COUNTING MAXIMAL ARITHMETIC SUBGROUPS

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Abstract

We study the growth rate of the number of maximal arithmetic subgroups of bounded covolumes in a semisimple Lie group using an extension of the method developed by Borel and Prasad.

1. Introduction

A classical theorem of Wang [W] states that a simple Lie group not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ contains only finitely many conjugacy classes of discrete subgroups of bounded covolumes. This theorem, which describes the distribution of lattices in the higher-rank Lie groups, also brings attention to the quantitative side of the distribution picture. To date, several attempts have been made toward a quantitative analogue of Wang's theorem, but with inconclusive results.

The problem of determining the number of discrete subgroups of bounded covolumes naturally splits into two parts: the first part is to count maximal lattices, and the second is to count subgroups of bounded index in a given lattice. A recent project initiated by Lubotzky [Lu] resulted in significant progress toward understanding the subgroup growth of lattices and also allowed general conjectures to be formulated on the asymptotic of the number of lattices of bounded covolumes in semisimple Lie groups (see [BGLM], [GLP], [LN], [LS]). In this article, we consider another aspect of the problem: counting maximal lattices in a given semisimple Lie group.

Let H be a product of groups $H_s(k_s)$, $s \in S$, where S is a finite set, each k_s is an archimedean local field (i.e., $k_s = \mathbb{R}$ or $\mathbb{C}$), and $H_s(k_s)$ is an absolutely almost simple k_s-group. Then H is a semisimple Lie group. Throughout this article, we consider only semisimple Lie groups of this form. We assume, moreover, that H is connected and that none of the factors $H_s(k_s)$ is compact or has type A_1. In particular, H can be a noncompact simple Lie group (real or complex) not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$.

Let $m_H^u(x)$ and $m_H^{nu}(x)$ denote the number of conjugacy classes of maximal cocompact irreducible arithmetic subgroups and the number of conjugacy classes of

DUKE MATHEMATICAL JOURNAL
Vol. 140, No. 1, © 2007
Received 6 January 2006. Revision received 14 March 2007.
2000 *Mathematics Subject Classification.* Primary 22E40; Secondary 20G30, 20E07.
Belolipetsky's work supported in part by a Hebrew University of Jerusalem Golda Meir Postdoctoral Fellowship.

maximal non-cocompact irreducible arithmetic subgroups in H of covolume less than x, respectively. If the real rank of H is greater than 1, then by Margulis's theorem [M, Th. 1, p. 2], these numbers are equal to the numbers of the conjugacy classes of maximal uniform and nonuniform irreducible lattices in H of covolume less than x. In the real rank 1 case, there may also exist nonarithmetic lattices that we do not consider here.

THEOREM 1

(A) *If H contains an irreducible cocompact arithmetic subgroup (or, equivalently, if H is isotypic), then there exist effectively computable positive constants A and B that depend only on the type of almost simple factors of H such that for sufficiently large x,*

$$x^A \leq m_H^u(x) \leq x^{B\beta(x)},$$

where $\beta(x)$ is a function that we define for an arbitrary $\epsilon > 0$ as $\beta(x) = C(\log x)^\epsilon$, $C = C(\epsilon)$ being a constant that depends only on ϵ.

(B) *If H contains a non-cocompact irreducible arithmetic subgroup, then there exist effectively computable positive constants A', which depend only on the type of almost simple factors of H, and B', which depends on H, such that for sufficiently large x,*

$$x^{A'} \leq m_H^{nu}(x) \leq x^{B'}.$$

Conjecturally, the function $\beta(x)$ in part (A) can also be replaced by a constant; the constant B' in part (B) depends only on the type of almost simple factors of H. This would require, in particular, a polynomial bound on the number of fields with a bounded discriminant. The existence of such a bound is an old conjecture in number theory which may derive from Linnik; it appears in a stronger form in Cohen's book [C, Conj. 9.3.5]. In fact, we can show an equivalence of the conjecture "$\beta(x) = \text{const}$" to Linnik's conjecture (see Sec. 6.6) as a corollary from the proof of Theorem 1.

Theorem 1 is motivated by the problem of distribution of lattices in semisimple Lie groups. An application of these results (and their corollaries) to the problem is part of a joint work in progress with Lubotzky [BL].

To conclude this introduction, let us briefly outline the proof of Theorem 1. Our method is based on the work of Borel and Prasad [BP]. What distinguishes our task from theirs is that besides proving the finiteness of the number of arithmetic subgroups of bounded covolumes, we give bounds or, at least, asymptotic bounds for the number. This requires certain modifications to the method on one side and some special number-theoretic results on the other. In Proposition 3.3, we improve a number-theoretic result from [BP, Sec. 6], which enables us to effectively count the possible

fields of definition of the maximal arithmetic subgroups. The proof of this proposition is technical and is safely skipped in the first reading. The key ingredient for a good upper bound for the number of fields is an elaboration of recent work of Ellenberg and Venkatesh [EV], which we formulate in Proposition 3.1 and for which a proof is given in the appendix. After bounding the number of possible fields of definition k, we count admissible k-forms, corresponding collections of local factors, and conjugacy classes of arithmetic subgroups. Here, we use some Galois cohomology techniques, the Hasse principle, and the basic number-theoretic Proposition 3.2. Altogether, these lead to the proof of the upper bounds in Theorem 1, which is given in Section 4. The lower bounds, which are easier, are established in Section 5.

2. Preliminaries on arithmetic subgroups

This section presents a short account of the fundamental results of Borel and Prasad ([BP], [P]) which are used in this article. We encourage the reader to look into the original articles cited above for a better understanding of the subject. Our modest purpose here is to fix the notation and to recall some formulas for future reference.

2.1

Throughout this article, k is a number field, $\mathcal{O}_k$ is its ring of integers, $V = V(k)$ is the set of places (valuations) of k which is the union of $V_\infty(k)$ archimedean and $V_f(k)$ nonarchimedean places, and $\mathbb{A} = \mathbb{A}_k$ is the ring of adèles of k. The number of archimedean places of k is denoted by $a(k) = \#V_\infty(k)$. Let $r_1(k)$, $r_2(k)$ denote the number of real and complex places of k, respectively, so $a(k) = r_1(k) + r_2(k)$. As usual, $\mathscr{D}_k$ and h_k stay for the absolute value of the discriminant of $k/\mathbb{Q}$ and the class number of k. For a finite extension l of k, $\mathscr{D}_{l/k}$ denotes the $\mathbb{Q}$-norm of the relative discriminant of l over k.

All logarithms in this article are in base 2, unless stated otherwise.

2.2

Let G/k be an algebraic group defined over a number field k so that there exists a continuous surjective homomorphism $\phi : \mathrm{G}(k \otimes_{\mathbb{Q}} \mathbb{R})^o \to H$ with a compact kernel. We call such fields k, and we call k-groups G *admissible*. If $S \subset V_\infty(k)$ is the set of archimedean places of k over which G is isotropic (i.e., noncompact), then ϕ induces an epimorphism $\mathrm{G}_S = \prod_{v\in S} \mathrm{G}(k_v)^o \to H$ whose kernel is a finite subgroup of G_S.

We consider G as a k-subgroup of $\mathrm{GL}(n)$ for large enough n. We define a subgroup Γ of $\mathrm{G}(k)$ to be *arithmetic* if it is commensurable with the subgroup of k-integral points $\mathrm{G}(k) \cap \mathrm{GL}(n, \mathcal{O}_k)$; that is, the intersection $\Gamma \cap \mathrm{GL}(n, \mathcal{O}_k)$ is of finite index in both Γ and $\mathrm{G}(k) \cap \mathrm{GL}(n, \mathcal{O}_k)$. The subgroups of H which are commensurable with $\phi(\Gamma)$ for some admissible G/k are called *arithmetic subgroups* of H defined over the field k.

We restrict ourselves to the irreducible lattices, which implies that in the definition of the arithmetic subgroups, it is enough to consider only simply connected, absolutely almost simple algebraic groups G (see [M, Chap. 9.1]).

2.3

A semisimple Lie group contains irreducible lattices if and only if all its almost simple factors have the same type (H is *isotypic*). For example, we can take $H = \mathrm{SL}(2,\mathbb{R})^a \times \mathrm{SL}(2,\mathbb{C})^b$ or $H = \mathrm{SO}(p_1, q_1) \times \mathrm{SO}(p_2, q_2)$ $(p_1 + q_1 = p_2 + q_2)$, but in $H = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$, all lattices are reducible. Sufficiency of this condition is provided by the Borel-Harder theorem [BH], and its necessity is discussed elsewhere (e.g., in [M, Chap. 9.4]). Note that, in general, the assumption that H is isotypic does not imply that H contains nonuniform irreducible lattices, as is shown in an example suggested by Prasad (see [Wi, Prop. 12.31]). This is the reason why we impose an additional assumption concerning the existence of nonuniform irreducible lattices in Theorem 1(B).

2.4

The methods of Borel and Prasad depend to a considerable degree on the Bruhat-Tits theory of reductive groups over local fields. We assume familiarity with the theory and recall only some basic definitions. An extensive account of what we need can be found in Tits's survey article [T].

Let K be a nonarchimedean local field of characteristic zero (a finite extension of the p-adic field $\mathbb{Q}_p$), and let G be an absolutely almost simple, simply connected K-group. The Bruhat-Tits theory associates to G/K a simplicial complex $\mathscr{B} = \mathscr{B}(\mathrm{G}/K)$ on which $\mathrm{G}(K)$ acts by simplicial automorphisms that are special. (This implies, in particular, that if an element of $\mathrm{G}(K)$ leaves a simplex of $\mathscr{B}$ stable, then it fixes the simplex pointwise.) The complex $\mathscr{B}$ is called the *affine building* of G/K. A *parahoric subgroup* P of $\mathrm{G}(K)$ is defined as a stabiliser of a simplex of $\mathscr{B}$. Every parahoric subgroup is compact and open in $\mathrm{G}(K)$ in the p-adic topology. Minimal parahoric subgroups, called *Iwahori*, are defined as subgroups of $\mathrm{G}(K)$ fixing chambers (i.e., maximal simplexes) in $\mathscr{B}$. All Iwahori subgroups are conjugate in $\mathrm{G}(K)$. Maximal parahoric subgroups are the maximal compact subgroups of $\mathrm{G}(K)$; they are characterised by the property of being stabilisers of the vertices of $\mathscr{B}$. A maximal parahoric subgroup is called *special* if it fixes a *special vertex* of $\mathscr{B}$. A vertex $x \in \mathscr{B}$ is special if the affine Weyl group W of $\mathrm{G}(K)$ is a semidirect product of the translation subgroup by the isotropy group W_x of x in W. In this case, W_x is canonically isomorphic to the (finite) Weyl group of the K-root system of G. If G is quasi-split over K and splits over an unramified extension of K, then $G(K)$ contains *hyperspecial* parahoric subgroups. (We refer to [T, Sec. 1.10] for the definition of hyperspecial parahorics.)

An important property of these subgroups is that they have maximal volumes among all parahoric subgroups (see [T, Sec. 3.8.2]).

2.5

We now define a Haar measure μ on H with respect to which the volumes of arithmetic quotients are computed. Of course, the final result then holds for any other normalization of the Haar measure on H. The definition and most of the subsequent facts come from [P] and [BP].

Let G be an admissible simply connected algebraic k-group. If $v \in V_f(k)$, we let μ_v be the Haar measure on $G(k_v)$ which assigns volume 1 to the Iwahori subgroups of $G(k_v)$. If v is archimedean, we first consider the case where $k_v = \mathbb{R}$. There exists a unique anisotropic $\mathbb{R}$-form G_{cpt} of G which has a natural Haar measure giving the group volume 1. This measure can be transferred to $G(k_v)$ in a standard way, and we define μ_v as its image; it is a canonical Haar measure on $G(\mathbb{R})$. In the case where $k_v = \mathbb{C}$, we have $G(k_v) = G_1(\mathbb{R})$ with $G_1 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}G$, and we define μ_v to be equal to the canonical measure on $G_1(\mathbb{R})$. The Haar measure μ_S on G_S is defined as a product of μ_v, $v \in S$. This also induces the measure μ on H, and it is easy to check that μ does not depend on a choice of G and the epimorphism $\phi : G_S \to H$.

2.6

A collection $P = (P_v)_{v \in V_f}$ of parahoric subgroups P_v of a simply connected k-group G is called *coherent* if $\prod_{v \in V_\infty} G(k_v) \cdot \prod_{v \in V_f} P_v$ is an open subgroup of the adèle group $G(\mathbb{A}_k)$. A coherent collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ defines an arithmetic subgroup $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$ of $G(k)$, which is called the *principal arithmetic subgroup* associated to P. The corresponding arithmetic subgroup $\Lambda' = \phi(\Lambda) \subset H$ is also called *principal*.

The covolume of a principal arithmetic subgroup with respect to the measure μ defined as above is given by Prasad's formula [P, Th. 3.7]:

$$\mu(H/\Lambda') = \mu_S(G_S/\Lambda) = \mathscr{D}_k^{\dim(G)/2}(\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s/2}\Big(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]}\tau_k(G)\,\mathscr{E}(P),$$

where

(i) $\dim(G)$ and m_i denote the dimension and Lie exponents of G;

(ii) l is a Galois extension of k defined as in [P, Sec. 0.2] (if G is not a k-form of type ${}^6D_4$, then l is the split field of the quasi-split inner k-form of G, and if G is of type ${}^6D_4$, then l is a fixed cubic extension of k contained in the corresponding split field; in all the cases, $[l : k] \leq 3$);

(iii) $s = s(G)$ is an integer defined in [P, Sec. 0.4], and in particular, $s = 0$ if G is an inner form of a split group, while $s \geq 5$ if G is an outer form;

(iv) $\tau_k(\mathrm{G})$ is the Tamagawa number of G over k since G is simply connected and k is a number field $\tau_k(\mathrm{G}) = 1$; and

(v) $\mathscr{E}(P) = \prod_{v \in V_f} e_v$ is an Euler product of the local factors $e_v = e(P_v)$; for $v \in V_f$, e_v is the inverse of the volume of P_v with respect to the Haar measure $\gamma_v \omega_v^*$ defined in [P, Secs. 1.3, 2.1].

The local factors can be computed using the Bruhat-Tits theory. (In particular, $e_v > 1$ for every $v \in V_f$; see [P, Prop. 2.10(iv)].)

2.7

Any maximal arithmetic subgroup Γ of H can be obtained as a normaliser in H of the image Λ' of some principal arithmetic subgroup of $\mathrm{G}(k)$ (see [BP, Prop. 1.4(iv)]). Moreover, the collections of parahoric subgroups which are associated to the maximal arithmetic subgroups have maximal types as shown in Rohlfs [R] (see also [RC]). So, in order to compute the covolume of a maximal arithmetic subgroup, we need to be able to compute the index of a principal arithmetic subgroup in its normaliser. In a general setting, the upper bound for the index was obtained in [BP, Sec. 2]:

$$[\Gamma : \Lambda'] \leq n^{\epsilon \# S} \cdot \#\mathrm{H}^1(k, \mathrm{C})_\xi \cdot \prod_{v \in V_f} \#\Xi_{\Theta_v}.$$

Here, n and ϵ are constants defined below, so $n^{\epsilon \# S}$ depends only on H and does not depend on the choice of $\mathrm{G}(k)$ and Λ. The group $\mathrm{H}^1(k, \mathrm{C})_\xi$ is a finite subgroup of the first Galois cohomology group of k with coefficients in the center of G, as is defined in [BP, Sec. 2.10]. The order of $\mathrm{H}^1(k, \mathrm{C})_\xi$ can be further estimated (see [BP, Sec. 5]); the following bound is a combination of [BP, Props. 5.1, 5.6]:

$$\#\mathrm{H}^1(k, \mathrm{C})_\xi \leq 2 h_l^{\epsilon'} n^{\epsilon a(k) + \epsilon' a(l) + \epsilon \# T} (\mathscr{D}_l / \mathscr{D}_k^{[l:k]})^{\epsilon''},$$

where

(i) $n = r + 1$ if G is of type A_r; $n = 2$ if G is of type B_r, C_r (r arbitrary), D_r (with r even), or E_7; $n = 3$ if G is of type E_6; $n = 4$ if G is of type D_r (with r odd); $n = 1$ if G is of type E_8, F_4 or G_2;

(ii) $\epsilon = 2$ if G is of type D_r (with r even), and $\epsilon = 1$ otherwise (so the center $\mathrm{C} = \mathrm{C}(\mathrm{G})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\epsilon$ and $\#\mathrm{C} = n^\epsilon$);

(iii) $\epsilon' = \epsilon$ if G is an inner form of a k-split group, and $\epsilon' = 1$ otherwise;

(iv) $\epsilon'' = 1$ if G/k is an outer form of type D_r (r even), and $\epsilon = 0$ otherwise; and

(v) T is the set of places $v \in V_f$ for which G splits over an unramified extension of k_v but is not quasi-split over k_v.

Finally, Ξ_{Θ_v} is a subgroup of the automorphism group of the affine Dynkin diagram which comes from the adjoint group and preserves the type Θ_v of P_v. In particular, $\#\Xi_{\Theta_v} \leq r + 1$, and $\#\Xi_{\Theta_v} = 1$ if P_v is special.

2.8

As a result, we have the following lower bound for the covolume of Γ:

$$\mu(H/\Gamma) \geq (2n^{\epsilon a(k)+\epsilon' a(l)} h_l^{\epsilon'})^{-1} \mathscr{D}_k^{\dim(\mathrm{G})/2} \Big(\frac{\mathscr{D}_l}{\mathscr{D}_k^{[l:k]}}\Big)^{s'} \Big(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]} \tau_k(\mathrm{G}) \mathscr{F},$$

where

(i) $s' = s/2-1$ if G/k is an outer form of type D_r, r even, but $s' = s/2$ otherwise; and

(ii) $\mathscr{F} = \prod_{v\in V_f} f_v$ with $f_v = e_v(\#\Xi_{\Theta_v})^{-1} = e_v$ if G is quasi-split over k_v and P_v is hyperspecial (which is true for almost all v), $f_v = e_v n^{-\epsilon}(\#\Xi_{\Theta_v})^{-1}$ if G splits over an unramified extension of k_v but is not quasi-split over k_v, and $f_v = e_v(\#\Xi_{\Theta_v})^{-1}$ in the rest of the cases.

Using the computations in [BP, Apps. A, C], it is not hard to check that $f_v > 1$ for every $v \in V_f$.

More details about this formula can be found in [BP, Secs. 5, 7].

3. Number-theoretic results

3.1

Let $N_{k,d}(x)$ be the number of k-isomorphism classes of extensions l of k such that $[l:k] = d$, $\mathscr{D}_{l/k} < x$, and let $N(x)$ be the number of isomorphism classes of number fields with discriminant less than x.

PROPOSITION

For large enough positive x, we have the following:

(i) *given a number field k and a fixed degree d, there exist constants $c, b_1, b_2 > 0$, depending only on d, such that $N_{k,d}(x) \leq c\mathscr{D}_k^{b_1} x^{b_2}$; and*

(ii) *for every $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$ such that $N(x) \leq x^{\beta(x)}$, $\beta(x) = C(\log x)^{\epsilon}$.*

Proof

Effectively, the proof

(i) follows, for example, from [EV, Th. 1.1]; and

(ii) it follows from the general method used in [EV] but requires some extra work, namely, that we have to know how the implicit constants in [EV, Th. 1.1] depend on the degree of the extensions in order to be sure that this does not change the expected upper bound; this is carried out in detail in the attached appendix provided by Ellenberg and Venkatesh. □

3.2

Let $Q_k(x)$ be the number of squarefree ideals of k of norm at most x.

PROPOSITION
For a number field k, we have
(i) $Q_k(x) = (\mathrm{Res}_{s=1}(\zeta_k)/\zeta_k(2))\, x + o(x)$ *for* $x \to \infty$*; and*
(ii) *there exist absolute constants* b_3, b_4 *(not depending on* k*) such that* $Q_k(x) \le \mathscr{D}_k^{b_3} x^{b_4}$.

Proof
Of the two items in the preceding proposition, we have the following.
(i) The proof is a known fact from analytic number theory. For a short and conceptual proof, we refer to [Se, Th. 14].
(ii) As far as we do not claim that $b_4 = 1$, the proof is easy. Consider the Dedekind zeta function of k:

$$\zeta_k(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where a_n is the number of ideals of k of norm n, $s > 1$.

Let $I_k(x)$ denote the number of ideals of k of norm less than x. We have

$$I_k(x) = a_1 + a_2 + \cdots + a_{[x]},$$
$$\zeta_k(s) \cdot x^s \ge I_k(x).$$

Taking $s = 2$, we obtain

$$Q_k(x) \le I_k(x) \le \zeta_k(2) \cdot x^2 \le \left(\frac{\pi^2}{6}\right)^{[k:\mathbb{Q}]} x^2 \le c_1^{\log \mathscr{D}_k} x^2 = \mathscr{D}_k^{b_3} x^{b_4}.$$

Here, we used inequalities $\zeta_k(2) \le \zeta(2)^{[k:\mathbb{Q}]}$ and for $k \ne \mathbb{Q}$, $[k:\mathbb{Q}] \le c \log \mathscr{D}_k$. The first inequality follows from the definition of the functions ζ and ζ_k, and the second is a well-known corollary of Minkowski's discriminant bound. □

3.3
Finally, we need an improved version of a number-theoretic result from [BP, Sec. 6]. The main idea is that instead of using $\mathscr{D}_k^{\dim(\mathrm{G})/2}$ to absorb the small factors in the volume formula, we use only part of it, saving the rest for a later occasion. This is easy to achieve for the groups of a large-enough absolute rank; when the rank becomes small, the estimates become much more delicate.

Let G/k be an absolutely almost simple, simply connected algebraic group of absolute rank $r \ge 2$, so that the numbers n, ϵ, ϵ', s' and $m_1 \le \cdots \le m_r$ are fixed and

defined as in Section 2. Let

$$B(\mathrm{G}/k) = \mathscr{D}_k^{\dim(\mathrm{G})/2} n^{-\epsilon a(k)-\epsilon' a(l)} h_l^{-\epsilon'} (\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s'} \Big(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \Big)^{[k:\mathbb{Q}]}.$$

Then by Section 2.8, we have $\mu(H/\Gamma) \geq (1/2)B(\mathrm{G}/k)\tau_k(\mathrm{G})\mathscr{F} \geq (1/2)B(\mathrm{G}/k)$ for every arithmetic subgroup Γ of H which is associated to G/k.

PROPOSITION
There exist positive constants δ_1, δ_2 depending only on the absolute type of G *such that $B(\mathrm{G}/k) \geq \mathscr{D}_k^{\delta_1} \mathscr{D}_{l/k}^{\delta_2}$ for almost all number fields k.*

Proof
Given an absolutely almost simple, simply connected algebraic group G of an absolute type T and rank r, we show that for almost all k,
(i) $B(\mathrm{G}/k) \geq \mathscr{D}_k^{\dim(\mathrm{G})/2-2} \mathscr{D}_{l/k}$ if $r \geq 30$;
(ii) $B(\mathrm{G}/k) \geq \mathscr{D}_k \mathscr{D}_{l/k}$ if $r < 30$ and T is not A_2, A_3, B_2;
(iii) $B(\mathrm{G}/k) \geq \mathscr{D}_k^{0.1} \mathscr{D}_{l/k}$ if T is A_3 or B_2;
(iv) $B(\mathrm{G}/k) \geq \mathscr{D}_k^{0.01} \mathscr{D}_{l/k}^{0.5}$ if T is A_2.
Clearly, these four inequalities all together imply the proposition.

First, assume that G is not a k-form of type ${}^6\mathrm{D}_4$. We have

$$[l:k] \leq 2,$$
$$n^{-\epsilon a(k)-\epsilon' a(l)} \geq n^{-\epsilon(a(k)+a(l))} \geq (r+1)^{-3[k:\mathbb{Q}]}.$$

It is known that

$$h_l \leq 10^2 \Big(\frac{\pi}{12}\Big)^{[l:\mathbb{Q}]} \mathscr{D}_l \tag{1}$$

(see [BP, proof of Prop. 6.1]; let us point out that this bound holds without any assumption on the degree of the field l);

$$\mathscr{D}_l/\mathscr{D}_k^{[l:k]} = \mathscr{D}_{l/k} \geq 1. \tag{2}$$

Combining the above inequalities, we obtain

$$\begin{aligned} B(\mathrm{G}/k) &= \mathscr{D}_k^{\dim(\mathrm{G})/2} n^{-\epsilon a(k)-\epsilon' a(l)} h_l^{-\epsilon'} (\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s'} \Big(\prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \Big)^{[k:\mathbb{Q}]} \\ &\geq 10^{-2\epsilon'} \mathscr{D}_k^{\dim(\mathrm{G})/2-2} \Big(\frac{\pi}{12}\Big)^{-\epsilon'[l:\mathbb{Q}]} \mathscr{D}_{l/k} \Big(\frac{1}{(r+1)^3} \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \Big)^{[k:\mathbb{Q}]} \\ &\geq 10^{-2\epsilon'} \mathscr{D}_k^{\dim(\mathrm{G})/2-2} \mathscr{D}_{l/k} \Big(\frac{1}{(\pi/12)(r+1)^3} \prod_{i=1}^{r} \frac{m_i!}{(2\pi)^{m_i+1}} \Big)^{[k:\mathbb{Q}]}. \end{aligned}$$

(If G is k-split, then $s' = 0$, $l = k$, $\mathscr{D}_{l/k} = 1$; in the nonsplit case, we use the fact that $s' > 2$.)

Since for i large enough, $m_i! \gg (2\pi)^{m_i+1}$, it is clear that for large enough r,

$$\frac{1}{(\pi/12)(r+1)^3}\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}} > 1.$$

An easy, direct computation shows that starting from $r = 30$,

$$10^{-2\epsilon'}\mathscr{D}_k^{\dim(\mathrm{G})/2-2}\mathscr{D}_{l/k}\Big(\frac{1}{(\pi/12)(r+1)^3}\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]} \geq \mathscr{D}_k^{\dim(\mathrm{G})/2-2}\mathscr{D}_{l/k}.$$

So, for $r \geq 30$, $\delta = \dim(\mathrm{G})/2 - 2$, and any field k, we have $B(\mathrm{G}/k) \geq \mathscr{D}_k^{\delta}\mathscr{D}_{l/k}$, the finite set of the exceptional fields is empty, and case (i) is proved.

To proceed with the argument, let us remark that

$$B(\mathrm{G}/k) \geq \mathscr{D}_k^{\dim(\mathrm{G})/2-2}\mathscr{D}_{l/k}c,$$

where $c > 0$ depends only on the absolute type of G and degree $d = [k : \mathbb{Q}]$. So, if the degree d is fixed, then for any $z > 0$ that is chosen later, we have

$$B(\mathrm{G}/k) \geq \mathscr{D}_k^{\dim(\mathrm{G})/2-2-z}\mathscr{D}_{l/k}\mathscr{D}_k^{z}c \geq \mathscr{D}_k^{\dim(\mathrm{G})/2-2-z}\mathscr{D}_{l/k} \tag{3}$$

for all k with $\mathscr{D}_k \geq c^{-z}$. Since there are only finitely many number fields with a bounded discriminant, (3) holds for all but finitely many k of degree d. Since we always have $\dim(\mathrm{G})/2 > 2$, this allows us to assume (at least when G is not ${}^6\mathrm{D}_4$) that the degree of k is large enough.

We now come to case (ii). Let G be not of type ${}^6\mathrm{D}_4$. By the previous remark, we can suppose that $[k : \mathbb{Q}]$ is large enough. Due to Odlyzko [O, Th. 1], we have the following lower bound for $\mathscr{D}_k$:

$$\text{if } [k : \mathbb{Q}] > 10^5, \text{ then } \mathscr{D}_k \geq 55^{r_1(k)}21^{2r_2(k)}. \tag{4}$$

So, for $[k : \mathbb{Q}] > 10^5$,

$$B(\mathrm{G}/k) \geq 10^{-2\epsilon'}\Big(\frac{21^{\dim(\mathrm{G})/2-2-\delta}}{(\pi/12)(r+1)^3}\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]}\mathscr{D}_k^{\delta}\mathscr{D}_{l/k}.$$

A direct case-by-case verification shows that for $\delta = 1$, the latter expression is at least $\mathscr{D}_k^{\delta}\mathscr{D}_{l/k}$. So, if we put $z = \dim(\mathrm{G})/2 - 3$ in (3), then we obtain that for all but finitely many k, $B(\mathrm{G}/k) \geq \mathscr{D}_k\mathscr{D}_{l/k}$.

Let now G/k be a triality form of type ${}^6\mathrm{D}_4$. We have

$$\epsilon = 2, \qquad \epsilon' = 1, \qquad n = 2, \qquad s' = 2.5, \qquad \{m_i\} = \{1, 3, 5, 3\},$$
$$[l : k] = 3, \qquad \text{and} \qquad a(l) \le 3[k : \mathbb{Q}].$$

So, if $[k : \mathbb{Q}] > 10^5$,

$$B(\mathrm{G}/k) \ge 10^{-2}\Big(\frac{21^{14-3-\delta}}{(\pi/12)\cdot 2^3}\cdot\frac{6\cdot 120\cdot 6}{(2\pi)^{16}}\Big)^{[k:\mathbb{Q}]}\mathscr{D}_k^{\delta}\mathscr{D}_{l/k}.$$

For $\delta = 1$, it is $\ge \mathscr{D}_k\mathscr{D}_{l/k}$. If $[k : \mathbb{Q}] \le 10^5$, we still have the inequality (3) (the precise formula for the constant c would be different, but it is not essential), so that, for all but finitely many k, again $B(\mathrm{G}/k) \ge \mathscr{D}_k\mathscr{D}_{l/k}$. The case (ii) is now settled completely.

Let G/k be of type A_3 or B_2. As before, we can assume $[k : \mathbb{Q}] > 10^5$. We have

$$B(\mathrm{G}/k) \ge 10^{-2}\Big(\frac{\pi}{12}\Big)^{-[l:\mathbb{Q}]}\Big(\frac{21^{\dim(\mathrm{G})/2-[l:k]-\delta}}{n^{1+[l:k]}}\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]}\mathscr{D}_k^{\delta}\mathscr{D}_{l/k}.$$

Now, $n = 4$ and $n = 2$ for the types A_3 and B_2, respectively; if $l \ne k$, then $[l : \mathbb{Q}] = 2[k : \mathbb{Q}]$. Using this, it is easy to check that if $\delta = 0.1$, then $B(\mathrm{G}/k) \ge \mathscr{D}_k^{\delta}\mathscr{D}_{l/k}$ in each of the possible cases.

It remains for us to consider (iv). This is the most difficult case; the proof almost repeats the argument of [BP, Prop. 6.1(vi)].

With the notation of [BP], for $[l : \mathbb{Q}] > 10^5$ we have

$$\begin{aligned}
B(\mathrm{G}/k) &= \mathscr{D}_k^4\cdot 3^{-a(k)-a(l)}h_l^{-1}(\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{5/2}\Big(\frac{1}{2^4\pi^5}\Big)^{[k:\mathbb{Q}]}\\
&= \mathscr{D}_k^{\delta}\mathscr{D}_l^{2-\delta/2}(\mathscr{D}_{l/k})^{1/2}3^{-a(k)-a(l)}h_l^{-1}\Big(\frac{1}{2^4\pi^5}\Big)^{[k:\mathbb{Q}]}\\
&\ge \mathscr{D}_k^{\delta}(\mathscr{D}_{l/k})^{1/2}\frac{0.02}{s(s-1)}\Big(\frac{55^{(4-\delta-s)/2}}{2\cdot 3^{3/2}\cdot\pi^{(6-s)/2}}\Big)^{r_1(l)}\Big(\frac{21^{(4-\delta-s)}}{2^{(4-s)}\cdot 3^2\cdot\pi^{(5-s)}}\Big)^{r_2(l)}\\
&\quad\times\exp\Big((3-\delta-s)Z_l(s) - (4-\delta-s)\Big(\frac{c_1}{2}+(s-1)^{-1}\Big)\\
&\qquad + \big(0.1-(c_3+c_4)(s-1)\big)a(l)\Big).
\end{aligned}$$

Now, let $\delta = 0.01$. Since

$$55^{2.99/2}(2\cdot 3^{3/2}\cdot\pi^{5/2})^{-1} > 2.19, \qquad 21^{2.99}(2^3\cdot 3^2\cdot\pi^4)^{-1} > 1.28,$$

and

$$\exp\big((3-\delta-s)Z_l(s)\big) \ge 1 \quad \text{if } s < 2-\delta,$$

by choosing $s > 1$ sufficiently close to 1, we obtain that there is an absolute constant c_6 such that

$$\mathscr{D}_k^4 \cdot 3^{-a(k)-a(l)} h_l^{-1} (\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{5/2} \Big(\frac{1}{2^4\pi^5}\Big)^{[k:\mathbb{Q}]} \geq \mathscr{D}_k^{0.01} (\mathscr{D}_{l/k})^{1/2} 2.19^{r_1(l)} 1.28^{r_2(l)} c_6.$$

The right-hand side is at least $\mathscr{D}_k^{0.01}(\mathscr{D}_{l/k})^{1/2}$ if $[l:\mathbb{Q}]$ is large enough, say, $[l:\mathbb{Q}] > d_l$ (and $d_l \geq 10^5$).

If $[l:\mathbb{Q}] < d_l$, then $[k:\mathbb{Q}] < d_l$, and by (3) for all but finitely many fields k, we have

$$B(\mathrm{G}/k) \geq \mathscr{D}_k \mathscr{D}_{l/k} \geq \mathscr{D}_k^{0.01} \mathscr{D}_{l/k}^{0.5}. \qquad \square$$

Remark. The proof provides explicit values of δ_1, δ_2 for each of the types; however, in many cases, the bound for $B(\mathrm{G}/k)$ can be improved. This requires more careful argument and is useful for particular applications.

4. Proof of Theorem 1: The upper bound

As before, H denotes a connected semisimple Lie group whose almost simple factors are all noncompact and have the same type different from A_1; G is an absolutely almost simple, simply connected k-group admissible in the sense that there exists a continuous surjective homomorphism $\mathrm{G}(k \otimes_{\mathbb{Q}} \mathbb{R})^o \to H$ with a compact kernel.

4.1. Counting number fields

For a (maximal) arithmetic subgroup Γ of H, we have (see Sec. 2.8)

$$\mu(H/\Gamma) \geq \frac{1}{2} B(\mathrm{G}/k) \tau_k(\mathrm{G}) \mathscr{F}, \tag{5}$$

where

(i) k is the field of definition of Γ,
(ii) G/k is a k-form from which Γ is induced (see Sec. 2.2),
(iii) $B(\mathrm{G}/k) = \mathscr{D}_k^{\dim(\mathrm{G})/2} n^{-\epsilon a(k) - \epsilon' a(l)} h_l^{-\epsilon'} (\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s'} \big(\prod_{i=1}^r m_i!/(2\pi)^{m_i+1}\big)^{[k:\mathbb{Q}]}$,
(iv) $\tau_k(\mathrm{G}) = 1$, and
(v) $\mathscr{F} = \prod_{v \in V_f} f_v > 1$ is considered later.

By Proposition 3.3, for all but finitely many number fields k,

$$\mu(H/\Gamma) \geq c_1 \mathscr{D}_k^{\delta_1} \mathscr{D}_{l/k}^{\delta_2},$$

where δ_1, δ_2 are the constants determined by the absolute type of G (which is the type of almost simple factors of H).

So, for large enough x, if $\mu_S(H/\Gamma) < x$, then $\mathscr{D}_k < (x/c_1)^{1/\delta_1}$, $\mathscr{D}_{l/k} < (x/c_1)^{1/\delta_2}$.

By Proposition 3.1(ii), the number of such fields k is at most

$$\Big(\frac{x}{c_1}\Big)^{\beta((x/c_1)^{1/\delta_1})} \le x^{c_2\beta(x)},$$

and by Proposition 3.1(i), for each k the number of such extensions l is at most

$$c\mathscr{D}_k^{b_1}\Big(\frac{x}{c_1}\Big)^{b_2/\delta_2} \le c\Big(\frac{x}{c_1}\Big)^{b_1/\delta_1}\Big(\frac{x}{c_1}\Big)^{b_2/\delta_2} \le x^{c_3}.$$

It follows that the number of all admissible pairs (k, l) is bounded by

$$x^{c_2\beta(x)+c_3},$$

and moreover, since $k \ne \mathbb{Q}$ implies $[k : \mathbb{Q}] \le c \log \mathscr{D}_k$, for all admissible k we have

$$a(k) \le c_4 \log x.$$

4.2. Non-cocompact case

If Γ is non-cocompact, the degree of the field of definition of Γ is bounded. Indeed, the non-cocompactness of Γ implies that the corresponding algebraic group G is k-isotropic, so G/k_v is noncompact for every $v \in V$. It follows that the number of infinite places of k is equal to the number $\#S$ of almost simple factors of H, so $[k : \mathbb{Q}] \le 2\#S$.

Now, in Section 4.1, we can consider only the number fields k with $[k : \mathbb{Q}] \le 2\#S$ and the number fields l with $[l : \mathbb{Q}] \le 3[k : \mathbb{Q}] \le 6\#S$. By Proposition 3.1(i), for large enough x the number of admissible pairs (k, l) is at most

$$x^{c_5}, \quad c_5 = c_5(\#S).$$

(In fact, here we can use a weaker result from Schmidt [S], who showed that the number of degree n extensions l of k with $\mathscr{D}_{l/k} < x$ is bounded by $C(n, k)x^{(n+2)/4}$.)

4.3. Counting k-forms

Given an admissible pair (k, l) of number fields, there exists a unique quasi-split k-form $\mathscr{G}$ for which l is the splitting field (or a certain subfield of the splitting field if $\mathscr{G}$ is of type ${}^6\mathrm{D}_4$ and $[l : \mathbb{Q}] = 3$). So, we have an upper bound for the number of quasi-split groups for which there can exist an inner form that defines an arithmetic subgroup of covolume less than x. We now fix a quasi-split k-form $\mathscr{G}$ and estimate the number of admissible inner forms. Since every inner equivalence class of k-forms contains a unique quasi-split form, this gives us a bound on the total number of admissible G/k.

By the assumption, $\prod_{v\in V_\infty(k)} \mathrm{G}(k_v)$ is isogenous to $H \times K$ (K is a compact Lie group), so the k_v-form of G is almost fixed at the infinite places of k. More

precisely, let c_h be the number of nonisomorphic almost simple factors of H. For each $v \in V_\infty(k)$, $\mathrm{G}(k_v)$ is isomorphic to one of c_h noncompact (simply connected) groups or is compact. Let n_h denote the number of places v at which $\mathrm{G}(k_v)$ is noncompact. By the assumption, $n_h = \#S$. This implies that the number of variants for $\mathrm{G}(k_v)$ at the infinite places of k is bounded by

$$c_h^{n_h}\binom{a(k)}{n_h} < \big(c_h a(k)\big)^{n_h} \le (\log x)^{c_6},$$

where $\binom{\cdot}{\cdot}$ denotes the binomial coefficient.

Let now v be a finite place of k. The inner k_v-forms of G correspond to the elements of the first Galois cohomology set $\mathrm{H}^1(k_v, \overline{\mathrm{G}})$, where $\overline{\mathrm{G}}$ is the adjoint group of G. The order of $\mathrm{H}^1(k_v, \overline{\mathrm{G}})$ can be computed from the cohomological exact sequence

$$\mathrm{H}^1(k_v, \mathrm{G}) \to \mathrm{H}^1(k_v, \overline{\mathrm{G}}) \xrightarrow{\delta} \mathrm{H}^2(k_v, \mathrm{C}),$$

which corresponds to the universal k_v-covering sequence of groups

$$1 \to \mathrm{C} \to \mathrm{G} \to \overline{\mathrm{G}} \to 1.$$

For a simply connected k_v-group G, the first cohomology $\mathrm{H}^1(k_v, \mathrm{G})$ is trivial by Kneser's theorem (see [K]), so δ is injective. Furthermore, the group $\mathrm{H}^2(k_v, \mathrm{C})$ can be identified with a subgroup of the Brauer group of k_v and then explicitly computed using results from the local class field theory (see [PR, Chap. 6] for details and explanations). As a corollary here, we have that the number of inner k_v-forms is bounded by n^ϵ in the notation of Section 2.7. (Recall that $n^\epsilon = \#\mathrm{C}$ is the order of the center of G.)

Let $T_1 \subset V_f(k)$ be a (finite) subset of the nonarchimedean places of k such that G is not quasi-split over k_v for $v \in T_1$. It follows from [P, Prop. 2.10] that there exists a constant $\delta > 0$, which depends only on the absolute type of G, such that for every $v \in T_1$,

$$f_v \ge n^{-\epsilon}(\#\Xi_{\Theta_v})^{-1} e_v \ge q_v^\delta \tag{6}$$

(q_v denotes the order of the residue field of k at v).

Indeed, we can take $\delta = \log(2^{r_v} n^{-\epsilon})$ if the absolute type of G is not A_2 and $\delta = \log(2^2 \cdot 3^{-1}) = 0.415\ldots$ for the type A_2, and we can then check that $\delta > 0$ and inequality (6) holds going through the case-by-case consideration in [BP, App. C.2].

To a set $T \subset V_f(k)$, we can assign an ideal $\mathscr{I}_T$ of $\mathscr{O}_k$ given by the product of prime ideals defining the places in T. Conversely, each squarefree ideal of $\mathscr{O}_k$ uniquely defines a subset T in $V_f(k)$ corresponding to its prime decomposition. Note also that $\prod_{v\in T} q_v = \mathrm{Norm}(\mathscr{I}_T)$.

Now, for an arithmetic subgroup Γ induced from G, we have

$$\mu(H/\Gamma) = \frac{1}{2} B(\mathrm{G}/k)\tau_k(\mathrm{G})\mathscr{F} \geq c_7 \prod_{v \in T_1} q_v^{\delta}.$$

This implies that if $\mu(H/\Gamma) \leq x$, then $\mathrm{Norm}(\mathscr{I}_{T_1}) = \prod_{v \in T_1} q_v \leq x^{c_8}$. By Proposition 3.2(ii), the number of variants for T_1 is bounded by x^{c_9}. Moreover, since for every $v \in V_f$, $q_v \geq 2$, for every such a set T_1 we have $\#T_1 \leq c_{10} \log x$.

Now, the Hasse principle implies that a k-form of G is uniquely determined by $(\mathrm{G}(k_v))_{v \in V(k)}$. The Hasse principle for semisimple groups is valid due to the work of Kneser, Harder, and Chernousov (see [PR, Chap. 6]). So, the number of the admissible k-forms is at most

$$(\log x)^{c_6} x^{c_8} n^{\epsilon c_{10} \log x} \leq x^{c_{11}}.$$

4.4. Counting collections of parahorics

For a given large enough x, we have defined a collection of G/k for which there exists a (centrally) k-isogenous group G' that may give rise to the arithmetic subgroups $\Gamma \subset H$ with $\mu(H/\Gamma) < x$. The number of such k-groups G is finite and can be bounded as in Section 4.3, but each G/k still defines countably many maximal arithmetic subgroups. We now fix a group G/k and estimate the number of coherent collections of parahoric subgroups of G which can give rise to the maximal arithmetic subgroups with covolumes less than x. In the classical language, what we are going to do in this section is count the number of admissible genera.

We use again the local-to-global approach. Let us fix a central k-isogeny $i : \mathrm{G} \to \mathrm{G}'$ with G' so that G'_S projects onto H. Every maximal arithmetic subgroup $\Gamma \subset \mathrm{G}'_S$ is associated to some coherent collection $P = (P_v)_{v \in V_f}$ of parahoric subgroups of G (see [BP, Prop. 1.4]):

$$\Gamma = N_{\mathrm{G}'}\big(i(\Lambda)\big), \quad \Lambda = \mathrm{G}(k) \cap \prod_{v \in V_f(k)} P_v.$$

The image of Γ in H is an arithmetic subgroup, and every maximal arithmetic subgroup of H can be obtained as a projection of some such Γ.

For almost all finite places v of k, G is quasi-split over k_v and splits over an unramified extension of k_v. Moreover, for almost all such v, P_v is hyperspecial. Any two hyperspecial parahoric subgroups of $\mathrm{G}(k_v)$ are conjugate under the action of the adjoint group $\overline{\mathrm{G}}(k_v)$ (see [T, Sec. 2.5]), so P is determined up to the action of $\overline{\mathrm{G}}(\mathbb{A}_f)$ by the types of P_v at the remaining places. Using this, we now count the number of P's.

As in Section 4.3, let T_1 denote the set of places of k for which G is not quasi-split. By the previous argument, we have

$$\#T_1 \le c_{10}\log x, \qquad \#(\text{variants for } T_1) \le x^{c_9} \quad \text{(see Sec. 4.3)}.$$

Let R denote the set of places for which G is quasi-split but is not split over an unramified extension of k_v. For such places $v \in V_f$, $l_v = l \otimes_k k_v$ is a ramified extension of k_v, and so, by the formula from [P, App.], each of such places contributes to $\mathscr{D}_{l/k}$ a power of q_v. Again, using Proposition 3.2(ii) and the inequality $\mathscr{D}_{l/k} \le x^c$ from Section 4.1, we obtain

$$\#R \le c_{12}\log x, \qquad \#(\text{variants for } R) \le x^{c_{13}}.$$

Finally, let $T_2 \subset V_f\backslash(T_1 \cup R)$ be the set of places for which P_v is not hyperspecial. If $v \in T_2$, then by [P, Prop. 2.10(iv)],

$$e_v \ge (q_v+1)^{-1} q_v^{r_v+1}.$$

Similarly to (6), this implies that

$$f_v \ge q_v^{\delta}.$$

By Proposition 3.2(ii) and the volume formula,

$$\#T_2 \le c_{14}\log x, \qquad \#(\text{variants for } T_2) \le x^{c_{15}}.$$

For a given $v \in V_f$, the number of the possible types of parahoric subgroups (parametrised by the subsets of the set of simple roots) is bounded by a constant c_t that depends only on the absolute type of G. We conclude that for a given G, the number of P's up to the action of $\overline{\mathrm{G}}(\mathbb{A}_f)$ is at most

$$c_t^{\#(T_1\cup R\cup T_2)}\#(\text{variants for } T_1\cup R\cup T_2) \le c_t^{(c_{10}+c_{12}+c_{14})\log x} x^{c_9+c_{13}+c_{15}} = x^{c_{16}}.$$

4.5. Counting conjugacy classes

In this final step, we give an upper bound for the number of conjugacy classes of arithmetic subgroups associated to a fixed group G'/k and a given $\overline{\mathrm{G}}(\mathbb{A}_f)$-orbit of collections of parahoric subgroups P of a simply connected group G centrally k-isogenous to G′. We are interested in the $\overline{\mathrm{G}}(k)$-conjugacy classes of maximal subgroups associated to P which are indexed by the double cosets $\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A})/\overline{\mathrm{G}}_\infty\overline{P}$, where $\overline{\mathrm{G}}_\infty = \prod_{v\in V_\infty}\overline{\mathrm{G}}(k_v)$, $\overline{P}_v$ is the stabiliser of P_v in $\overline{\mathrm{G}}(k_v)$, and $\overline{P} = \prod_{v\in V_f}\overline{P}_v$ is a compact open subgroup of $\overline{\mathrm{G}}_f = \prod_{v\in V_f}\overline{\mathrm{G}}(k_v)$ (see [BP, Prop. 3.10]). The number $c(\overline{P})$ of the double cosets is called the *class number* of $\overline{\mathrm{G}}$ with respect to $\overline{P}$. The

argument is similar to [BP, proof of Prop. 3.9], except that we need to get an explicit upper bound for $c(\overline{P})$.

Let ω be a nonzero invariant exterior k-form of top degree on $\overline{\mathrm{G}}$; such a form is unique up to multiplication by an element of k^* and is called a *Tamagawa form*. We denote by $|\omega|$ the Haar measure on the adèle group $\overline{\mathrm{G}}(\mathbb{A})$ determined by ω. The natural embedding of k into $\mathbb{A}$ gives an embedding of $\overline{\mathrm{G}}(k)$ in $\overline{\mathrm{G}}(\mathbb{A})$; it is well known that the image of $\overline{\mathrm{G}}(k)$ is a lattice in $\overline{\mathrm{G}}(\mathbb{A})$. By the product formula, its covolume with respect to the measure $|\omega|$ does not depend on the choice of the form ω; thus the number $\tau_k(\overline{\mathrm{G}}) := \mathscr{D}_k^{-\dim(\overline{\mathrm{G}})/2}|\omega|(\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A}))$ is correctly defined. It is called the *Tamagawa number* of $\overline{\mathrm{G}}/k$. By a theorem of Ono in [On], $\tau_k(\overline{\mathrm{G}})$ is bounded by a constant multiple of the order of the center of the simply connected covering group G multiplied by $\tau_k(\mathrm{G})$. According to the Weil conjecture, the Tamagawa number of a simply connected group is equal to 1; this has been proved completely for the groups over number fields, thanks to the work of many people (see [P, Sec. 3.3] for a short discussion). Therefore, we have

$$|\omega|\big(\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A})\big) = \tau_k(\overline{\mathrm{G}})\mathscr{D}_k^{\dim(\overline{\mathrm{G}})/2} \le c_{17}\mathscr{D}_k^{\dim(\overline{\mathrm{G}})/2}, \tag{7}$$

where c_{17} depends only on the absolute type of G.

Coming back to the problem of bounding the class number $c(\overline{P})$, we recall that the double cosets $\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A})/\overline{\mathrm{G}}_\infty\overline{P}$ correspond bijectively to the orbits of $\overline{\mathrm{G}}_\infty\overline{P}$ on $\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A})$, which are open. Given an upper bound for $|\omega|(\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A}))$, in order to give a bound for $c(\overline{P})$ it is enough to obtain a uniform lower bound for the $|\omega|$-volumes of these orbits. The double cosets are represented by elements of $\overline{\mathrm{G}}_f$, so it is sufficient to consider the orbit of the image of $a \in \overline{\mathrm{G}}_f$ which is isomorphic to $\Gamma_a\backslash\overline{\mathrm{G}}_\infty a\overline{P}a^{-1}$, $\Gamma_a = \overline{\mathrm{G}}(k) \cap \overline{\mathrm{G}}_\infty a\overline{P}a^{-1}$. Let Γ'_a be the projection of Γ_a to $\overline{\mathrm{G}}_\infty$ with respect to the decomposition $\overline{\mathrm{G}}(\mathbb{A}) = \overline{\mathrm{G}}_\infty \times \overline{\mathrm{G}}_f$. As $a\overline{P}a^{-1}$ is a compact open subgroup of $\overline{\mathrm{G}}_f$, Γ'_a is an arithmetic subgroup of $\overline{\mathrm{G}}_\infty$. We have

$$|\omega|(\Gamma_a\backslash\overline{\mathrm{G}}_\infty a\overline{P}a^{-1}) = |\omega|_\infty(\Gamma'_a\backslash\overline{\mathrm{G}}_\infty)|\omega|_f(\overline{P}),$$

where $|\omega|_\infty$, $|\omega|_f$ denote the product measures on $\overline{\mathrm{G}}_\infty$, $\overline{\mathrm{G}}_f$ corresponding to ω.

In order to estimate the factors in the right-hand side of the formula, for each $v \in V(k)$ we relate the measure $|\omega|$ to the canonical measure $|\omega_{\overline{\mathrm{G}}_v}|$ on $\overline{\mathrm{G}}(k_v)$ defined in [G, Secs. 4, 11]. In particular, if G is simply connected, then the measure $|\omega_{\mathrm{G}_v}|$ coincides with the measure $\gamma_v\omega_v^*$ that is used for the local computations in [P]; for $v \in V_\infty(k)$, $|\omega_{\overline{\mathrm{G}}_v}|$ is equal to the measure μ on $\overline{\mathrm{G}}(k_v)$ defined as in Section 2.5; and for all but finitely many v, $|\omega_{\overline{\mathrm{G}}_v}| = |\omega|_v$. Let γ_v denote the ratio $|\omega_{\overline{\mathrm{G}}_v}|/|\omega|_v$, which by the previous remark is equal to 1 for all but finitely many places v. Hence,

$$|\omega|(\Gamma_a\backslash\overline{\mathrm{G}}_\infty a\overline{P}a^{-1}) = \mu_\infty(\Gamma'_a\backslash\overline{\mathrm{G}}_\infty)\prod_{v\in V_f}|\omega_{\overline{\mathrm{G}}_v}|(\overline{P}_v)/\prod_{v\in V}\gamma_v.$$

We now recall the main result of [BP], which implies that covolumes of arithmetic subgroups of $\overline{\mathrm{G}}_\infty$ with respect to the measure μ are bounded from below by a universal constant, and thus, $\mu_\infty(\Gamma'_a\backslash\overline{\mathrm{G}}_\infty) \geq \mu_0$.

The crucial ingredient that allows us to carry out the required estimates is the product formula for γ_v. It was obtained in [P, Th. 1.6] for the simply connected groups and later extended by Gross to arbitrary reductive groups defined over number fields (see also [Ku] for the groups over global function fields). Thus, by [G, Th. 11.5], we have

$$\prod_{v\in V}\gamma_v = (\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s/2}\Big(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]}.$$

Finally, we make use of the following inequality.

CLAIM
We have $|\omega_{\overline{\mathrm{G}}_v}|(\overline{P}_v) \geq |\omega_{\mathrm{G}_v}|(P_v) = e(P_v)^{-1}$.

The proof of this claim, which is given below, is quite technical but not conceptually new; related questions were studied in detail and full generality in [G] and [Ku]. The argument falls into several steps.

Proof
Let $K = k_v$ be a nonarchimedean local field, let $\mathscr{O}$ be its ring of integers, let G be a simply connected semisimple K-group, let $i : \mathrm{G} \to \mathrm{G}'$ be a central K-isogeny (we actually need only the case $\mathrm{G}' = \overline{\mathrm{G}}$), and let $X = X(\mathrm{G})$ denote the Bruhat-Tits building of G/K.

(1) We assume first that the groups G and G′ are quasi-split over K. Let $x \in X$ be a special vertex in X chosen as in [G, Sec. 4] (see also [P, Sec. 1.2]). The Bruhat-Tits theory assigns to G'/K and $x \in X(\mathrm{G})$ a smooth affine group scheme $\underline{\mathrm{G}}'^0_x$ over $\mathscr{O}$. Its generic fiber is isomorphic to G'/K, and its special fiber $\overline{\mathrm{G}}'^0_x$ is connected. Let $P_x = \underline{\mathrm{G}}_x(\mathscr{O})(= \underline{\mathrm{G}}^0_x(\mathscr{O}))$, $P'_x = \underline{\mathrm{G}}'^0_x(\mathscr{O})$. Then P_x (resp., P'_x) is an open compact subgroup of $\mathrm{G}(K)$ (resp., $\mathrm{G}'(K)$), P_x is the stabiliser of x in $\mathrm{G}(K)$, and P'_x is contained in the stabiliser of x in $\mathrm{G}'(K)$ with finite index. Recall also that the measure $|\omega_{\mathrm{G}}|$ (resp., $|\omega_{\mathrm{G}'}|$) corresponds to a differential ω_{G} (resp., $\omega_{\mathrm{G}'}$) of top degree on G (resp., G′) over K which has good reduction (see [G, Sec. 4]). This brings us to the conditions of [Oe, Prop. 1.2.5], which implies

$$|\omega_{\mathrm{G}}|(P_x) = \#\overline{\mathrm{G}}_x(\mathrm{F}_q)q^{-\dim \mathrm{G}}, \qquad |\omega_{\mathrm{G}'}|(P'_x) = \#\overline{\mathrm{G}}'^0_x(\mathrm{F}_q)q^{-\dim \mathrm{G}'}.$$

(F_q denotes the residue field of K.)

Since G and G′ are isogenous, $\overline{\mathrm{G}}_x$ and $\overline{\mathrm{G}}'_x$ are isogenous. Hence, $\dim \mathrm{G} = \dim \mathrm{G}'$, and by Lang's theorem (see [L]), $\#\overline{\mathrm{G}}_x(\mathrm{F}_q) = \#\overline{\mathrm{G}}'^0_x(\mathrm{F}_q)$. Thus, we obtain $|\omega_{\mathrm{G}}|(P_x) = |\omega_{\mathrm{G}'}|(P'_x)$.

(2) Now, let C be a chamber of X which contains x, and let Ω be a subset of C. Denote by I_C (resp., I'_C) the Iwahori subgroup of $\mathrm{G}(K)$ (resp., $\mathrm{G}'(K)$) corresponding to C. Note that by definition, I'_C is the preimage in $\mathrm{G}'(K)$ of a Borel subgroup $\overline{B}'$ of $\overline{\mathrm{G}}'^0_x(\mathrm{F}_q)$. Let P_Ω (resp., P'_Ω) be the parahoric subgroup of $\mathrm{G}(K)$ (resp., $\mathrm{G}'(K)$) associated to Ω; so $P_\Omega = \underline{\mathrm{G}}_\Omega(\mathscr{O})$, $P'_\Omega = \underline{\mathrm{G}}'^0_\Omega(\mathscr{O})$, and any parahoric subgroup of $\mathrm{G}(K)$ is conjugate to some P_Ω. The inclusion $\Omega \subset C$ induces a group scheme homomorphism $\rho_{\Omega C} : \underline{\mathrm{G}}'^0_C \to \underline{\mathrm{G}}'^0_\Omega$ whose reduction maps the group $\overline{\mathrm{G}}'^0_C$ onto a Borel subgroup $\overline{B}'$ of $\overline{\mathrm{G}}'^0_\Omega$. Therefore, we have

$$[P'_\Omega : I'_C] = [\overline{\mathrm{G}}'^0_\Omega(\mathrm{F}_q) : \overline{B}'] = [\overline{\mathrm{G}}_\Omega(\mathrm{F}_q) : \overline{B}] = [P_\Omega : I_C],$$

as $\overline{\mathrm{G}}'^0_\Omega$ is isogenous to $\overline{\mathrm{G}}_\Omega$, $\overline{B}'$ is isogenous to $\overline{B}$, and all the groups are connected. It follows that $|\omega_{\mathrm{G}'}|(P'_\Omega) = |\omega_{\mathrm{G}}|(P_\Omega)$.

(3) We finally note that $P'_\Omega \subset \overline{P}'_\Omega$ ($\overline{P}'_\Omega$ denotes the stabiliser of Ω in $\mathrm{G}'(K)$), and thus, $|\omega_{\mathrm{G}'}|(\overline{P}'_\Omega) \geq |\omega_{\mathrm{G}'}|(P'_\Omega) = |\omega_{\mathrm{G}}|(P_\Omega)$, which implies the desired inequality in the quasi-split case.

(4) In order to extend this result to the general case, we have to recall the definition of the canonical measure $|\omega_{\mathrm{G}}|$ for the general G by pullback from the quasi-split inner form (see [G, p. 294]) and its interpretation in terms of the volume form ν_G associated to an Iwahori subgroup of $\mathrm{G}(K)$ described in [G, pp. 294–295]. The latter allows us to apply the argument similar to step (1) to Iwahori subgroups I_C and I'_C corresponding to a chamber C of $X(\mathrm{G})$, proving $|\omega_{\mathrm{G}}|(I_C) = |\omega_{\mathrm{G}'}|(I'_C)$. All the rest of the proof does not depend on the quasi-split assumption, and the claim follows. □

Let us collect together the results of this section. We obtain

$$\begin{aligned} c(\overline{P}) &\leq \frac{|\omega|(\overline{\mathrm{G}}(k)\backslash\overline{\mathrm{G}}(\mathbb{A}))\prod_{v\in V}\gamma_v}{\mu_0\prod_{v\in V_f} e(P_v)^{-1}} \\ &\leq \frac{1}{\mu_0}\mathscr{D}_k^{\dim(\overline{\mathrm{G}})/2}(\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s/2}\Big(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]}\tau_k(\overline{\mathrm{G}})\mathscr{E}(P). \end{aligned} \tag{8}$$

This formula can be viewed as an extension of the upper bound for the class number from [P, Th. 4.3].

We now bound the right-hand side of (8). By Sections 4.1 and 4.2, we have $\mathscr{D}_k < (x/c_1)^{1/\delta_1}$, $\mathscr{D}_{l/k} < (x/c_1)^{1/\delta_2}$, and $[k : \mathbb{Q}] \leq c \log \mathscr{D}_k$. By (7), $\tau_k(\overline{\mathrm{G}}) \leq c_{17}$. From Sections 4.3 and 4.4, it follows that if $\mu(H/\Gamma) \leq x$, then $\prod_{v\in V_f} e_v \leq x^{c_{18}}$ for

some constant c_{18} that depends only on the type of almost simple factors of H. Hence, it follows from (8) that there exists a constant c_{19} such that

$$c(\overline{P}) \leq x^{c_{19}}.$$

4.6. The upper bounds

It remains to combine the results of the previous sections to get the upper bounds. By Sections 4.1, 4.3, 4.4, and 4.5,

$$m_H^u(x) \leq x^{c_2\beta(x)+c_3} x^{c_{11}} x^{c_{16}} x^{c_{19}} \leq x^{B\beta(x)},$$

and constant B depends only on the type of almost simple factors of H.

By Sections 4.2, 4.3, 4.4, and 4.5,

$$m_H^{nu}(x) \leq x^{c_5} x^{c_{11}} x^{c_{16}} x^{c_{19}} \leq x^{B'},$$

and constant B' depends on the type and the number of almost simple factors of H.

5. Proof of Theorem 1: The lower bound

5.1. Cocompact case

A theorem of Borel and Harder [BH] implies that a semisimple group over a local field of characteristic zero contains cocompact arithmetic lattices. The method of [BH] actually proves the existence of such lattices defined over a given field k, which satisfies a natural admissibility condition, for any isotypic semisimple Lie group. So, if H has a_1 real and a_2 complex almost simple factors (all of the same type) and k is a number field with greater than a_1 real and precisely a_2 complex places, then H contains a cocompact arithmetic subgroup Γ_1 defined over k.

Let Γ_0 be a maximal arithmetic subgroup of H which contains Γ_1. There exists an absolutely almost simple, simply connected k-group G and a principal arithmetic subgroup Λ_0 of G such that $\Gamma_0 = N_H(\phi(\Lambda_0))$.

We assume that x is large enough and estimate the number of principal arithmetic subgroups $\Lambda \subset \mathrm{G}(k)$ which are associated to the coherent collections of parahoric subgroups of $\mathcal{O}$-maximal types (see [R], [RC]) and such that $\mu_S(\mathrm{G}_S/\Lambda) < x$. Then for $\Gamma = N_H(\phi(\Lambda))$, we also have $\mu_S(\mathrm{G}_S/\Gamma) < x$. Moreover, by Rohlfs's theorem, each such Γ is a maximal arithmetic subgroup of H and all maximal arithmetic subgroups of H are obtained as the normalizers of the images of the principal arithmetic subgroups corresponding to $\mathcal{O}$-maximal collections of parahorics.

The condition of maximality for the type of a collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ is a local condition on the types of P_v at each $v \in V_f$, while $\mathcal{O}$-maximality requires an additional global restriction that is needed to further narrow

down the set of admissible collections of parahoric subgroups of maximal types. We do not give precise definitions here, referring the reader to the articles [R] and [RC]. What is important for our argument is that given $P_0 = (P_{0,v})_{v\in V_f}$, a collection of parahoric subgroups of $\mathcal{O}$-maximal type, for every $v_0 \in V_f$ there exists another $\mathcal{O}$-maximal collection $P = (P_v)_{v\in V_f}$ such that for $v \neq v_0$, $P_v = P_{0,v}$ and $P_{v_0} \ncong P_{0,v_0}$. This is clearly true. For the groups of the absolute rank greater than one (which is our standing assumption), it is enough to consider the maximal types corresponding to single vertices of the affine Dynkin diagram, and for such types, $\mathcal{O}$-maximality can be easily checked.

We have

$$\begin{aligned}\mu_S(\mathrm{G}_S/\Lambda) &= \mathscr{D}_k^{\dim(\mathrm{G})/2}(\mathscr{D}_l/\mathscr{D}_k^{[l:k]})^{s/2}\Big(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\Big)^{[k:\mathbb{Q}]}\tau_k(\mathrm{G})\,\mathscr{E}(P)\\ &= c_1\prod_{v\in T} e(P_v)/e(P_{0,v})\\ &\le c_1\prod_{v\in T} e(P_v), \quad c_1 = \mu_S(\mathrm{G}_S/\Lambda_0),\end{aligned}$$

where P_v (resp., $P_{0,v}$) is the closure of Λ (resp., Λ_0) in $\mathrm{G}(k_v)$, $v \in V_f$; T is a finite subset of the nonarchimedean places of k for which $P_v \ncong P_{0,v}$; and the constant c_1 depends on G/k and Λ_0 but does not depend on the choice of Λ.

If $\prod_{v\in T} e(P_v) < x/c_1$, then $\mu_S(\mathrm{G}_S/\Lambda) < x$. There exists a constant δ determined by the absolute type of G such that for every $v \in V_f$ and every parahoric subgroup $P_v \subset \mathrm{G}(k_v)$, $e(P_v) \le q_v^{\delta}$ (e.g., take $\delta = \dim(\mathrm{G})$). This implies

$$\prod_{v\in T} e(P_v) \le \prod_{v\in T} q_v^{\delta}.$$

Hence, $\prod_{v\in T} q_v < (x/c_1)^{1/\delta}$ is sufficient for $\mu_S(\mathrm{G}/\Lambda) < x$. The number of variants for such sets T is controlled via Proposition 3.2(i). (Note that the field k is fixed.) We obtain that for large enough x, there are at least

$$c_2\Big(\frac{x}{c_1}\Big)^{1/\delta} \ge x^A$$

variants for T, where the constant $A > 0$ is determined by δ and, thus, depends only on the absolute type of G.

It remains to recall that for each T, there exists a collection of parahoric subgroups $P = (P_v)_{v\in V_f}$ such that $P_v = P_{0,v}$ for $v \in V_f \setminus T$, $P_v \ncong P_{0,v}$ for $v \in T$, and P has $\mathcal{O}$-maximal type. Each such collection defines a maximal arithmetic subgroup of H of

covolume less than x, and subgroups corresponding to different T's are not conjugate. The number of maximal arithmetic subgroups obtained this way is at least x^A with $A > 0$, a constant depending only on the absolute type of G. This proves the lower bound for $m_H^u(x)$.

Note that all the maximal arithmetic subgroups constructed in this section are commensurable. It is also possible to construct different commensurability classes that contain arithmetic subgroups of covolumes less than x. This may enlarge the constant A in our asymptotic inequality but, since it follows from the first part of the proof and the conjecture on the number of isomorphism classes of fields with discriminant less than x, would hardly change the type of the asymptotic.

5.2. Non-cocompact case

Let now Γ_1 be a nonuniform irreducible lattice in H which exists by the assumption of Theorem 1(B), and let G be a corresponding algebraic k-group. Arithmetic subgroups of H which are induced from $G(k)$ are all noncocompact. (They are actually commensurable with Γ_1.) To prove the lower bound for $m_H^{nu}(x)$, it remains to repeat the argument of Section 5.1 for the group G.

Note that contrary to the compact case, the existence of non-cocompact arithmetic lattices in H generally is not guaranteed by the condition that H is isotypic (for a counterexample, see Sec. 2.3). The conditions under which such examples can be constructed are rather exceptional; in most cases, isotypic groups contain both cocompact and non-cocompact arithmetic subgroups.

The theorem is now proved. □

6. Corollaries, conjectures, remarks

6.1

COROLLARY

There exists a constant C_1 that depends only on the type of almost simple factors of H such that if Λ is a principal arithmetic subgroup of H and $\Gamma = N_H(\Lambda)$ has covolume less than x, then $[\Gamma : \Lambda] \le x^{C_1}$.

Proof

By [BP] (see Sec. 2.7 for the notation and precise references),

$$[\Gamma : \Lambda] \le n^{\epsilon \# S} \cdot 2h_l^{\epsilon'} n^{\epsilon a(k) + \epsilon' a(l) + \epsilon \# T} (\mathscr{D}_l / \mathscr{D}_k^{[l:k]})^{\epsilon''} \cdot \prod_{v \in V_f} \# \Xi_{\Theta_v}. \tag{9}$$

Now, since $\mu(H/\Gamma) \le x$, the group Γ has to satisfy the conditions on the subgroups of covolume less than x obtained in the proof of the upper bound of Theorem 1. (In Theorem 1, only maximal arithmetic subgroups are considered, but the proof of the

upper bound applies without a change to arbitrary principal arithmetic subgroups and their normalisers, thus providing a somewhat stronger result to which we appeal here.) We have

$$\mathscr{D}_l/\mathscr{D}_k^{[l:k]} = \mathscr{D}_{l/k} \le x^{c_1}, \qquad a(k) \le c_2 \log x \quad \text{(Sec. 4.1)},$$
$$\#T \le \#T_1 \le c_3 \log x \quad \text{(Sec. 4.3)},$$
$$\{v \in V_f,\ \#\Xi_{\Theta_v} \ne 1\} \subset T_1 \cup R \cup T_2$$

as for the rest of v, P_v is special, so

$$\#\{v \in V_f,\ \#\Xi_{\Theta_v} \ne 1\} \le \#(T_1 \cup R \cup T_2) \le c_4 \log x \quad \text{(Sec. 4.4)}.$$

Also, recall that $\#\Xi_{\Theta_v} \le r+1$, r is the absolute rank of G; $h_l \le c^{[l:\mathbb{Q}]}\mathscr{D}_l \le x^{c_5}$ (see, e.g., the proof of Prop. 3.3); and $a(l) \le 3a(k)$ (as $[l:k] \le 3$). Altogether, these imply the corollary. □

6.2

For some particular cases, the bound in Corollary 6.1 can be improved. Let us assume that the degrees of the fields of definition of the arithmetic subgroups are bounded. Thus,

$$[k:\mathbb{Q}] \le d, \tag{10}$$

which is the case, for example, if we consider only nonuniform lattices in H.

Assumption (10) implies that the number m of different prime ideals $\mathscr{P}_1, \ldots, \mathscr{P}_m$ of $\mathscr{O}_k$ such that $\mathrm{Norm}(\mathscr{P}_1 \cdots \mathscr{P}_m) \le x$ is bounded by $c \log x / \log\log x$, $c = c(d)$ (instead of the bound $\log x$ that we used for the general case). Indeed, $k = \mathbb{Q}$ follows from the prime number theorem, and the case of arbitrary k of bounded degree can be easily reduced to the rational case.

Therefore, assumption (10) implies that most of the terms in (9) are at most $c^{\log x/\log\log x}$ with $c = c(d)$. What remains is $\mathscr{D}_{l/k} = \mathscr{D}_l/\mathscr{D}_k^{[l:k]}$ (for type D_r, r even) and h_l (or h_k if $l = k$). The former, in fact, appears in the formula as an upper bound for $2^{\#R}$ (see [BP, Sec. 5.5]), which again can be improved to $c^{\log x/\log\log x}$ by the same argument. What remains is the class number.

Going back to [BP, Sec. 5, Prop. 0.12], we see that what occurs in the formula is not h_l but the order of the group $C_n(l)$, which consists of the elements of the class group $C(l)$ whose orders divide n. (As before, n is a constant determined by the type of H.) Instead of using the trivial bound $\#C_n(l) \le h_l$, let us keep it as it is. We now come to the following formula:

$$[\Gamma : \Lambda] \le c^{\log x/\log\log x} \#C_n(l), \quad x \ge \mu(H/\Gamma). \tag{11}$$

If $n = p$ is a prime, let $\rho_p(l)$ denote the p-rank of $C(l)$. Then, clearly, $\#C_p(l) \leq p^{\rho_p(l)}$, and in general, for $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, $\#C_n(l) \leq p_1^{\alpha_1 \rho_{p_1}(l)} \cdots p_m^{\alpha_m \rho_{p_m}(l)}$. So, we are interested in the upper bounds for p-ranks of the class groups.

Apparently, even though this and related questions have been much studied, there are very few results beyond Gauss's celebrated theorem, which can be applied in our case. We have the fact that

(i) if $[l : \mathbb{Q}] = 2$, then $\rho_2(l) \leq t_l - 1$ (by Gauss);

(ii) if $[k : \mathbb{Q}] = 2$ and $[l : k] = 2$, then $\rho_2(l) \leq 2(t_l + t_k - 1)$ (by [Co, Th. 2]),

where t_k (resp., t_l) denotes the number of primes ramified in $k/\mathbb{Q}$ (resp., l/k).

From this, we obtain

$$\#C_n(l) \leq n^{c \log x / \log\log x} \tag{12}$$

if n is a power of 2 and l is as in (i) or (ii).

Similar results for other n and other fields can only be conjectured; even $\rho_3(k)$ for quadratic fields k seems to be out of reach with the currently available methods. Nevertheless, estimates (11) and (12) imply the following corollary.

6.3

COROLLARY

Let H be a simple Lie group of type $A_{2^\alpha - 1}$ ($\alpha > 1$), B_r, C_r, D_r ($r \neq 4$), E_7, E_8, F_4, *or* G_2. *There exists a constant C_2 that depends only on the type of H such that if Λ is a non-cocompact principal arithmetic subgroup of H and if $\Gamma = N_H(\Lambda)$ has covolume less than x, then* $[\Gamma : \Lambda] \leq C_2^{\log x / \log\log x}$.

Proof

Indeed, the assumption that H has one of the given types implies that n is a power of 2 (see the definition of n in Sec. 2.7). Since H is simple and the arithmetic subgroup is noncompact, its field of definition k is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$, depending on whether H is a real or complex Lie group (see also Sec. 4.2). Finally, the fact that the type of H is not D_4 implies that $[l : k] \leq 2$. The corollary now follows from the discussion in Section 6.2. □

We expect similar estimates to be valid for the nonuniform lattices in other groups, but we do not know how to prove it.

6.4

Remark. Concerning the general case, let us point out that if the degrees of the fields are a priori not bounded, then we cannot expect a $(\log x / \log\log x)$-bound for the p-rank of the class group. An example of a sequence of fields k_i for which $\rho_2(k_i)$ grows as $\log \mathscr{D}_{k_i}$ was constructed by Hajir [H, Sec. 5]. The fields k_i in Hajir's example form an infinite class field tower. This remark together with the previous estimates motivates the following question:

Is the estimate in Corollary 6.1 sharp?; that is, given a group H, is there a constant $C_0 = C_0(H) > 0$ such that there exists an infinite sequence of pairwise nonconjugate principal arithmetic subgroups Λ_i in H for which $[\Gamma_i : \Lambda_i] \geq \mu(H/\Gamma_i)^{C_0}$, where $\Gamma_i = N_H(\Lambda_i)$ and μ is a Haar measure on H?

Corollaries 6.1 and 6.3 are important in [BL], in which the growth rate of the number of irreducible lattices in semisimple Lie groups is studied.

6.5

Remark. Groups of type A_1 have been consistently excluded here. It is not feasible to use the formula from Section 2.8 combined with an analogue of Proposition 3.3 for this case, even to prove a finiteness result. However, one can follow another method, also due to Borel, and employ geometric bounds for the index of a principal arithmetic subgroup in a maximal arithmetic. This indeed allows us to establish the finiteness of the number of arithmetic subgroups of bounded covolume in $\mathrm{SL}(2,\mathbb{R})^a \times \mathrm{SL}(2,\mathbb{C})^b$ (see [B]). The problem is that the quantitative bounds that can be obtained this way are only exponential. We suppose that the true bounds should be similar to the general case (and conjecturally polynomial), although we do not know how to prove this conjecture and leave it as an open problem.

Problem

Find the growth rate of the number of maximal arithmetic subgroups for the semisimple Lie groups whose almost simple factors have type A_1 or obtain a better-than-exponential upper bound for the growth.

6.6

The following two conjectures were mentioned in the introduction.

CONJECTURE 1

There exists an absolute constant B such that for large enough x, the number of isomorphism classes of number fields with discriminants less than x is at most x^B.

CONJECTURE 2

Given a connected semisimple Lie group H without almost simple factors of type A_1 and without compact factors, there exists a constant $B_H > 0$ which depends only on the type of almost simple factors of H such that for large enough x, the number of conjugacy classes of maximal irreducible arithmetic subgroups of H of covolumes less than x is at most x^{B_H}.

We now prove the following proposition.

PROPOSITION
Conjectures 1 and 2 are equivalent.

Proof
The implication that $1 \to 2$ follows directly from the proof of the upper bounds in Theorem 1. Consider the implication that $2 \to 1$. Assume that Conjecture 2 is true but that Conjecture 1 is false (i.e., $m_H^u(x) + m_H^{nu}(x) \le x^{B_H}$ for every $x > x_0$, and for an arbitrary C, there exists $x > x_0$ such that $N(x) > x^C$). Some additional assumptions on x_0 are needed, a fact that becomes clear later; these could have been imposed from the beginning. So, let us fix $C > 1$, and let $x > x_0$ be such that $N(x) > x^C$.

Let $N_{i,j}(x)$ denote the number of extensions of $\mathbb{Q}$ of discriminant less than x which have precisely i real and j complex places. We have

$$N(x) = \sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} N_{i,j}(x).$$

The condition that the discriminants of the fields are less than x implies, by Minkowski's theorem, that the degrees of the extensions are bounded by $c \log x$ for an absolute constant c, and so the number of summands is less than $(c \log x)^2$. By Dirichlet's box principle, there exists a pair (i, j) such that $N_{i,j}(x) > x^C/(c \log x)^2 \ge x^{C-1}$. (This inequality requires $(c \log x)^2 \le x$, which is true for large enough x and gives a first condition on x_0.) Let $\mathscr{K}$ be the set of such number fields, $\#\mathscr{K} = N_{i,j}(x) > x^{C-1}$. Consider a simply connected semisimple Lie group H that has i split real simple factors and j complex simple factors all of the same type. For each $k \in \mathscr{K}$, let G/k be a simply connected, absolutely simple split group of the same absolute type as the simple factors of H, defined over k. Let $P = (P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups of G which are all hyperspecial (such a collection exists since G splits over k), and let Λ be the principal arithmetic subgroup of H defined by P. We have

(a) for $S = V_\infty(k)$, $\mathrm{G}_S \cong H$;
(b) $\mu(H/\Lambda) = \mathscr{D}_k^{\dim(\mathrm{G})/2} \big(\prod_{i=1}^r m_i!/(2\pi)^{m_i+1} \big)^{[k:\mathbb{Q}]} \mathscr{E}(P)$ by Prasad's formula.

Using the orders of finite groups of Lie type, the Euler product $\mathscr{E}(P)$ can be expressed as a product of the Dedekind zeta function of k, and certain Dirichlet L-functions at the integers $m_i + 1$, m_i are the Lie exponents of G (see [P, Rem. 3.11]). Obvious inequalities $L(s, \chi) \le \zeta_k(s)$ and $\zeta_k(s) \le \zeta(s)^{[k:\mathbb{Q}]}$, for $s \ge 2$, imply that there exists a constant c_1 that depends only on the type of simple factors of H and such that each zeta or L-function in the product is bounded from above by $c_1^{[k:\mathbb{Q}]}$. Since $[k : \mathbb{Q}] \le c \log x$, we have $\mathscr{E}(P) \le (c_1^{c \log x})^r$. By definition of the set $\mathscr{K}$, $k \in \mathscr{K}$

implies $\mathscr{D}_k \leq x$. Therefore, we obtain

$$\mu(H/\Lambda) \leq x^{\dim(G)/2} c_2^{c \log x} (c_1^{c \log x})^r \leq x^{\delta},$$

where δ is greater than 1 and depends only on the type of simple factors of H. The latter inequality may require that x be larger than a certain value that depends on the type of simple factors of H and that gives us the second condition on x_0. Clearly, both conditions do not depend on C, and both could be imposed from the beginning.

For each $k \in \mathscr{K}$, we have at least one maximal arithmetic subgroup of H of covolume less then x^{δ}. Now, if we take $C = \delta B_H + 1$, we arrive at a contradiction with Conjecture 2 for H and $x^{\delta} > x_0$. □

Let us note that in the proof, Conjecture 2 is used only for non-cocompact arithmetic subgroups of semisimple groups H which have simple factors of a fixed type. It then implies Conjecture 1, which in turn implies Conjecture 2 in the whole generality. It is possible to specify further the relation between two conjectures, but we do not go into details. What we emphasise is that our result provides a new geometric interpretation for a classical number-theoretic problem. An optimistic expectation would be that study of the distributions of lattices in semisimple Lie groups can give a new insight on the number fields and their discriminants.

Appendix

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A.1

Let $N(X)$ denote the number of isomorphism classes of number fields with discriminant less than X.

THEOREM

For every $\epsilon > 0$, there is a constant $C(\epsilon)$ such that $\log N(X) \leq C(\epsilon)(\log X)^{1+\epsilon}$ *for every $X \geq 2$.*

In fact, we prove the more precise upper bound that

$$\log N(X) \leq C_6 \log X \exp(C_7 \sqrt{\log \log X})$$

for absolute constants C_6, C_7.

This theorem (almost) follows from [EV, Th. 1.1], the only point being to control the dependence of implicit constants on the degree of the number field.

We refer to [EV] for further information and for some motivational comments about the method. In the proof, $C_1, C_2, \ldots$ denote certain *absolute* constants.

A.2

Let K be an extension of $\mathbb{Q}$ of degree $d \geq 200$. Denote by $\Sigma(K)$ the set of embeddings of K into $\mathbb{C}$ ($\#\Sigma(K) = d$), and denote by $\overline{\Sigma}(K)$ a set of representatives for $\Sigma(K)$ modulo complex conjugation (in the notation of the article, $\overline{\Sigma}(K) = V_\infty(K)$). We regard the ring of integers $\mathcal{O}_K$ as a lattice in $K \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{\sigma \in \Sigma(K)} K_\sigma$. We endow the real vector space $K \otimes_{\mathbb{Q}} \mathbb{R}$ with the supremum norm (i.e., $\|(x_\sigma)\| = \sup_\sigma |x_\sigma|$). Here, $|\cdot|$ denotes the standard absolute value on $\mathbb{C}$. In particular, we obtain a "norm" on $\mathcal{O}_K$ by restriction. Explicitly, for $z \in \mathcal{O}_K$, we have $\|z\| = \sup_{\sigma \in \Sigma(K)} |\sigma(z)|$.

We denote by $M_d(\mathbb{Z})$ (resp., $M_d(\mathbb{Q})$) the algebra of d by d matrices over $\mathbb{Z}$ (resp., $\mathbb{Q}$).

By *trace form* we mean the pairing $(x, y) \mapsto \mathrm{Tr}_{K/\mathbb{Q}}(xy)$. It is a symmetric nondegenerate $\mathbb{Q}$-bilinear pairing on K^2.

Let s be a positive integer that can be specified later. We denote by $\mathbf{y} = (y_1, y_2, \ldots, y_s)$ an ordered s-tuple of elements of $\mathcal{O}_K$, and we write $\|\mathbf{y}\| := \max(\|y_1\|, \ldots, \|y_s\|)$. For $\mathbf{y} = (y_1, \ldots, y_s) \in \mathcal{O}_K^s$ and $l \geq 1$, we set

$$\begin{aligned} S(l) &= \left\{(k_1, \ldots, k_s) \in \mathbb{Z}^s : k_1 + \cdots + k_s \leq l, k_1, \ldots, k_s \geq 0\right\}, \\ S(\mathbf{y}, l) &= \left\{y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s} : (k_1, \ldots, k_s) \in S(l)\right\} \subset \mathcal{O}_K. \end{aligned} \tag{13}$$

If S is a subset of $S(l)$, we denote by $S(\mathbf{y})$ the set $\{y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s} : (k_1, \ldots, k_s) \in S\}$.

A.3

LEMMA

Let S be a subset of $S(l)$ such that $S(\mathbf{y})$ spans a $\mathbb{Q}$-linear subspace of K with dimension strictly greater than $d/2$. Let $S + S$ be the set of sums of two elements of S. Then $(S + S)(\mathbf{y})$ spans K over $\mathbb{Q}$.

Proof (see [EV, Lem. 2.1])

Suppose that there existed $z \in K$ which was perpendicular, with respect to the trace form, to the $\mathbb{Q}$-span of $(S + S)(\mathbf{y})$. Since $(S + S)(\mathbf{y})$ consists precisely of all products $\alpha\beta$, with $\alpha, \beta \in S(\mathbf{y})$, it follows that

$$\mathrm{Tr}_{K/\mathbb{Q}}(z\alpha\beta) = 0 \quad (\alpha, \beta \in S(\mathbf{y})). \tag{14}$$

Call $W \subset K$ the $\mathbb{Q}$-linear span of $S(\mathbf{y})$. Then (14) implies that zW is perpendicular to W with respect to the trace form, contradicting $\dim(W) > d/2$. □

A.4

LEMMA

Let $\mathscr{C} \subset \mathcal{O}_K$ be a finite subset containing 1 and generating K as a field over $\mathbb{Q}$. Let $z_1, z_2, \ldots, z_d$ be a $\mathbb{Q}$-linear basis for K. For each $u \in \mathscr{C}$, let $M(u) = (\mathrm{Tr}_{K/\mathbb{Q}}(uz_iz_j))_{1\le i,j\le d} \in M_d(\mathbb{Q})$. Then the $\mathbb{Q}$-subalgebra of $M_d(\mathbb{Q})$ generated by $M(u)M(1)^{-1}$, as u ranges over $\mathscr{C}$, is isomorphic to K.

Proof (see [EV, Lem. 2.2])

In fact, $M(u)M(1)^{-1}$ gives the matrix of *multiplication by* u, in the basis $\{z_i\}$. □

A.5

We denote by $\mathscr{D}_K$ the absolute value of the discriminant of K.

LEMMA

There is an absolute constant $C_1 \in \mathbb{R}$ such that for any K as above, there exists a basis $\gamma_1, \gamma_2, \ldots, \gamma_d$ for $\mathcal{O}_K$ over $\mathbb{Z}$ such that

$$\|\gamma_j\| \le \|\gamma_{j+1}\|, \qquad \prod_{i=1}^{d} \|\gamma_i\| \le \mathscr{D}_K^{1/2} C_1^d, \qquad \|\gamma_i\| \le (C_1^d \mathscr{D}_K^{1/2})^{1/(d-i)} \quad (i < d). \tag{15}$$

Proof

This is Minkowski's second theorem, applied exactly as in [EV, Prop. 2.5]. The final statement of (15) follows from the preceding statements, in view of the fact that $\|\gamma_j\| \ge 1$ for each j. □

A.6

Let r, l be integers so that $d/2 < r \le |S(l)| = \binom{l+s}{s}$.

LEMMA

Suppose that $W \subset K$ is a $\mathbb{Q}$-linear subspace of dimension r, and let $S \subset S(l)$ be a subset of size r. Then there exists $\mathbf{y} = (y_1, y_2, \ldots, y_s) \in W^s$ such that the elements of $S(\mathbf{y})$ are $\mathbb{Q}$-linearly independent.

Proof

This is precisely [EV, Lem. 2.3]. □

A.7

LEMMA

Let $\Lambda = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 + \cdots + \mathbb{Z}\gamma_r$, and let $S \subset S(l)$ be a subset of size r. Then there is $\mathbf{y} = (y_1, y_2, \ldots, y_s) \in \Lambda^s$ such that the elements of $S(\mathbf{y})$ are linearly independent over $\mathbb{Q}$, and $\|\mathbf{y}\| \le r^2 l (C_1^d \mathscr{D}_K^{1/2})^{1/(d-r)}$.

Proof
Considering Λ^s as a $\mathbb{Z}$-module of rank rs, the proof of [EV, Lem. 2.3] shows that there is a polynomial F of degree at most rl in the rs variables so that the elements of $S(\mathbf{y})$ are linearly independent over $\mathbb{Q}$ whenever $F(\mathbf{y}) \neq 0$. Lemma 2.4 of [EV] then shows that we can choose such a $\mathbf{y}$ whose coefficients are at most $(1/2)(rl+1) \leq rl$. It follows that

$$\|y_i\| \leq r^2 l (C_1^d \mathscr{D}_K^{1/2})^{1/(d-r)}$$

for $i = 1, 2, \ldots, s$. □

A.8
LEMMA
The number of number fields with degree $d \geq 200$ and discriminant of absolute value at most X is at most

$$(C_3 d)^{d \exp(C_4 \sqrt{\log d})} X^{\exp(C_5 \sqrt{\log d})}.$$

Proof
Fix once and for all a total ordering of $S(2l)$. We denote the order relation as $(k_1, \ldots, k_s) \prec (k_1', \ldots, k_s')$. Choose $S \subset S(l)$ of cardinality r as above.

Let K have degree d over $\mathbb{Q}$ and satisfy $\mathscr{D}_K < X$. Choose $\mathbf{y}$ as in Lemma A.7. By Lemma A.3, $S(2l)(\mathbf{y})$ spans K over $\mathbb{Q}$. It follows that there exists a subset $\Pi \subset S(2l)$ of size d such that $\{z_1, \ldots, z_d\} := \{y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s} : (k_1, k_2, \ldots, k_s) \in \Pi\}$ forms a $\mathbb{Q}$-basis for K and such that the ordering $z_1, \ldots, z_d$ conforms with the specified ordering on $\Pi \subset S(2l)$.

We apply Lemma A.4 to $\{z_1, \ldots, z_d\}$ and $\mathscr{C} = (1, y_1, y_2, \ldots, y_s)$. Then each product $u z_i z_j$ ($u \in \mathscr{C}$, $1 \leq i, j \leq d$) is contained in $S(4l+1)$.

Put $\mathbf{A} = (\mathrm{Tr}_{K/\mathbb{Q}}(y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s}))_{(k_1,k_2,\ldots,k_s) \in S(4l+1)}$. For each K, the collection of matrices $M(u)$ is determined by $\mathbf{A}$ and Π. Since $|\mathrm{Tr}_{K/\mathbb{Q}}(z)| \leq d\|\mathbf{y}\|^{4l+1}$ for any $z \in S(\mathbf{y}, 4l+1)$, the number of possibilities for $\mathbf{A}$ is at most $(d\|\mathbf{y}\|^{4l+1})^{|S(4l+1)|}$; since Π is a subset of $|S(2l)|$, the number of possibilities for Π is at most $2^{|S(2l)|}$.

Lemma A.4 now yields that the number of possibilities for the isomorphism class of K is at most $2^{|S(2l)|}(d\|\mathbf{y}\|^{4l+1})^{|S(4l+1)|}$. By our bound on $\|\mathbf{y}\|$, we now have that the number of possibilities for K is at most

$$2^{|S(2l)|}\left(d(r^2 l (C_1^d \mathscr{D}_K^{1/2})^{1/(d-r)})^{4l+1}\right)^{|S(4l+1)|}. \tag{16}$$

Note that $|S(4l+1)| = \binom{s+4l+1}{s}$.

Now, just as in the paragraph following [EV, (2.6)], we choose s to be the greatest integer less than $\sqrt{\log d}$ and l to be the least integer greater than $(ds!)^{1/s}$. Note

that $l < \exp(C_2\sqrt{\log d})$. Now, $|S(l)| = \binom{s+l}{s}$ is at least d, so we may choose r between $d/2$ and $3d/4$. In particular, $r^2 l < d^3$. Also, $\binom{s+4l+1}{s}$ is at most $10^s d$ and $|S(2l)| = \binom{s+2l}{s} \leq 6^s d$. Finally, $s < 2\sqrt{\log d}$.

Substituting these values into (16), we get the fact that the number of possible K is at most

$$2^{6^s d}\left(d(d^3(C_1^d X^{1/2})^{4/d})^{5\exp(C_2\sqrt{\log d})}\right)^{10^s d},$$

which is in turn at most

$$(C_3 d)^{d\exp(C_4\sqrt{\log d})} X^{\exp(C_5\sqrt{\log d})}.$$

□

A.9

PROPOSITION

There are absolute constants C_6, C_7 with

$$\log N(X) \leq C_6 \log X \exp(C_7\sqrt{\log\log X}).$$

Proof

By Minkowski's discriminant bound, there is an absolute constant $C_6 > 1$ such that $\mathscr{D}_K > C_6^{[K:\mathbb{Q}]}$ for any extension $K/\mathbb{Q}$. Therefore, we may take d to be bounded by a constant multiple of $\log X$. From Lemma A.8, it now follows that the logarithm of the number of extensions $K/\mathbb{Q}$ with $\mathscr{D}_K < X$ and $[K : \mathbb{Q}] \geq 200$ is bounded by $C_6 \log X \exp(C_7\sqrt{\log\log X})$. Trivial bounds suffice to show that the number of K with $\mathscr{D}_K < X$ and $[K : \mathbb{Q}] < 200$ is at most $C_8 X^{200}$. □

Acknowledgments. I am grateful to Alex Lubotzky for the lecture that inspired this project, and for many helpful discussions. I thank Tsachik Gelander, Alireza Salehi Golsefidy, Gopal Prasad, and the referees for their critical remarks and suggestions. Jordan Ellenberg and Akshay Venkatesh contributed valuable number-theoretic support to this project.

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