $\overline{E}_\mu$ *on all gambles;*

(ix) $\mathcal{J}_{(\mathbb{V},\mu)} = \mathcal{J}_\mu$;

(x) $\mathcal{L}_{(\mathbb{V},\mu)} = \mathcal{L}_\mu$;

(xi) $\mathcal{F} \subseteq \mathcal{J}_{(\mathbb{V},\mu)}$.

Proof. It is clear from conjugacy considerations that (i) and (ii) are equivalent. So are, respectively, (iii) and (iv), (v) and (vi), and (vii) and (viii). Moreover, (ix) and (x) are also easily seen to be equivalent if we look at Proposition 9 and its immediate counterpart for μ-integrability. This nearly halves the number of things to prove. We now give of circular proof for the remaining statements.

We first prove that (i) implies (iii). Consider any A in $\mathcal{F}$, and assume that $\underline{\mu}_{\mathbb{V}}(A) = \mu(A)$. Then it follows from Eq. (9) that for every $\epsilon < 0$ there is some $\mathcal{V}_{\epsilon,A}$ in $\mathbb{V}$ such that

$$\sum_{V\in\mathcal{V}_{\epsilon,A}}\underline{P}_V(A)\mu(V) > \mu(A)-\epsilon.$$

Now observe that $\underline{P}_V(A)$ equals one if $V\subseteq A$ and is zero otherwise.

To prove that (iii) implies (v), consider $A\subseteq\Omega$ and $\epsilon > 0$. It follows from the definition of $\mu_*(A)$ that there is some B_ϵ in $\mathcal{F}$ such that $B_\epsilon\subseteq A$ and $\mu(B_\epsilon) > \mu_*(A)-\epsilon$, and from (iii) we deduce that there is some $\mathcal{V}_{\epsilon,B_\epsilon}$ in $\mathcal{V}$ such that $\sum_{V\in\mathcal{V}_{\epsilon,B_\epsilon}V\subseteq B_\epsilon}\mu(V) > \mu(B_\epsilon)-\epsilon$ and consequently $\sum_{V\in\mathcal{V}_{\epsilon,B_\epsilon},V\subseteq A}\mu(V) > \mu_*(A)-2\epsilon$. Moreover, it follows from Lemma 5 that $\underline{\mu}_{\mathbb{V}}(A) \geq \sum_{V\in\mathcal{V}_{\epsilon,B_\epsilon},V\subseteq A}\mu(V)$, whence $\underline{\mu}_{\mathbb{V}}(A) > \mu_*(A)-2\epsilon$. Since this holds for any $\epsilon > 0$, we get $\underline{\mu}_{\mathbb{V}}(A)\geq\mu_*(A)$, and the converse inequality follows from Proposition 15.

It is immediate that (v) implies (vii), because $\underline{E}_\mu$ is a Choquet functional with respect to μ_*, and $(\mathbb{V})\underline{\int}\cdot\,\mathrm{d}\mu$ a Choquet functional with respect to $\underline{\mu}_{\mathbb{V}}$.

That (vii) implies (ix) follows almost by definition of $\mathcal{J}_{(\mathbb{V},\mu)}$ and $\mathcal{J}_\mu$.

If (ix) holds, then (xi) follows, since always $\mathcal{F}\subseteq\mathcal{J}_\mu$ (because μ is monotone, so μ and μ_* and μ^* coincide on $\mathcal{F}$).

If (xi) holds, then $\underline{\mu}_{\mathbb{V}}$ coincides with $\overline{\mu}_{\mathbb{V}}$ on $\mathcal{F}$, so it follows from Proposition 15 that it also coincides with μ_* and therefore also with μ on $\mathcal{F}$. Therefore (i) follows. □

4.2.3. *Examples of representation.* If the set $\mathbb{V}$ is made up of all finite partitions of Ω whose elements belong to $\mathcal{F}$, as is the case for the Lebesgue, Young–Stieltjes, Dunford, and S-integral, described in Sections 3.2.3–3.2.5, then it is easy to see that Condition (iii) is trivially satisfied, and so, therefore, are all the others. The corresponding lower $\mathbb{V}$-integrals can therefore all be used as expressions for the natural extension $\underline{E}_\mu$ of a positive bounded charge μ.

In case of Riemann integration, or more generally, Riemann–Stieltjes integration with continuous F (see Sections 3.2.1 and 3.2.2), Condition (iii) is again trivially satisfied, if we take for $\mathcal{F}$ the field consisting of all finite unions of intervals in $[a,b]$ (note that this is *not* a σ-field), and μ the unique charge on $\mathcal{F}$ that satisfies $\mu([x,y]) = F(y)-F(x)$ for every $a\leq x<y\leq b$. In the next section, we shall prove a more general representation theorem that allows us to also treat the case of discontinuous F.

Let us also take a look at the consequences of Theorem 16 for Kadane and O'Hagan's proposal for a uniform distribution on $\mathbb{N}_0$, discussed in Section 3.2.6. Since the $\mathcal{V}_m$ defined there are finite partitions of $\mathbb{N}_0$, Condition (iii) of Theorem 16 is trivially satisfied, and therefore, all the other statements of the theorem hold as well. In particular, the lower $\mathbb{V}$-integral associated with the probability charge μ on $\mathcal{F}$, given by Eq. (7), is a completely monotone coherent lower prevision. Its restriction $\underline{\mu}_{\mathbb{V}}$ to events, given by Eq. (8), is a completely monotone coherent lower probability, and it coincides with the inner set function μ_* of μ. This also tells us that the $\mathbb{V}$-integral coincides everywhere with the natural extension $\underline{E}_\mu$ of the probability charge μ, and in particular, we deduce that it is the lower envelope of all linear previsions (probability charges) that coincide with μ on $\mathcal{F}$, or equivalently, that assign the value $\frac{1}{m}$ to all the residue sets R_m^r for $r = 0, \dots, m-1$ and $m \in \mathbb{N}$. This generalises Kadane and O'Hagan's Theorem 6 in [12].

4.2.4. *A more general representation theorem.* The representation result established in Theorem 16 allows us to express most of the instances of lower $\mathbb{V}$-integrals as the natural extension of a finitely additive set function μ defined on a field. It cannot, however, be applied to the Riemann-Stieljes integral associated with a discontinuous F on a compact interval $[a, b]$. Indeed, the charge μ (defined on a field) induced by such F does not satisfy $\mathcal{F} \subseteq \mathcal{J}_{\mathbb{V},\mu}$, where, from Section 3.2.2, $\mathbb{V}$ is the collection of all finite collections of closed intervals that overlap only on their borders and whose union is $[a, b]$. To see that $\mathcal{F} \not\subseteq \mathcal{J}_{(\mathbb{V},\mu)}$, assume for instance that $F(x+) \neq F(x)$. Then $\underline{\mu}_{\mathbb{V}}([a, x]) = F(a) - F(x)$ but $\overline{\mu}_{\mathbb{V}}([a, x]) = F(a) - F(x+)$, and hence, $[a, x]$ belongs to $\mathcal{F}$, but not to $\mathcal{J}_{(\mathbb{V},\mu)}$. As a consequence, Theorem 16 is not applicable.

In this section, we remedy this situation by proving a simple generalised version of Theorem 16, in terms of 2-monotone set functions on lattices of events, rather than bounded positive charges on fields. First, we establish a number of simple preliminary results.

Lemma 17. *Let μ be a bounded positive charge on a field $\mathcal{F}$, and let η be a 2-monotone set function defined on all events, such that $\eta \leq \mu$. Then $\mathcal{H} := \{A \in \mathcal{F} \colon \eta(A) = \mu(A)\}$ is a lattice of events.*

Proof. Consider V_1 and V_2 in $\mathcal{H}$. Then $\eta(V_1 \cap V_2) + \eta(V_1 \cup V_2) \leq \mu(V_1 \cap V_2) + \mu(V_1 \cup V_2) = \mu(V_1) + \mu(V_2) = \eta(V_1) + \eta(V_2) \leq \eta(V_1 \cap V_2) + \eta(V_1 \cup V_2)$, where the last inequality is a consequence of the 2-monotonicity of η. Hence, $\eta(V_1 \cap V_2) = \mu(V_1 \cap V_2)$ and $\eta(V_1 \cup V_2) = \mu(V_1 \cap V_2)$, and consequently $V_1 \cup V_2$ and $V_1 \cap V_2$ belong to $\mathcal{H}$. We deduce that $\mathcal{H}$ is closed under finite intersections and unions, so it is a lattice of events. □

Now consider any $\mathbb{V}$ associated with the bounded positive charge μ on $\mathcal{F}$, and satisfying ($\mathbb{V}$1)–($\mathbb{V}$4). Also consider the corresponding collection $\mathfrak{V}_{\mathbb{V}}$ of all events V in elements $\mathcal{V}$ of $\mathbb{V}$:

$$\mathfrak{V}_{\mathbb{V}} := \{V \colon (\exists \mathcal{V} \in \mathbb{V})(V \in \mathcal{V})\}$$

Since an arbitrary intersection of lattices of events is still a lattice of events, we can consider the lattice of events $\mathfrak{L}_{\mathbb{V}}$ generated by $\mathfrak{V}_{\mathbb{V}}$, i.e., the smallest lattice of events that includes $\mathfrak{V}_{\mathbb{V}}$. Observe that $\mathfrak{V}_{\mathbb{V}} \subseteq \mathfrak{L}_{\mathbb{V}} \subseteq \mathcal{F}$. The previous lemma now allows us to deduce the following somewhat surprising result.

Corollary 18. *$\underline{\mu}_{\mathbb{V}}$ and μ agree on the lattice of events $\mathfrak{L}_{\mathbb{V}}$ generated by $\mathfrak{V}_{\mathbb{V}}$.*

Proof. It follows from Proposition 15 that $\underline{\mu}_{\mathbb{V}} \leq \mu$ on $\mathcal{F}$, and we also know that $\underline{\mu}_{\mathbb{V}}$ is in particular 2-monotone. We may therefore infer from the previous lemma that $\underline{\mu}_{\mathbb{V}}$ and μ agree on a lattice of events. If we can show that they agree on $\mathfrak{V}_{\mathbb{V}}$, the proof is therefore complete. So consider any $\mathcal{V}_0$ in $\mathbb{V}$ and $V_0 \in \mathcal{V}_0$. Then since $\mathbb{V}$ is directed [($\mathbb{V}$1)], it is easy to see that

$$\underline{\mu}_{\mathbb{V}}(V_0) = \lim_{\mathcal{V} \in \mathbb{V}} \sum_{V \in \mathcal{V}, V \subseteq V_0} \mu(V) = \lim_{\mathcal{V} \succeq \mathcal{V}_0} \sum_{V \in \mathcal{V}, V \subseteq V_0} \mu(V).$$

Now if we apply ($\mathbb{V}2$) and ($\mathbb{V}3$) to the set $\{V \in \mathcal{V} \colon V \subseteq V_0\}$ we find that for any $\mathcal{V} \succeq \mathcal{V}_0$,

$$\sum_{V \in \mathcal{V}, V \subseteq V_0} \mu(V) = \mu\left(\bigcup \{V \in \mathcal{V} \colon V \subseteq V_0\}\right) = \mu(V_0),$$

and consequently $\underline{\mu}_{\mathbb{V}}(V_0) = \mu(V_0)$. □

Lemma 19. *Let ν be a monotone set function defined on a lattice of events $\mathcal{G}$ that includes $\emptyset$ and Ω, and let μ be a bounded positive charge defined on a field $\mathcal{F}$. Consider the inner set function ν_* of ν, defined on $\wp(\Omega)$. Then the following statements hold.*

(a) $\nu_* \leq \underline{\mu}_{\mathbb{V}}$ if and only if $\nu \leq \underline{\mu}_{\mathbb{V}}$;

(b) $\nu_* \geq \underline{\mu}_{\mathbb{V}}$ if and only if ν_* dominates $\underline{\mu}_{\mathbb{V}}$ on the lattice $\mathfrak{L}_{\mathbb{V}}$ generated by $\mathfrak{V}_{\mathbb{V}}$.

Proof. We begin with a proof of the first statement. The direct implication is trivial. Conversely, assume that $\nu \leq \underline{\mu}_{\mathbb{V}}$. Then for any subset A of Ω:

$$\nu_*(A) = \sup_{B \subseteq A, B \in \mathcal{G}} \nu(B) \leq \sup_{B \subseteq A, B \in \mathcal{G}} \underline{\mu}_{\mathbb{V}}(B) \leq \underline{\mu}_{\mathbb{V}}(A),$$

where the inequalities follow from the assumption and the monotonicity of $\underline{\mu}_{\mathbb{V}}$.

We now turn to the proof of the second statement. Again, the direct implication is trivial. Conversely, assume that ν_* dominates $\underline{\mu}_{\mathbb{V}}$ on the lattice $\mathfrak{L}_{\mathbb{V}}$ generated by $\mathfrak{V}_{\mathbb{V}}$. Then for any subset A of Ω:

$$\begin{aligned}\underline{\mu}_{\mathbb{V}}(A) &= \sup_{\mathcal{V} \in \mathbb{V}} \sum_{V \in \mathcal{V}, V \subseteq A} \mu(V) = \sup_{\mathcal{V} \in \mathbb{V}} \mu\left(\bigcup \{V \in \mathcal{V} \colon V \subseteq A\}\right) \\ &= \sup_{\mathcal{V} \in \mathbb{V}} \underline{\mu}_{\mathbb{V}}\left(\bigcup \{V \in \mathcal{V} \colon V \subseteq A\}\right) \leq \sup_{\mathcal{V} \in \mathbb{V}} \nu_*\left(\bigcup \{V \in \mathcal{V} \colon V \subseteq A\}\right) \leq \nu_*(A),\end{aligned}$$

where the second equality follows from ($\mathbb{V}3$), the third equality from Corollary 18, the first inequality from the assumption, and the last inequality from the monotonicity of ν_*. □

The following theorem generalises Theorem 16. It gives necessary and sufficient conditions for the natural extension $\underline{E}_\nu$ of a 2-monotone set function ν to coincide with the lower $\mathbb{V}$-integral generated by some bounded positive charge μ. Such a 2-monotone set function obviously has to be *completely* monotone (this follows for instance from (iv) below), amongst other things.

Theorem 20. *Let ν be a 2-monotone set function defined on a lattice of events $\mathcal{G}$ that contains $\emptyset$ and Ω, and let μ be a positive bounded charge defined a field $\mathcal{F}$. Then the following statements are equivalent:*

(i) ν is a restriction of $\underline{\mu}_{\mathbb{V}}$, and for all $A \subseteq \Omega$, and all $\epsilon > 0$, there is a $B_\epsilon \in \mathcal{G}$ such that $B_\epsilon \subseteq A$ and $\underline{\mu}_{\mathbb{V}}(A) - \underline{\mu}_{\mathbb{V}}(B_\epsilon) < \epsilon$.

(ii) $\underline{\mu}_{\mathbb{V}}$ coincides with ν_* on all events;

(iii) $(\mathbb{V})\underline{\int} \cdot \, \mathrm{d}\mu$ coincides with $\underline{E}_\nu$ on all gambles;

(iv) ν is a restriction of $\underline{\mu}_{\mathbb{V}}$ and ν_* and μ coincide on the lattice of events $\mathfrak{L}_{\mathbb{V}}$ generated by $\mathfrak{V}_{\mathbb{V}}$.

Proof. (i) $\implies$ (ii). From Lemma 19, we deduce that $\underline{\mu}_{\mathbb{V}} \geq \nu_*$ on all events. Let $A \subseteq \Omega$, $\epsilon > 0$. By assumption, there is a subset $B_\epsilon \in \mathcal{G}$ of A such that $\underline{\mu}_{\mathbb{V}}(A) - \underline{\mu}_{\mathbb{V}}(B_\epsilon) < \epsilon$. Therefore, $\underline{\mu}_{\mathbb{V}}(A) < \nu(B_\epsilon) + \epsilon \leq \nu_*(A) + \epsilon$, for all $\epsilon > 0$, and consequently $\underline{\mu}_{\mathbb{V}}(A) \leq \nu_*(A)$. Hence, $\underline{\mu}_{\mathbb{V}} = \nu_*$.

(ii) $\implies$ (i). If $\underline{\mu}_{\mathbb{V}}$ coincides with ν_* on all events, then obviously ν must be a restriction of $\underline{\mu}_{\mathbb{V}}$. By definition of ν_*, for every $A \subseteq \Omega$ and every $\epsilon > 0$, there is a subset $B_\epsilon \in \mathcal{G}$ of A such that $\nu_*(A) - \nu(B_\epsilon) < \epsilon$. Now, $\nu_*(A) = \underline{\mu}_{\mathbb{V}}(A)$ and $\nu(B_\epsilon) = \nu_*(B_\epsilon) = \underline{\mu}_{\mathbb{V}}(B_\epsilon)$ by assumption, whence $\underline{\mu}_{\mathbb{V}}(A) - \underline{\mu}_{\mathbb{V}}(B_\epsilon) < \epsilon$.

(ii) $\Longrightarrow$ (iv). Given $A \in \mathcal{G}$, we have that $\nu(A) = \nu_*(A) = \underline{\mu}_{\mathbb{V}}(A)$, hence the first statement holds. For the second statement, simply use Corollary 18.

(iv) $\Longrightarrow$ (ii). Immediate by Lemma 19.

The proof of the equivalence between (ii) and (iii) is similar to that in Theorem 16. □

Note that we can trivially use conjugacy considerations to establish similar equivalences involving $\overline{\mu}_{\mathbb{V}}, \nu^*$ and $(\mathbb{V})\overline{\int} \cdot \,\mathrm{d}\mu$, in the manner of Theorem 16.

One of the advantages of this theorem over Theorem 16 is its applicability to *any* lower $\mathbb{V}$-integral. Moreover, we can always choose ν to be finitely additive, as the following corollary shows.

Corollary 21. *Let μ be a bounded positive charge defined on a field $\mathcal{F}$ of subsets of Ω, and let ν be its restriction to the lattice of events $\mathfrak{L}_{\mathbb{V}}$ generated by $\mathfrak{V}_{\mathbb{V}}$. Then for μ and ν the equivalent statements in Theorem 20 hold.*

Proof. Check that Theorem 20(iv) holds. This follows immediately from Corollary 18. □

Hence, also Riemann–Stieltjes integrals associated with a discontinuous F on a compact interval $[a, b]$ can be represented by the natural extension of a finitely additive set function *defined on a lattice*. Consider the finitely additive set function ν, defined on the lattice generated by all closed intervals, and uniquely determined by $\nu([x, y]) = F(y) - F(x)$ for any $a \leq x \leq y \leq b$. Obviously, $\nu([x, y]) = \underline{\mu}_{\mathbb{V}}([x, y])$, and hence, the natural extension of ν does coincide with the lower Riemann–Stieltjes integral with respect to F, regardless of the continuity properties of F.

5. Conclusions

The notions of lower and upper integral for positive bounded charges that we have introduced in this paper subsume as particular cases most of the existing notions in the literature. As such, we think that they are general enough to capture the ideas underlying these different notions, while keeping at the same time interesting mathematical properties. For instance, we see from our results that most (if not all) of the lower integrals defined in the literature are actually instances of completely monotone exact functionals. As a consequence, they are representable as a Choquet functional with respect to a completely monotone set function (the restriction to events).

Moreover, the use of completely monotone set functions brings together a number of fields that may seem apart at first sight. For instance, our results are related to game theory through the use of exact functionals. Also, the vacuous lower previsions that we use in our definition of the lower $\mathbb{V}$-integral can be seen as the Choquet integrals with respect to a unanimity game. The equivalence between 2-monotone and comonotone additive functionals relates our results to the field of economics. And finally, the main results in this paper indicate that exact functionals (and therefore also the coherent lower previsions encountered in the theory of imprecise probabilities) have important things to say in the field of classical measure theory.

The representation theorems we have given tell us that we can use most notions of lower integral to calculate the natural extension of bounded charges, and of some finitely additive set functions. This natural extension is moreover the smallest exact functional which extends the probability charge to all gambles. Finally, let us remark that the second, and more general representation theorem has been possible because we have not required completely monotone set functions to be defined on fields of events, but only on lattices.

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